

OBSERVER-BASED STABILIZATION OF SYSTEMS WITH MONOTONIC NONLINEARITIES

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ABSTRACT

We design an observer-based control law for a class of systems that include monotonic nonlinearities of the unmeasured states. Our observer results in nonlinear error dynamics which can be represented as the feedback interconnection of a linear system and a time-varying multivariable sector nonlinearity. The convergence of the estimation error is guaranteed by an observer matrix that renders the linear part passive, and is computable with LMI software. The feedback design is completed by combining the observer with a control law that renders the plant input-to-state stable with respect to the state estimation error.

KeyWords: Output feedback, nonlinear observers, backstepping.

I. INTRODUCTION

Most of the existing nonlinear observer designs restrict the system to be linear in the unmeasured states. When the system equations depend nonlinearly on the unmeasured states, global stabilization by output feedback may be impossible as shown by the counterexamples of Mazenc *et al.* (1994).

Krener and Isidori (1983) and Bestle and Zeitz (1983) introduced *output injection observers* for systems in the form

$$\dot{x} = A_o x + \gamma(y, u), \quad y = C_o x, \quad (1)$$

where A_o and C_o are in observer canonical form:

$$A_o = \begin{bmatrix} 0 & & & \\ \vdots & I & & \\ 0 & \cdots & 0 & \end{bmatrix}, \quad C_o = [1 \ 0 \ \cdots \ 0]. \quad (2)$$

Krener and Isidori (1983) and Marino and Tomei (1991) have derived differential geometric conditions that characterize systems which can be transformed into (1). These results have been extended to the multi-output case

by Krener and Respondek (1985). Gauthier *et al.* (1992) and Tsinias (1991) have designed observers for globally Lipschitz nonlinear systems. For all these observers, the estimation error dynamics are linear or linearly bounded making linear stability analysis techniques applicable. For ship models, which satisfy linear bounds in the unmeasured states, Fossen and Strand (1999) have designed passive observers.

Even when an observer is available for a nonlinear system, the design of output feedback control laws is more difficult than the full-state feedback design because certainty equivalence does not hold. For a special structure of $\gamma(y, u)$, Marino and Tomei (1991) and Kanellakopoulos *et al.* (1992) have designed control laws with output injection observers. These control laws are not of the certainty equivalence type but, instead, guarantee input-to-state stability (ISS) with respect to the state estimation error.

In this paper we extend the applicability of observer-based designs to systems

$$\dot{x} = A_o x - \psi(x) + \gamma(y, u), \quad y = C_o x, \quad (3)$$

where the nonlinear dependence of $\psi(x)$ on the unmeasured states is monotonic. For such systems, we design observers that result in nonlinear error dynamics which can be represented as the feedback interconnection of a linear system and a time-varying sector nonlinearity. Then the convergence of the estimation error is guaranteed by choosing an observer matrix that renders the linear part passive. While the existence of such a matrix is not known *a priori*, by posing the problem as a linear matrix inequality (LMI), a numerical solution is efficiently computed if it exists.

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Our observer is used together with an ISS control law obtained by extending the backstepping design of Kanellakopoulos *et al.* (1992). Even when the estimation error system is only globally stable without being convergent, the resulting closed-loop system is globally stable thanks to the ISS property of the control law.

In Section 2 we characterize the allowable nonlinearities which yield the desired form of the estimation error dynamics. In Section 3 we design an observer matrix to guarantee stability of the error dynamics. In Section 4 we design ISS control laws by backstepping.

Notation and Definitions

- $\|\cdot\|$ denotes the Euclidean norm and for any measurable function $v(t)$, $\|v\|_\infty$ denotes $\text{ess.sup}_{t \geq 0} \{|v(t)|\}$. L_∞ denotes the Banach space of essentially bounded functions with norm $\|\cdot\|_\infty$.
- A function $\theta: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be class- K if it is continuous, increasing, and $\theta(0) = 0$. It is called class- K_∞ if, in addition, $\lim_{t \rightarrow +\infty} \theta(t) = +\infty$.
- A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be class- KL if for each $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is class- K and for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is decreasing and $\lim_{t \rightarrow +\infty} \beta(s, t) = 0$.
- A signal $x(t)$ is said to satisfy an input-to-state stability (ISS) bound with respect to w if there exist a class- KL function $\beta(\cdot, \cdot)$ and a class- K function $\theta(\cdot)$ such that

$$|x(t)| \leq \max \{ \beta(|x(0)|, t), \theta(\|w\|_\infty) \}.$$

The system $\dot{x} = f(x, w)$, $f(0, 0) = 0$, is said to be ISS with respect to w if $x(t)$ satisfies an ISS bound with respect to w .

II. STATE ESTIMATION ERROR DYNAMICS

The observer design in this paper is feasible for systems in the form (3) where $x \in \mathbb{R}^n$, $\psi(0) = 0$, A_o and C_o are as in (2), $\gamma(\cdot, \cdot)$ and $\psi(\cdot)$ are locally Lipschitz functions in the form

$$\psi(x) = \begin{bmatrix} 0 \\ k_2 x_2 + \tilde{\psi}_2(x_2) \\ \vdots \\ k_n x_n + \tilde{\psi}_n(x_n) \end{bmatrix}, \quad \gamma(y, u) = \begin{bmatrix} \gamma_1(y, u) \\ \gamma_2(y, u) \\ \vdots \\ \gamma_n(y, u) \end{bmatrix}, \quad (4)$$

where, for $i = 2, \dots, n$, $k_i \geq 0$ and $\tilde{\psi}_i(\cdot)$ are restricted to be monotone increasing.

Assumption 1. $\tilde{\psi}_i(\cdot)$ is either identically zero or strictly increasing, that is,

$$\sigma[\tilde{\psi}_i(x_i) - \tilde{\psi}_i(x_i - \sigma)] > 0 \quad \forall \sigma \neq 0, \quad \forall x_i \in \mathbb{R} \quad (5)$$

For system (3) we design the observer

$$\dot{\hat{x}} = A_o \hat{x} + L(y - C_o \hat{x}) - \psi(\hat{x}) + \gamma(y, u), \quad (6)$$

where $L = [l_1 \dots l_n]^T$ is yet to be determined.

To decompose $\psi(x)$ into its linear and nonlinear parts, we introduce

$$\tilde{\psi}(x) := [0 \quad \tilde{\psi}_2(x) \quad \dots \quad \tilde{\psi}_n(x)]^T, \quad (7)$$

and, by adding and subtracting $[0 \quad k_2 x_2 \quad \dots \quad k_n x_n]^T$, we rewrite (3) as

$$\dot{x} = A_k x - \tilde{\psi}(x) + \gamma(y, u), \quad (8)$$

where $A_k := A_o - \text{diag}(0, k_2, \dots, k_n)$. Similarly (6) becomes

$$\dot{\hat{x}} = A_k \hat{x} + L(y - C_o \hat{x}) - \tilde{\psi}(\hat{x}) + \gamma(y, u). \quad (9)$$

Then, the dynamics of the state estimation error

$$e := x - \hat{x} \quad (10)$$

are governed by

$$\dot{e} = (A_k - LC_o)e - B(\tilde{\psi}(x) - \tilde{\psi}(\hat{x})). \quad (11)$$

The $n \times n$ matrix B is obtained from the identity matrix by zeroing the columns that correspond to the zero entries in the vector $\tilde{\psi}$, that is,

$$b_{ij} = 0 \text{ if } i \neq j, \quad b_{ii} = \begin{cases} 0 & \text{if } i = 1 \text{ or } \tilde{\psi}_i(x_i) \equiv 0, \\ 1 & \text{otherwise.} \end{cases} \quad (12)$$

By this construction $\tilde{\psi}(x)$ depends only on Bx and, hence, $\tilde{\psi}(x) - \tilde{\psi}(\hat{x}) = \tilde{\psi}(x) - \tilde{\psi}(x - e)$ depends on Bx and $Be =: z$. To be able to analyze the error dynamics (11) independently of x , we view $\tilde{\psi}(x) - \tilde{\psi}(x - e)$ as a time-varying function of e ,

$$\phi(t, z) := \phi(t, Be) = \tilde{\psi}(x(t)) - \tilde{\psi}(x(t) - e), \quad (13)$$

which is well defined as long as $x(t)$ exists, that is on its maximal interval of existence $[0, t_f)$. On that interval we rewrite (11) as

$$\dot{e} = (A_k - LC_o)e - B\phi(t, z),$$

$$z = Be. \quad (14)$$

Assumption 1 guarantees that $\phi(t, z)$ has the property

$$z^T \phi(t, z) > 0 \quad \forall z \neq 0, \quad \forall t \in [0, t_f). \quad (15)$$

This sector property allows us to conclude stability of (14) via the Positive Real Lemma which requires $PB = B$ with $P = P^T > 0$. Here, the fact that B is singular by construction (12) provides the freedom for $P \neq I$.

Proposition 1. Consider the error system (14) with the sector property (15). If there exists $P = P^T > 0$ such that

$$(A_k - LC_o)^T P + P(A_k - LC_o) \leq 0 \text{ and } PB = B, \quad (16)$$

then the state estimation error $e(t)$ satisfies

$$|e(t)| \leq \kappa |e(0)| \quad \forall t \in [0, t_f], \quad (17)$$

where $\kappa := \sqrt{\lambda_{\max}(P) / \lambda_{\min}(P)}$.

Proof. With $V(e) = \frac{1}{2} e^T P e$ as a Lyapunov function, (14) and (16) imply

$$\dot{V} \leq -e^T B \phi(t, Be) = -z^T \phi(t, z) \leq 0 \quad \forall t \in [0, t_f], \quad (18)$$

and, hence,

$$e(t)^T P e(t) \leq e(0)^T P e(0) \quad \forall t \in [0, t_f], \quad (19)$$

from which (17) follows. ■

For output feedback stabilization, observer (6) is to be used with a control law $u = \alpha(y, \hat{x})$ which employs the available measurement y and the state estimates \hat{x} . Since certainty equivalence does not hold, we design a control law $u = \alpha(y, \hat{x})$ which renders the plant (3) input-to-state stable (ISS) with respect to the estimation error e . At this point we assume that such an ISS control law exists. We address its design in Section 4.

Assumption 2. There exists a locally Lipschitz control law $u = \alpha(y, \hat{x})$ such that the solutions of (3) satisfy

$$|x(t)| \leq \max \{ \beta(|x(0)|, t), \theta(\|e\|_\infty) \} \quad \forall t \geq 0, \quad (20)$$

with class- KL function $\beta(\cdot, \cdot)$ and a class- K function $\theta(\cdot)$.

Under this assumption

$$\sup_{t \in [0, t_f]} |x(t)| \leq \max \{ \beta(|x(0)|, 0), \theta(\kappa |e(0)|) \}, \quad (21)$$

that is, $x(t)$ is bounded on its maximal interval of existence. Thus, $t_f = \infty$ and

$$\|x(t)\|_\infty \leq \max \{ \beta(|x(0)|, 0), \theta(\kappa |e(0)|) \}. \quad (22)$$

By (17) and (22), the closed-loop formed of the plant (3),

observer (6) and the ISS control law $u = \alpha(y, \hat{x})$ is globally stable (GS).

We now summarize our results and show that global asymptotic stability (GAS) is achieved with an additional detectability condition.

Theorem 1. Suppose that L and $P = P^T > 0$ satisfy (16) and an ISS control law $u = \alpha(y, \hat{x})$ is designed as in Assumption 2. Then the closed-loop system (3), (6) is GS. If, in addition, the pair $(B, A_k - LC_o)$ is detectable, then the closed-loop system is GAS.

Proof. Global stability has already been proven. Since $x(t)$ and $\hat{x}(t)$ are in L_∞ , we conclude from (3) and (6) that $\dot{x}(t)$ and $\dot{\hat{x}}(t)$ are also in L_∞ . Therefore, $x(t)$ and $\hat{x}(t)$ are uniformly continuous on $[0, \infty)$. By the continuity of $\tilde{\psi}(\cdot)$ and the boundedness and uniform continuity of $x(t)$ and $\hat{x}(t)$, it follows from (13) that $z(t)^T \phi(t, z(t))$ is also uniformly continuous on $[0, \infty)$. Noting from (18) that

$$\lim_{T \rightarrow \infty} \int_0^T z(t)^T \phi(t, z(t)) dt \quad (23)$$

exists and is finite, it follows from Barbalat's lemma (Khalil, 1996, Lemma 4.2) that $z(t)^T \phi(t, z(t)) \rightarrow 0$ as $t \rightarrow \infty$. Then, from (15), $z(t) \rightarrow 0$ as $t \rightarrow \infty$. To conclude GAS, we note that the plant (3) and the observer (6) taken together form an autonomous system with bounded solutions. Therefore, LaSalle's invariance principle applies and solutions $x(t)$, $\hat{x}(t)$ converge to the largest invariant set $\Omega \subset \{(x, \hat{x}(t)) | z = Be = 0\}$. From (14) and detectability of $(B, A_k - LC_o)$, it follows that $e = 0$ in Ω . Then, we conclude from (20) that $\Omega = \{0\}$ which establishes GAS. ■

III. OBSERVER DESIGN

With Theorem 1, the observer design for system (3) is reduced to the problem of finding an observer matrix L such that (16) is satisfied with $P = P^T > 0$. In general, such L may not exist. However, by formulating the problem as a LMI, we can use the efficient numerical tools available for LMI's to determine whether the problem is feasible and, if so, to obtain a solution for L .

To prepare for the LMI formulation, we rewrite (16) as a matrix inequality

$$\begin{bmatrix} (A_k - LC_o)^T P + P(A_k - LC_o) & PB - B \\ BP - B & 0 \end{bmatrix} \leq 0, \quad (24)$$

which we want to solve for L and $P = P^T > 0$. This inequality is an LMI in P and $\Gamma = L^T P$:

$$\begin{bmatrix} A_k^T P + PA_k - C_o^T \Gamma - \Gamma^T C_o & PB - B \\ BP - B & 0 \end{bmatrix} \leq 0,$$

$$P = P^T > 0. \quad (25)$$

Example 1. For the open-loop unstable system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= x_3 - x_2(1 + x_2^2) + u, \\ \dot{x}_3 &= 2u, \\ y &= x_1, \end{aligned} \quad (26)$$

a root-locus argument for the linearization shows that the equilibrium $x = 0$ cannot be stabilized by static output feedback. An attempt to transform this system into the form (1) fails because the necessary conditions of Krener and Isidori (1983) are not satisfied. However, (26) is of the form (3) with

$$\psi(x) = \begin{bmatrix} 0 \\ x_2(1 + x_2^2) \\ 0 \end{bmatrix}, \quad \gamma(y, u) = \begin{bmatrix} y^2 \\ u \\ 2u \end{bmatrix}. \quad (27)$$

We observe that $\psi(x)$ can be written as in (4) with $k_2 = 1$, $\tilde{\psi}_2(x_2) = x_2^3$, $k_3 = 0$ and $\tilde{\psi}_3(x_3) \equiv 0$. Since $\tilde{\psi}_2(\cdot)$ is strictly increasing, Assumption 1 is satisfied, therefore, the observer (6) is applicable:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(y - \hat{x}_1) + y^2, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + l_2(y - \hat{x}_1) - \hat{x}_2(1 + \hat{x}_2^2) + u, \\ \dot{\hat{x}}_3 &= l_3(y - \hat{x}_1) + 2u. \end{aligned} \quad (28)$$

To design an L that satisfies Theorem 1, we solve the LMI (25) with

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$C_o = [1 \ 0 \ 0].$$

A solution is

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}^T. \quad (29)$$

Then, $L = P^{-1}\Gamma^T = [3 \ 3 \ 1]^T$ and the pair $(B, A_k - LC_o)$ is detectable. Thus, we conclude from Theorem 1 that the observer (28) with $l_1 = 3$, $l_2 = 3$ and $l_3 = 1$ guarantees GAS of the closed-loop system (26), (28) if an ISS control law $u = \alpha(y, \hat{x})$ can be designed.

IV. ISS CONTROL LAW DESIGN

ISS control laws have been designed by Marino and Tomei (1991) and Kanellakopoulos *et al.* (1992) for sys-

tems like (3), but with $\psi(x) \equiv 0$ and with $\gamma(y, u)$ restricted to be in the form

$$\gamma(y, u) = \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \vdots \\ \varphi_r(y) + p_r \rho(y)u \\ \varphi_{r+1}(y) + p_{r+1} \rho(y)u \\ \vdots \\ \varphi_n(y) + p_n \rho(y)u \end{bmatrix}, \quad (30)$$

where $p_r \neq 0$,

$$p(s) = p_r s^{n-r} + p_{r+1} s^{n-r-1} + \dots + p_{n-1} s + p^n \quad (31)$$

is a Hurwitz polynomial, $\varphi_i(0) = 0 \ i = 1, \dots, n$, and $\rho(y) \neq 0 \ \forall y \in \mathbb{R}$.

We now extend the backstepping design of Kanellakopoulos *et al.* (1992) to systems (3) with nonzero $\psi(x)$ in the form

$$\psi(x) = [0 \ \psi_2(x_2) \ \dots \ \psi_r(x_r) \ 0 \ \dots \ 0]^T, \quad (32)$$

where $\psi_i(0) = 0 \ i = 2, \dots, r$. We keep the restrictions (30) and (31) on $\gamma(y, u)$ and assume $\psi_i(\cdot)$ and $\varphi_i(\cdot)$ are smooth.

Substituting (30), we rewrite the plant (3) and observer (6) as

$$\begin{aligned} \dot{y} &= x_2 + \varphi_1(y), \\ \dot{x}_2 &= x_3 - \psi_2(x_2) + \varphi_2(y), \\ &\vdots \\ \dot{x}_r &= x_{r+1} - \psi_r(x_r) + \varphi_r(y) + p_r \rho(y)u, \\ \dot{x}_{r+1} &= x_{r+2} + \varphi_{r+1}(y) + p_{r+1} \rho(y)u, \\ &\vdots \\ \dot{x}_n &= \varphi_n(y) + p_n \rho(y)u, \end{aligned} \quad (33)$$

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(y - \hat{x}_1) + \varphi_1(y), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + l_2(y - \hat{x}_1) - \psi_2(\hat{x}_2) + \varphi_2(y), \\ &\vdots \\ \dot{\hat{x}}_r &= \hat{x}_{r+1} + l_r(y - \hat{x}_1) - \psi_r(\hat{x}_r) + \varphi_r(y) + p_r \rho(y)u, \\ \dot{\hat{x}}_{r+1} &= \hat{x}_{r+2} + l_{r+1}(y - \hat{x}_1) + \varphi_{r+1}(y) + p_{r+1} \rho(y)u, \\ &\vdots \\ \dot{\hat{x}}_n &= l_n(y - \hat{x}_1) + \varphi_n(y) + p_n \rho(y)u. \end{aligned} \quad (34)$$

An ISS control law is obtained by backstepping as follows:

Step 1. We set

$$\chi_1 = y, \quad (35)$$

and, using $x_2 = \hat{x}_2 + e_2$, we rewrite the first equation of (33) as

$$\dot{\chi}_1 = \hat{x}_2 + \varphi_1(y) + e_2. \quad (36)$$

Adding and subtracting the 'virtual' control law

$$\alpha_1(y) = -c_1\chi_1 - d_1\chi_1 - \varphi_1(y) \quad c_1 > 0, d_1 > 0, \quad (37)$$

in (36), we obtain

$$\dot{\chi}_1 = -c_1\chi_1 + \chi_2 - d_1\chi_1 + e_2, \quad (38)$$

where

$$\chi_2 := \hat{x}_2 - \alpha_1(y). \quad (39)$$

Step 2. From (33) and (34),

$$\begin{aligned} \dot{\chi}_2 = & \hat{x}_3 + l_2(y - \hat{x}_1) - \psi_2(\hat{x}_2) + \varphi_2(y) \\ & - \frac{\partial \alpha_1}{\partial y}(\hat{x}_2 + \varphi_1(y)) - \frac{\partial \alpha_1}{\partial y} e_2. \end{aligned} \quad (40)$$

We let $c_2 > 0, d_2 > 0$ and

$$\begin{aligned} \alpha_2(y, \hat{x}_1, \hat{x}_2) = & -\chi_1 - c_2\chi_2 - d_2 \left(\frac{\partial \alpha_1}{\partial y} \right)^2 \chi_2 - l_2(y - \hat{x}_1) \\ & + \psi_2(\hat{x}_2) - \varphi_2(y) + \frac{\partial \alpha_1}{\partial y}(\hat{x}_2 + \varphi_1(y)). \end{aligned} \quad (41)$$

Then, (40) becomes

$$\dot{\chi}_2 = -\chi_1 - c_2\chi_2 + \chi_3 - d_2 \left(\frac{\partial \alpha_1}{\partial y} \right)^2 \chi_2 - \frac{\partial \alpha_1}{\partial y} e_2, \quad (42)$$

where

$$\chi_3 := \hat{x}_3 - \alpha_2(y, \hat{x}_1, \hat{x}_2). \quad (43)$$

Step i. ($3 \leq i \leq r$) We proceed as in Step 2 and use the formulas

$$\begin{aligned} \alpha_i(y, \hat{x}_1, \hat{x}_2, \dots, \hat{x}_i) = & -\chi_{i-1} - c_i\chi_i - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \chi_i \\ & - l_i(y - \hat{x}_1) + \psi_i(\hat{x}_i) - \varphi_i(y) + \frac{\partial \alpha_{i-1}}{\partial y}(\hat{x}_2 + \varphi_1(y)) \\ & + \frac{\partial \alpha_{i-1}}{\partial \hat{x}_1}[\hat{x}_2 + l_1(y - \hat{x}_1) + \varphi_1(y)] \\ & + \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}[\hat{x}_{j+1} + l_j(y - \hat{x}_1) - \psi_j(\hat{x}_j) + \varphi_j(y)], \end{aligned} \quad (44)$$

$$\chi_{i+1} := \hat{x}_{i+1} - \alpha_i(y, \hat{x}_1, \hat{x}_2, \dots, \hat{x}_i), \quad (45)$$

where $c_i > 0, d_i > 0$ in each step. Our final control law is

$$\begin{aligned} u = \alpha(y, \hat{x}) = & \frac{-\chi_{r+1}}{p_r \rho(y)} \\ = & \frac{1}{p_r \rho(y)}[-\hat{x}_{r+1} + \alpha_r(y, \hat{x}_1, \dots, \hat{x}_r)], \end{aligned} \quad (46)$$

which is an ISS control law as we now prove.

Theorem 2. For system (33) with Hurwitz polynomial $p(s)$ in (31), the control law (46) obtained recursively from (35), (37), (39), (41), (43), (44), (45) satisfies the ISS Assumption 2.

Proof. With (46), the dynamics of χ are given by

$$\begin{aligned} \dot{\chi}_1 = & -c_1\chi_1 + \chi_2 - d_1 g_1(\chi)^2 \chi_1 + g_1(\chi) e_2, \\ & \vdots \\ \dot{\chi}_i = & -\chi_{i-1} - c_i\chi_i + \chi_{i+1} - d_i g_i(\chi)^2 \chi_i + g_i(\chi) e_2, \\ & \vdots \\ \dot{\chi}_r = & -\chi_{r-1} - c_r\chi_r - d_r g_r(\chi)^2 \chi_r + g_r(\chi) e_2, \end{aligned} \quad (47)$$

where $g_1(\chi) = 1, g_i(\chi) = -\frac{\partial \alpha_{i-1}}{\partial y}$ for $i = 2, \dots, r$. For (47), the derivative of $V(\chi) = \chi_1^2 + \dots + \chi_n^2$ satisfies, upon completion of squares,

$$\dot{V} \leq -cV + b(e), \quad (48)$$

where $c := \min_{i=1, \dots, r} \{c_i\}$, and $b(e) = \left(\frac{1}{4d_1} + \dots + \frac{1}{4d_r}\right) e_2^2$. From standard comparison theorems (see e.g. (Khalil, 1996, Lemma 2.5)), (47) is ISS with respect to e . Since $\chi(t)$ satisfies an ISS bound with respect to e , then $x_i = x_1$ and the application of (45) and $x_i = \hat{x}_i + e_i$ for $i = 2, \dots, r$ show that the first r components ($x_1(t), \dots, x_r(t)$) of $x(t)$ satisfy an ISS bound with respect to e .

To complete the proof we need to show that $x(t)$ satisfies an ISS bound with respect to e . To this end, we define the variables

$$\eta_i = x_{r+i} - \frac{p_{r+i}}{p_r} x_r, \quad i = 1, \dots, n-r, \quad (49)$$

governed by

$$\dot{\eta} = F\eta + G(x_1, x_r), \quad (50)$$

where

$$F = \begin{bmatrix} -\frac{p_{r+1}}{p_r} & & & \\ \vdots & & I & \\ -\frac{p_n}{p_r} & 0 & \dots & 0 \end{bmatrix},$$

$$G(x_1, x_r) = \begin{bmatrix} G_1(x_1, x_r) \\ \vdots \\ G_{n-r}(x_1, x_r) \end{bmatrix} \quad (51)$$

and, for $i = 1, \dots, n - r$,

$$G_i(x_1, x_r) = \varphi_{r+i}(y) - \frac{p_{r+i}}{p_r} \varphi_r(y) + \frac{p_{r+i}}{p_r} \psi_r(x_r) \\ + \left(\frac{p_{r+i+1}}{p_r} - \frac{p_{r+i} p_{r+1}}{p_r^2} \right) x_r.$$

From (50), the η -subsystem is driven by the $(x_1(t), \dots, x_r(t))$ -subsystem which is ISS with respect to e . The eigenvalues of F coincide with the roots of $p(s)$ in (31) therefore, F is Hurwitz and, hence, the η -subsystem is ISS with respect to $(x_1(t), x_r(t))$. Recalling that the cascade interconnection of ISS systems is ISS (Sontag, 1989, Proposition 7.2), we conclude that $x(t)$ satisfies an ISS bound with respect to e . ■

We now complete the observer-based control design of Example 1 by constructing an ISS control law.

Example 2. The system (26) has the form (33) with $r = 2$, $\psi_2(x_2) = x_2(1 + x_2^2)$, $\varphi_1(y) = y^2$, $\varphi_2(y) = \varphi_3(y) = 0$, $\rho(y) = 1$, and $p(s) = s + 2$ therefore, the above backstepping procedure applies.

By (35), (37) and (39), we let $c_1 > 0$, $d_1 > 0$ and obtain

$$\alpha_2(y) = -c_1 y - d_1 y - y^2, \quad \chi_2 = \hat{x}_2 + c_1 y + d_1 y + y^2.$$

Then, by (41),

$$\alpha_2(y, \hat{x}_1, \hat{x}_2) = -y - c_2 \chi_2 - d_2 (c_1 + d_1 + 2y)^2 \chi_2 \\ - 3(y - \hat{x}_1) + \hat{x}_2 (1 + \hat{x}_2^2) \\ - (c_1 + d_1 + 2y)(\hat{x}_2 + y^2),$$

where $c_2 > 0$, $d_2 > 0$. From (46), we obtain

$$u = -\hat{x}_3 + \alpha_2(y, \hat{x}_1, \hat{x}_2), \quad (52)$$

which is an ISS control law by theorem 2. Combining it with the observer designed in Example 1, we conclude that (26) is globally asymptotically stabilized by the observer-based control law (52), (28).

V. CONCLUSION

An observer-based stabilization design is presented for systems with monotonic nonlinearities in the unmeasured states. Our observer makes use of the fact that the estimation error system can be represented as the feedback interconnection of a linear system and a multivariable

sector nonlinearity. The design of this observer is constructive in the sense that the issues of existence and the evaluation of a passivating observer matrix L are resolved by efficient LMI computations. Global asymptotic stability of the closed-loop system is achieved by combining the observer with an ISS control law designed by backstepping. An extension to tracking is currently being developed.

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