TWO DECENTRALIZED HEADING CONSENSUS ALGORITHMS FOR NONLINEAR MULTI-AGENT SYSTEMS

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ABSTRACT

Inspired by Vicsek’s model, in this paper we propose two decentralized heading consensus algorithms for nonlinear multi-agent systems. The first algorithm, called WHCA, can be seen as a weighted Vicsek’s model. The second algorithm, named LBHCA, is a leader-follower strategy based on the WHCA. It is proved that, under a well-known connectivity assumption, the algorithm WHCA can ensure almost global consensus of the headings, except for the situation where they are initially balanced. For the LBHCA, the global heading consensus is guaranteed under the same connectivity assumption. Simulation results are provided to justify the proposed algorithms.

Key Words: Decentralized algorithms, multi-agent systems, consensus, Vicsek’s model.

I. INTRODUCTION

Recently, the rapid advances in communication, computation and miniaturization technologies have sparked enormous interest in building networked man-made multi-agent systems in numerous industrial and military fields. Application examples include mobile vehicle groups for various purposes, traffic control and management, rescue systems, information systems, network security systems, search teams, object recognition systems, cooperative decision making mechanism, path planning, task assignment, and production scheduling. The cooperation in these multi-agent systems can either improve the performance of the system with less cost or realize some goals which otherwise cannot be achieved. As we know, information sharing plays an important role in the coordination of a group of agents. In order to achieve common group objectives or react to unexpected external changes, the information, or the data, of all the agents in the group need to reach a common value or consensus [1]. Normally, the consensus of a multi-agent system is said to be reached if the states of interest of all the agents asymptotically converge to a common value.

In [2], the authors propose a simple model, now widely known as Vicsek’s model, that gives an amazing phase (heading) synchronization phenomenon of a group of self-driven particles. Among the early efforts towards explaining Vicsek’s model, [3] studies its linearized version and presents a sufficient condition for the consensus of the group in both the leaderless and leader-follower cases. Following this work is an intensive study of the well-known averaging consensus protocols [1, 4–12] and its generalized version [13–15]. However, much earlier contributions to averaging consensus problems have been done in the 1980s [16–18], where the studies were on more general settings which are motivated by different contexts, such as parallel computation, distributed optimization, and distributed signal processing [19]. Application of dynamic graph theory to multi-agents has been independently carried out by D. Siljak; see [20] and references therein. Also note that many of the above works rely heavily upon the

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results on the convergence of infinite product of non-negative matrices or stochastic matrices [21].

Recently, the original nonlinear Vicsek’s model has attracted a lot of research activities, which mostly focus on the global convergence issue. In [10, 13], it has been shown that the consensus of the group is guaranteed under a joint connectivity assumption if the initial headings of the agents are all confined to be in \((-\pi/2 + \beta, \pi/2 + \beta)\), where \(\beta\) is an arbitrary real number. This is not surprising in that when the initial headings of the agents are all within an interval of length \(\pi\), Vicsek’s model can be shown to be essentially an averaging consensus algorithm. In [10, 22], however, the authors present some examples to illustrate that the headings of a group may not achieve consensus if the above restriction for the initial headings is not imposed. In other words, Vicsek’s model is not a global heading consensus algorithm for multi-agent systems. Very recently, contributions to the construction of heading consensus algorithms with a global attractivity property are made by the authors of [23] and [24], in which a decentralized algorithm based on realizing a virtual centralized control is proposed. The main idea is to add an additional state to each agent, which, under some connectivity condition, achieves consensus asymptotically by linear averaging protocol. This state serves as the common heading reference for each agent and eventually all the headings of the group will converge to it almost surely. But there are two problems associated with this algorithm. First, the singularity is not guaranteed to be avoided. Secondly, although the algorithm can ensure the almost sure global consensus, one cannot specify the cases in which the consensus cannot be reached.

In this work, we propose two synchronous, but decentralized, algorithms that (almost) ensure the consensus of the headings: weighted heading consensus algorithm (WHCA) and leader based heading consensus algorithm (LBHCA). The WHCA can be seen as a weighted Vicsek model, where each agent in the group actively assigns time-varying weighting factors to its neighbors at each time instant, and determines its heading for next time instant by taking the weighted average headings of its neighbors. The proposed algorithm only uses local information in a local coordinate frame. Under a well-known connectivity assumption, this algorithm guarantees the consensus of headings of the group if the summation of the velocity vectors of all the agents does not vanish at the initial time instant. The LBHCA combines the weighted heading consensus strategy with the leader-follower idea. A leader is one agent in the group who has constant heading all the time, and is seen as an ordinary agent by the other agents. Each agent, except the leader, implements a weighted heading averaging algorithm similar to the WHCA. We prove that the LBHCA can ensure the global convergence of the headings of the group under the same connectivity assumption made for the WHCA. In particular, the counter-example for noiseless Vicsek’s model in [22] can be handled.

The rest of the paper is organized as follows: in Section II, some preliminaries and Vicsek’s model are reviewed. Our main results are introduced in Sections III and IV, where the WHCA and LBHCA algorithms are developed and analyzed, respectively. Simulation results are reported in Section V. Finally, some brief concluding remarks are offered in Section VI.

II. PRELIMINARIES AND VICSEK’S MODEL

2.1 Preliminaries

2.1.1 Notations

Throughout this paper, \(\mathbb{Z}^+ \subset \mathbb{Z}\) and \(\mathbb{N} \subset \mathbb{Z}\) are used to denote the integer sets \([0, 1, \ldots]\) and \([1, 2, \ldots]\), respectively. In addition, we use \(\mathbb{Z}^+_{k_1}, k_1, k_2 \in \mathbb{Z}, k_2 \geq k_1\) to denote the integer set \(\{k_1, k_1 + 1, \ldots, k_2\}\), and \(\mathbb{Z}^+_{k_1}, k_1 \in \mathbb{Z}\), to denote \(\{k_1, k_1 + 1, \ldots\}\).

2.1.2 Graph theory

To make this paper self-contained, we recall some basics of graph theory from the past literature, see, e.g., [25]. A directed graph \(G(V, E)\) consists of a vertex set \(V\) and an edge set \(E \subseteq V \times V\). For any \(i, j \in V\), the ordered pair \((i, j) \in E\) if and only if \(i\) is a neighbor of \(j\). Vertex \(i\) is said to have a self edge if \((i, i) \in E\). A directed path from vertex \(i\) to \(j\) is a sequence of directed edges \((u_1, u_2), (u_2, u_3), \ldots, (u_{n-2}, u_{n-1}), (u_{n-1}, v_n)\), where \(n \geq 1\), \(v_1 = i\), \(v_n = j\), and \(v_1, \ldots, v_n\) are distinct. A directed graph is said to be strongly connected if and only if for any \(i \in V\), there exists a directed path from \(i\) to any other vertex. A directed graph is said to have a spanning tree if and only if there exists a vertex \(i \in V\), called root, such that there is a directed path from \(i\) to any other vertex. A graph \(G(V, E)\) is undirected if and only if for any \(i, j \in V\), \((i, j) \in E\) implies \((j, i) \in E\), i.e., each edge in \(E\) is undirected. A path in an undirected graph is defined similarly to a directed path in a directed graph. An undirected graph is said to be connected if and only if there is a path between any pair of vertices.

In this paper, \(G(V, E(t)), t \in \mathbb{Z}\) is used to denote a time-dependent undirected graph at time instant \(t\) which consists of a time-invariant vertex set \(V\) and time-varying edge set \(E(t)\). \(\bigcup G(V, E(t))\) denotes
the graph composed of node set $V$ and edge set $\bigcup_{i \in \mathbb{Z}_{i_1}^N} E(t)$. For any $i, j \in V$, $(i, j)$ is said to be an edge of $\bigcup_{i \in \mathbb{Z}_{i_1}^N} G(V, E(t))$ if and only if $(i, j) \in \bigcup_{i \in \mathbb{Z}_{i_1}^N} E(t)$. A collection of graphs $\{G(V, E(t_1)), G(V, E(t_1 + 1)), \ldots, G(V, E(t_2))\}$, $t_2 \geq t_1$ is said to be jointly connected if and only if, for any $i, j \in V$, there is a path between $i$ and $j$ in $\bigcup_{i \in \mathbb{Z}_{i_1}^N} G(V, E(t))$. In this case, it is also said that all the vertices in $V$ are jointly connected across $\mathbb{Z}_{i_1}^N$.

2.1.3 Matrix notations

In this paper, we use $A(i, j), i = 1, \ldots, m, j = 1, \ldots, n$ to denote the element on the $i$th row and $j$th column of the matrix $A \in \mathbb{R}^{m \times n}$. For $A, B \in \mathbb{R}^{m \times n}$, $A \succ B$ if and only if $A(i, j) > B(i, j), i = 1, \ldots, m, j = 1, \ldots, n$. Let $C \in \mathbb{R}^{N \times N}$, $N \in \mathbb{N}$, be a non-negative matrix. We use $\mathcal{G}(C)$ to denote the graph whose vertex set is $\{1, \ldots, N\}$ and edge set is $\{(i, j) : C(j, i) > 0, i, j \in \{1, \ldots, N\}\}$. $C$ is said to be stochastic if $\sum_{i=1}^{N} C(i, j) = 1$ for all $i = 1, \ldots, N$, and doubly stochastic if, in addition, $\sum_{i=1}^{N} C(i, j) = 1$ for all $j = 1, \ldots, N$ [26]. Let $A, B \in \mathbb{R}^{N \times N}$ be two non-negative matrices, we use $A \sim B$ to represent the relation that $A(i, j) \neq 0$ if and only if $B(i, j) \neq 0$. $A$ is said to be type-symmetric if $A \sim A^T$ [10].

Throughout this paper, we use diag$(d_1, d_2, \ldots, d_N)$ to denote the $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_N$, respectively, and $1_N$ and $0_N$ to denote the $N \times 1$ column vector $[1, 1, \ldots, 1]^T$ and $[0, 0, \ldots, 0]^T$, respectively.

2.2 Vicsek’s model

In [2], Vicsek and his co-authors propose a simple discrete-time rule that describes the motion of a group of particles moving in a plane. ‘At each time step a given particle driven with a constant velocity assumes the average direction of motions of the particles in its neighborhood of radius $r$ with some random perturbation added’ [2]. In the following, we call these particles ‘agents’. Two agents are said to be neighbors if the distance between them is less than or equal to a positive constant $r$, particularly each agent is a neighbor of itself. In this paper, we are interested in the noiseless Vicsek’s model, which can be put as the follows: For all $i = 1, \ldots, N$ and $t \in \mathbb{Z}_{t_0}^{+\infty}$

$$x_i(t + 1) = x_i(t) + v_i(t)$$

(1)

$$v_i(t + 1) = \frac{\sum_{j \in \mathcal{N}_i(t)} v_j(t)}{\|\sum_{j \in \mathcal{N}_i(t)} v_j(t)\|}$$

(2)

where $x_i(t) \in \mathbb{R}^2$ and $v_i(t) \in \mathbb{R}^2$ are the position and velocity of agent $i$ at time $t$, respectively; $v$ denotes the constant speed of each agent; $N$ is the number of agents in the group; $\mathcal{N}_i(t)$ denotes the set of the neighbors of agent $i$ at time $t$. Clearly, the velocity vector $v_i(t)$ can be represented by $v_i(t) = [v \cos \theta_i(t), v \sin \theta_i(t)]^T$, where $\theta_i(t)$ is the angle between $v_i(t)$ and the $x$ axis of some given coordinate system. In the following, we say the headings of the group are balanced at time $t \in \mathbb{Z}_{t_0}^{+\infty}$ if $\sum_{i=1}^{N} \theta_i(t) = 0$. In addition, we use $n_i(t)$ to denote the number of agents in $\mathcal{N}_i(t)$.

But the model (1)–(2) has some problems. The first is the singularity problem: assume at some time instant, agent 1 has agents 1, 2, 3 as its neighbors. If the headings of these three agents are, respectively, $\theta_1(t) = 0, \theta_2(t) = 2\pi/3, \theta_3(t) = -2\pi/3$, then (2) is not well defined for agent 1 since $\sum_{i=1}^{3} \theta_i(t) = 0$. The second problem is pointed out independently by the authors of [10] and [22], i.e. Vicsek’s model does not guarantee the global convergence of the headings of the group. This can be illustrated in the following Example 1.

Example 1. (Example 1 in [22]) The parameters of a group of 12 agents are $0 < v \leq 0.1, r = 0.8$. The initial positions and headings of these agents are:

$$(x_i(t_0), y_i(t_0)) = \left(\cos \frac{(i - 1)\pi}{6}, \sin \frac{(i - 1)\pi}{6}\right),$$

$$i = 1, 2, \ldots, 12$$

$$\theta_i(t_0) = [(16 - i + 3(-1)^i)/6] \bmod(2\pi),$$

$$i = 1, 2, \ldots, 12.$$  

It is analyzed in [22] that, by Vicsek’s model, the headings of the group will keep balanced all the times while every agent reverses its heading at each time instant.

Another interesting thing is about a leader-follower case of Vicsek’s model. Unlike the corresponding result in the study of the linearized model [3], a group of agents with one (or more) leader(s), which has (have) the same constant heading(s), do not necessarily reach heading consensus under some well-known connectivity conditions that are sufficient for the consensus of the linearized model. This can be illustrated by Example 2.

Example 2. The initial positions and headings of the four agents in Fig. 1 are:

$$(x_1(t_0), y_1(t_0)) = (-1, 0), \quad \theta_1(t_0) = 0,$$

$$(x_2(t_0), y_2(t_0)) = (1, 0), \quad \theta_2(t_0) = -\pi,$$

$$(x_3(t_0), y_3(t_0)) = (1, 1), \quad \theta_3(t_0) = -\pi,$$

$$(x_4(t_0), y_4(t_0)) = (1, -1), \quad \theta_4(t_0) = -\pi.$$
Assume that agent 1 is the leader of the group; the parameters $r$ and $v$ are chosen as 1.5 and 1 respectively. By imposing the periodic boundary condition (see [2]) over a square $[-2, 2] \times [-2, 2]$, the group has periodic joint connectivity. However, it is easy to check that no agent will change its headings at any time.

For Vicsek’s model, the group can be described as a time-dependent undirected graph $G(V, E(t)), t \in \mathbb{Z}^+$. The agents, numbered from 1 to $N$, are the vertices in the vertex set $V := \{1, \ldots, N\}$. For any $i, j \in V$, $(i, j) \in E(t)$ if and only if agents $i$ and $j$ are neighbors at time $t$, i.e., agents $i$ and $j$ are within distance $r$ at time $t$. In the rest of this paper, we call such a graph ‘the graph of the group’. Note that, since the graph of the group is undirected and every vertex has self edges all the time, we have that if $C(t) \in \mathbb{R}^{N \times N}$ is the non-negative matrix such that $G(C(t)) = G(V, E(t))$, then $C(t)$ is type-symmetric and all the diagonal entries of $C(t)$ are positive.

III. A WEIGHTED AVERAGING HEADING CONSENSUS ALGORITHM

As mentioned in Section I, Vicsek’s model may not lead to consensus even if the joint connectivity of the group is satisfied. In this section, we study a weighted averaging heading consensus algorithm which can be seen as a weighted Vicsek’s model. The time-varying weights assigned are calculated with the help of a companion state associated with each agent. Under some joint connectivity assumption for the group, the proposed algorithm works for almost all initial conditions except when the initial headings are balanced.

3.1 Construction of the algorithm

By weighting, the original model (2) is modified to the following: $\forall t \in \mathbb{Z}^+_0$,

$$
v_i(t+1) = \frac{\sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t)v_j(t)}{\|\sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t)v_j(t)\|}
$$

(3)

where $\forall t \in \mathbb{Z}^+_0, \forall i \in V$, $\forall j \in \mathcal{N}_i(t)$, $\alpha_{ij}(t) > 0$. Note that if $\forall t \in \mathbb{Z}^+_0, \forall i \in V$, $\forall j \in \mathcal{N}_i(t)$, $\alpha_{ij}(t) = 1$, then (3) reduces to Vicsek’s model.

We rewrite (3) in the form as follows: $\forall i \in V$ and $t \in \mathbb{Z}^+_0$,

$$
\cos \theta_i(t+1) = \frac{\sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t) \cos \theta_j(t)}{q_i^w(t)},
$$

(4)

$$
\sin \theta_i(t+1) = \frac{\sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t) \sin \theta_j(t)}{q_i^w(t)}
$$

where $q_i^w(t)$ is defined as

$$
q_i^w(t) = \frac{\|\sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t)v_j(t)\|}{v} = \left(\sum_{k, l \in \mathcal{N}_i(t)} \alpha_{il}(t)\alpha_{lj}(t) \cos(\theta_k(t) - \theta_l(t))\right)^{1/2}.
$$

(5)

Letting $v_i^1(t) = \cos \theta_i(t), v_i^2(t) = \sin \theta_i(t)$, (4) can be further written in compact matrix form: $\forall t \in \mathbb{Z}^+_0$,

$$
V_x(t+1) = M_{\mathrm{c}}(t)V_x(t),
$$

$$
V_y(t+1) = M_{\mathrm{w}}(t)V_y(t)
$$

(6)

where

$$
V_x(t) = [v_1^1(t), v_2^1(t), \ldots, v_N^1(t)]^\top,
$$

$$
V_y(t) = [v_1^2(t), v_2^2(t), \ldots, v_N^2(t)]^\top.
$$

(7)
Next, decompose \(q \in \mathbb{R}^N \) where

\[
M_w(t) = [m_{ij}^w(t)] \in \mathbb{R}^{N \times N} \quad \text{with}
\]

\[
m_{ij}^w(t) = \begin{cases} \bar{q}_{ij}(t) \frac{z_{ij}(t)}{q_{ij}^w(t)} & \text{for } i \in V, \ j \in \mathcal{N}_i(t) \\ 0, & \text{otherwise}. \end{cases} \tag{8}
\]

The following lemma shows that the model (6) is invariant under the rotation of the coordinate system. For want of space, we omit its proof, which is straightforward.

**Lemma 1.** For any angle \(\beta\), let \(\tilde{v}_i^w(t) = \cos(\beta(t_i(t) + \beta)), \tilde{v}_j^w(t) = \sin(\beta(t_i(t) + \beta)), \forall i \in V, \forall t \in \mathbb{Z}^+\), and \(\tilde{V}_y(t) = [\tilde{v}_i^w(t), \ldots, \tilde{v}_N^w(t)]^\top\). Then \(\forall t \in Z_{+}^\infty\),

\[
\tilde{V}_y(t + 1) = M_w(t) \tilde{V}_y(t),
\]

where \(M_w(t)\) is defined as in (8).

Now, decompose the matrix \(M_w(t_0)\) as \(M_w(t_0) = D_t^w A_t^w\) where

\[
D_t^w = \text{diag}\left(\frac{\sum_{j \in \mathcal{V}_i(t_0)} z_{ij}(t_0)}{q_{ij}^w(t_0)}, \ldots, \frac{\sum_{j \in \mathcal{V}_N(t_0)} z_{Nj}(t_0)}{q_{Nj}^w(t_0)}\right),
\]

\[
A_t^w(i, j) = \begin{cases} \frac{z_{ij}(t_0)}{\sum_{j \in \mathcal{V}_i(t_0)} z_{ij}(t_0)} & j \in \mathcal{N}_i(t_0), \\ 0, & \text{otherwise}. \end{cases} \tag{10}
\]

Next, decompose \(M_w(t_0 + 1)D_t^w = D_{t+1}^w A_{t+1}^w\), where

\[
D_{t+1}^w = \text{diag}\left(\frac{\sum_{j \in \mathcal{V}_i(t_0 + 1)} z_{ij}(t_0 + 1)}{q_{ij}^w(t_0 + 1)}, \ldots, \frac{\sum_{j \in \mathcal{V}_N(t_0 + 1)} z_{Nj}(t_0 + 1)}{q_{Nj}^w(t_0 + 1)}\right),
\]

\[
A_{t+1}^w(i, j) = \begin{cases} \frac{z_{ij}(t_0 + 1)}{\sum_{j \in \mathcal{V}_i(t_0 + 1)} z_{ij}(t_0 + 1)} & j \in \mathcal{N}_i(t_0 + 1), \\ 0, & \text{otherwise}. \end{cases} \tag{11}
\]

so that we have \(M_w(t_0 + 1)M_w(t_0) = M_w(t_0 + 1)D_t^w A_t^w\).

Repeating the above procedures, the model (6) is turned into: \(\forall t \in Z^+_{\mathbb{Z}_0^+}\),

\[
V_i(t + 1) = D_t^w A_t^w \cdots A_{t_0 + 1}^w A_{t_0}^w V_i(t_0),
\]

\[
V_j(t + 1) = D_t^w A_t^w \cdots A_{t_0 + 1}^w A_{t_0}^w V_j(t_0),
\]

where

\[
D_t^w = \text{diag}(d_{1c}(t + 1), \ldots, d_{Nc}(t + 1)), \quad A_t^w(i, j) = \begin{cases} \frac{z_{ij}(t)}{\sum_{j \in \mathcal{N}_i(t)} z_{ij}(t)d_{ij}(t)}, & j \in \mathcal{N}_i(t) \\ 0, & \text{otherwise}. \end{cases} \tag{16}
\]

with the updating rule for the positive column vector \(d_{cw}(t) := [d_{1c}(t), \ldots, d_{Nc}(t)]^\top\) as

\[
d_{cw}(t_0) = 1_N,
\]

\[
d_{cw}(t + 1) = M_w(t)d_{cw}(t), \quad \forall t \in Z^+_{\mathbb{Z}_0^+}. \tag{17}
\]

Note that for any \(t \in Z^+_{\mathbb{Z}_0^+}\), \(A_t^w\) in (16) is a stochastic matrix. Here, our idea is to design the weighting factors \(z_{ij}(t)\) in a decentralized way such that \(\forall t \in Z^+_{\mathbb{Z}_0^+}\), \(A_t^w\)'s are doubly stochastic matrices with the nonzero entries uniformly lower bounded by a positive number. By doing this, we can guarantee that the heading consensus of the group will be reached under the following connectivity assumption if the initial headings of the group are not balanced.

**Assumption 1.** For any \(t \in \mathbb{Z}\), there exists \(T \in \mathbb{N}\), such that \(\forall K \in \mathbb{Z}^+, \) all the agents in the group are jointly connected across \(Z_{i + K+1}^+\).
To proceed further, we need a technical lemma on the convergence of product of stochastic matrices.

**Lemma 2.** [27] Suppose \( \{C(t)\}_{t=0}^{\infty} \) is a sequence of type-symmetric stochastic matrices in \( \mathbb{R}^{N \times N} \) with positive diagonal entries; and the nonzero entries of \( C(t), \forall t \in \mathbb{Z}_0^{\infty} \) are uniformly lower bounded by a positive real number \( \mu \). Furthermore, if the collection of graphs \( \{G(C(t) + KT)\}, \ldots, G(C(t) + (K + 1)T - 1)\} \) are jointly connected for some \( T \in \mathbb{N} \) and any \( K \in \mathbb{Z}^+ \), then \( \lim_{t \to \infty} C(t) \cdots C(t_0 + KT) = 1_N \eta K, \forall K \in \mathbb{Z}^+ \), where \( \gamma = 1 - \frac{\mu N + (K + 1)}{2} \) (19).

**Remark 1.** A similar result was obtained in Lemma 5.2.1 in [18] for an asynchronous consensus algorithm at the price of not being able to provide a value for the rate of convergence \( \gamma \).

**Proposition 1.** Suppose

- Assumption 1 holds,
- \( \forall t \in \mathbb{Z}_0^{\infty} \), \( A_t \)’s in (13), (14) are doubly stochastic matrices whose diagonal entries are positive, and nonzero entries are uniformly lower bounded by a positive number,
- the weighting factors \( z_{ij}(t), j \in N_i(t) \) are uniformly bounded above zero.

Then, all the headings of the group converge to a common value if the initial conditions satisfy \( \sum_{i=1}^{N} v_i(0) \neq 0 \).

**Proof.** By the assumptions, using Lemma 2, \( \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_x(t_0) \) and \( \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_y(t_0) \) exist. Let us denote

- \( \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_x(t_0) = \xi_x \top \top I_N \),
- \( \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_y(t_0) = \xi_y \top \top I_N \),

where \( \xi_x, \xi_y \in \mathbb{R} \). Since \( A_t^w \) is a doubly stochastic matrix for all \( t \in \mathbb{Z}_0^{\infty} \), which leads to \( I_N \top \top A_t^w = I_N \top \top \), we have

\[
\begin{align*}
I_N \top \top \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_x(t_0) &= I_N \top \top V_x(t_0) = I_N \top \top (\xi_x \top \top I_N) = N_\xi_x, \\
I_N \top \top \lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w V_y(t_0) &= I_N \top \top V_y(t_0) = I_N \top \top (\xi_y \top \top I_N) = N_\xi_y.
\end{align*}
\]

Now, from \( \sum_{i=1}^{N} v_i(t_0) \neq 0 \), we can conclude that \( \xi_x + \xi_y \) cannot be both zero. Hence, there exists an angle \( \beta \) such that \( \xi_x \cos(\beta) - \xi_y \sin(\beta) > 0 \). But from (19), it directly follows that

\[
\lim_{t \to +\infty} A_t^w \cdots A_{t_0+1}^w \tilde{V}_x(t_0) > 0_N
\]

which shows that, at some time instant \( t_1 \in \mathbb{Z}_0^{+\infty} \),

\[
\tilde{V}_x(t_1) = M_v(t_1 - 1) \cdots M_v(t_0 + 1) M_v(t_0) \tilde{V}_x(t_0)
\]

which is uniformly bounded by \( \min_{i=1}^{N} \tan(\tilde{\theta}_i(t+1)) = \sum_{j \in N_i(t)} \frac{z_{ij}(t) \cos(\tilde{\theta}_j(t))}{\sum_{j \in N_i(t)} z_{ij}(t) \cos(\tilde{\theta}_j(t))} \times \tan(\tilde{\theta}_j(t)), \quad i \in V
\]

Clearly, by (25), \( \min_{i=1}^{N} \tan(\tilde{\theta}_i(t)) \) is non-decreasing and \( \max_{i=1}^{N} \tan(\tilde{\theta}_i(t)) \) is non-increasing, which gives that \( \forall i, V \) and \( \forall i \in \mathbb{Z}_0^{\infty} \), \( \cos(\tilde{\theta}_i(t)) \) is uniformly lower bounded by \( \min_{i=1}^{N} \cos(\tilde{\theta}_i(t)) > 0 \). In virtue of Lemma 2, this, together with the uniform boundedness of \( \xi_{ij}(t) \) and Assumption 1, shows that \( \forall i, V \), \( \tan(\tilde{\theta}_i(t)) \to c \) as \( t \to +\infty \), where \( c \in \mathbb{R} \). This means that the headings of the group reach consensus.

In the rest of this section, we present a decentralized algorithm for designing the weighting factors \( \xi_{ij}(t) \), in which \( d_{ii}(t) \) as \( d_{ii}(t) \) as in (17) is considered as a companion state of agent \( i \) at time \( t \in \mathbb{Z}_0^{\infty} \). The main idea is to select some desired value, in a decentralized way, for the matrix \( A_t^w \) in (16), which can be achieved by assigning the weights using only local information.
Weighted Heading Consensus Algorithm (WHCA)

At each time instant \( t \in \mathbb{Z}_0^{\infty} \), \( \forall i \in V \), agent \( i \) takes
the following steps:

1. If \( t = t_0 \), sets the companion state \( d_{cw}^i(t_0) = 1 \).
2. Selects the desired nonzero entries of the matrix \( A^w_i \),
   which are denoted by \( \hat{A}^w_i(i, j), j \in \mathcal{N}_i(t) \)
   a. Sends the data \( 1/n_i(t) \) to all its neighbors; (because
   the inter-agent communication is bidirectional, agent \( i \) receives
   the data \( 1/n_j(t) \) from agent \( j \in \mathcal{N}_i(t) \).
   b. Chooses \( \hat{A}^w_i(i, j), j \in \mathcal{N}_i(t) \setminus \{i\} \) as

   \[
   \hat{A}^w_i(i, j) = \min\{1/n_i(t), 1/n_j(t)\} \quad (26)
   \]
   and sets \( \hat{A}^w_i(i, i) \) as

   \[
   \hat{A}^w_i(i, i) = 1 - \sum_{j \in \mathcal{N}_i(t), j \neq i} \hat{A}^w_i(i, j). \quad (27)
   \]
3. Determines the weighting factors \( \varpi_{ij}(t), j \in \mathcal{N}_i(t) \)
   a. Sends the data \( d_{cw}^j(t) \) to all its neighbors. (Receives \( d_{cw}^j(t) \)
   from agent \( j \in \mathcal{N}_i(t) \).
   b. Sets \( \hat{\varpi}_{ij}(t) = 1 \), and according to (16), computes
   \( \hat{\varpi}_{ij}(t), j \in \mathcal{N}_i(t) \setminus \{i\} \) as

   \[
   \hat{\varpi}_{ij}(t) = \frac{\hat{A}^w_i(i, j)d_{cw}^j(t)}{\hat{A}^w_i(i, i)d_{cw}^j(t)} \hat{\varpi}_{ij}(t) = \frac{\hat{A}^w_i(i, j)d_{cw}^j(t)}{\hat{A}^w_i(i, i)d_{cw}^j(t)} \hat{\varpi}_{ij}(t) \quad (28)
   \]
   c. Calculates \( q_i^w(t) \) according to (5) using \( \hat{\varpi}_{ij}(t), j \in \mathcal{N}_i(t) \). If \( q_i^w(t) = 0 \), i.e. singularity
   occurs, then stop the algorithm (we will discuss the singularity
   avoidance issue in the following); otherwise, set \( \varpi_{ij}(t) = \hat{\varpi}_{ij}(t), j \in \mathcal{N}_i(t) \).
4. Updates heading and the companion state \( d_{cw}^i(t) \)
   a. Updates its heading according to (3).
   b. By (8) and (17), updates \( d_{cw}^i(t) \) as

   \[
   d_{cw}^i(t + 1) = \sum_{j \in \mathcal{N}_i(t)} \frac{\varpi_{ij}(t)}{q_i^w(t)} d_{cw}^j(t). \quad (29)
   \]

By (28), we also have

\[
q_i^w(t + 1) = \frac{1}{q_i^w(t)} \sum_{j \in \mathcal{N}_i(t)} \frac{\hat{\varpi}_{ij}(t)}{\hat{A}^w_i(i, j)} \hat{A}^w_i(i, i) \hat{\varpi}_{ij}(t) d_{cw}^j(t) \frac{\hat{A}^w_i(i, j)}{\hat{A}^w_i(i, i)} d_{cw}^i(t). \quad (30)
\]

Remark 2. Here, some points should be noted:

- WHCA is a totally decentralized weighting factors assignment algorithm. Each agent, say agent \( i \), needs to acquire from its neighbor \( j \) the data \( 1/n_i(t) \), \( d_{cw}^j(t) \) and the relative headings of \( j \) with respect to itself.
- If \( q_i^w(t) \neq 0 \), WHCA can make the matrix \( A^w_i \), \( t \in \mathbb{Z}_0^{\infty} \) in (16) doubly stochastic. To see this, note that
(26) implies that \( A^w_i \) is a symmetric matrix with
nonzero entries on the \( i \)th row and \( j \)th column if
agent \( j \) is a neighbor of \( i \) at time \( t \), and (27) shows
that all the row sums and column sums of \( A^w_i \) are 1.
In addition, it is very easy to justify that the nonzero
entries of \( A^w_i \) are uniformly lower bounded by \( 1/N \).
- In Step 3, we pick \( \hat{\varpi}_{ij}(t) \) equal to 1 because the model
(3) is invariant if each agent factors all the weights
for its neighbors by the same positive constant.
- WHCA is a synchronous algorithm that requires synchro-
nized clock for every agent in the group. Further
it is assumed that the communication between
neighboring agents can be completed in negligible
time. Relaxation of these two conditions is a topic
of our current research.

According to Lemma 1, there is another
important issue to discuss, that is, we should guarantee
that the weighting factors \( \varpi_{ij}(t), j \in \mathcal{N}_i(t) \), are
uniformly bounded above zero. In the following lemma,
we show that this is the case if \( \sum_{j=1}^{N} v_j(t_0) \neq 0 \) and
\( \forall t \in \mathbb{Z}_0^{\infty}, \forall i \in V, q_i^w(t) \neq 0 \).

Lemma 3. Suppose Assumption 1 holds and \( \sum_{j=1}^{N} v_j(t_0) \neq 0 \). If further, \( \forall t \in \mathbb{Z}_0^{\infty} \) and \( \forall i \in V, q_i^w(t) \neq 0 \),
then by WHCA, \( \forall i \in V, \forall t \in \mathbb{Z}_0^{\infty} \) the weighting
factors \( \varpi_{ij}(t) \) satisfy

\[
\varpi_i \leq \varpi_{ij}(t) \leq \varpi_i, \quad j \in \mathcal{N}_i(t) \quad (31)
\]
where \( 0 < \varpi_i, \varpi_i < +\infty \).
Proof. From (30), we know that \(d_{ij}^w(t) < +\infty\) if \(q_i^w(t) \neq 0\), which, by (28), leads to \(z_{ij}(t) > 0\), \(\forall t \in Z_0^+\), \(\forall i \in V\), \(j \in \mathcal{N}_i(t)\). In addition, in the proof of Proposition 1, we see that in (19), \(r_x^w\) and \(c_x^w\) cannot be both zero if \(\sum_{i}^{N} v_i(t_0) \neq 0\). Without loss of generality, we assume \(r_x^w \neq 0\). Let \(\psi(t) = A_t^w \cdots A_{t_0+1}^w V_i(t_0)\), the fact \(\psi_x^w \neq 0\) implies that there exists a time instant \(t_1 \in Z_0^+\) such that \(\forall t \geq t_1\), \(\psi_x(t)\) has the same sign for each \(i \in V\) and \(|\psi_x(t)| \geq |c_x^w|/2\). Thus, by (13), we have \(\forall t > t_1\), \(d_{ij}^w(t)\) is uniformly upper bounded, which also says that \(d_{ij}^w(t)\) is uniformly upper bounded for all \(t \in Z_0^+\). On the other hand, it is not difficult to see from (8), (5) that \(\forall i \in V\), \(\forall t \in Z_0^+\), \(\sum_{j=1}^{M} m_{ij}(t) \geq 1\), which, combined with (17), gives \(\min_{i=1}^{N} d_{ijw}(t)\) is non-decreasing. Hence, \(d_{ijw}(t)\) is uniformly lower and upper bounded by positive numbers for all \(i \in V\) and \(t \in Z_0^+\). Finally, by (28), (31) holds. \(\square\)

Based on Lemma 3 and Proposition 1, we can now make the following claim:

Proposition 2. Suppose Assumption 1 holds and \(\sum_{i=1}^{N} v_i(t_0) \neq 0\). By WHCA, all the headings of the group converge to a common value if in Step 3, \(q_i^w(t) \neq 0\) for all \(i \in V\) and \(t \in Z_0^+\).

3.2 Singularity avoidance

Indeed, the possibility of the occurrence of singularity can be avoided by perturbing the weighting factors. Suppose that at some time \(t \in Z_0^+\), agent \(i\) has \(q_i^w(t) = 0\) in Step 3c of WHCA. Then if \(z_{ij}(t)\) (i.e. the weight agent \(i\) assigns to itself) is perturbed with a positive number (while keeping \(z_{ij}(t), j \in \mathcal{N}_i(t)\) unchanged) we will have the newly calculated \(q_i^w(t) \neq 0\) no matter what the number is. This idea gives the following modified WHCA:

Modified WHCA

At each time instant \(t \in Z_0^+\), for any \(i \in V\), agent \(i\) takes the following four steps:

1. Step 1 of WHCA.
2. Step 2 of WHCA.
3. Initially determines the weighting factors \(z_{ij}(t), j \in \mathcal{N}_i(t)\)
   a. Step 3a of WHCA.
   b. Step 3b of WHCA.
   c. Calculates \(q_i^w(t)\) according to (5) using \(z_{ij}(t), j \in \mathcal{N}_i(t)\). If \(q_i^w(t) \neq 0\), set \(z_{ij}(t) = z_{ij}(t), j \in \mathcal{N}_i(t)\) and goes to Step 5; otherwise, goes to Step 4.
4. Sets \(z_{ij}(t) = z_{ij}(t) + \delta_z\) and \(z_{ij}(t) = z_{ij}(t), j \in \mathcal{N}_i(t)\), \(\forall i \in \mathcal{N}_i(t)\)\{\[\delta_x\] and \(z_{ij}(t) = z_{ij}(t), j \in \mathcal{N}_i(t)\), \(\forall i \in \mathcal{N}_i(t)\).
5. Step 4 of WHCA, but here only (29) can be used.

In the following proposition, we prove that the singularity issue in the last subsection can be suppressed by the modified WHCA presented above (referred to as WHCA later on). Note that if singularity happens on agent \(i \in V\) at time \(t \in Z_0^+\) for the initially chosen weighting factors \(z_{ij}(t), j \in \mathcal{N}_i(t)\), it can be seen from (3) that, by WHCA we have \(v_i(t+1) = v_i(t)\) for any perturbation \(\delta_z > 0\). This allows us to prove the consensus result only for sufficiently small \(\delta_z\).

Proposition 3. Suppose Assumption 1 holds. By WHCA, all the headings of the group converge to a common value if \(\sum_{i=1}^{N} v_i(t_0) \neq 0\).

Proof. By WHCA, if \(q_i^w(t) \neq 0\), \(\forall i \in V\), \(\forall t \in Z_0^+\) in Step 3c, then the headings consensus result of the group follows from Proposition 2.

Let \(\hat{A}_i^w\) be the matrix given by (26) and (27). Clearly, we have \(A_i^w = \hat{A}_i^w\) when \(t \in \{\tau \in Z_0^+ : q_i^w(t) \neq 0\}\) in Step 3c of WHCA, \(\forall i \in V\). Now, if there are some \(t \in Z_0^+\) and some \(i \in V\) such that \(q_i^w(t) = 0\) in Step 3c of WHCA, then by Step 4 of WHCA, \(z_{ij}(t) = \hat{z}_{ij}(t) + \delta_z\), where \(\delta_z > 0\); and \(z_{ij}(t) = \hat{z}_{ij}(t), j \in \mathcal{N}_i(t)\). By (16), this implies \(A_i^w(i, j) = \hat{A}_i^w(i, j) + \delta_{ij}(t), \forall i \in \mathcal{N}_i(t)\), where the perturbations \(\delta_{ij}(t)\)’s satisfy

\[
\sum_{j \in \mathcal{N}_i(t)} \delta_{ij}(t) = 0, \quad (32)
\]

\[
\hat{z}_{ij}(t) = (\hat{A}_i^w(i, j) + \delta_{ij}(t))d_{ijw}(t) - (\hat{A}_i^w(i, i) + \delta_{ii}(t))d_{ijw}(t)\]

\[
\quad j \in \mathcal{N}_i(t)\{i\}. \quad (33)
\]

Indeed, \(\delta_{ij}(t), j \in \mathcal{N}_i(t)\) can be solved directly from (28), (32) and (33) as

\[
\delta_{ii}(t) = \frac{(1 - \hat{A}_i^w(i, i))\hat{A}_i^w(i, i)\delta_x}{1 + \hat{A}_i^w(i, i)\delta_x}, \quad (34)
\]

\[
\delta_{ij}(t) = \frac{-\hat{A}_i^w(i, j)\hat{A}_i^w(i, i)\delta_x}{1 + \hat{A}_i^w(i, i)\delta_x}, \quad j \in \mathcal{N}_i(t)\{i\}. \quad (35)
\]

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\[
|\delta_{ij}(t)| \leq \delta_{2}, \quad \forall j \in \mathcal{N}_i(t).
\]

From (34) and (35) we have that
\[
|\delta_{ij}(t)| \leq \delta_{2}, \quad \forall j \in \mathcal{N}_i(t).
\]

Let \( \hat{\psi}(t) = [\hat{\psi}_1(t), \ldots, \hat{\psi}_N(t)]^\top = \hat{A}_t^w \cdots \hat{A}_{t_0+1}^w \hat{A}_{t_0}^w V_x(t_0). \) Then, from the proof of Proposition 1, without loss of generality, we can assume that
\[
\lim_{t \to +\infty} \hat{\psi}(t) = \hat{\psi}_x(t) = 0.
\]

Hence there exists a time instant \( t_1 \in \mathbb{Z}_{t_0}^{+\infty} \) such that \( \forall t \geq t_1, \hat{\psi}(t) \) has the same sign for each \( i \in V \) and \( |\hat{\psi}_i(t)| \geq |\hat{\psi}_x(t)|/2. \) Furthermore, by Lemma 2, \( t_1 \) can be taken to depend only on the upper bound of the initial condition \( V_x(t_0) \) and the lower bound of the nonzero entries of \( \hat{A}_t^w \), which can be chosen as \( 1/N. \) Now, let
\[
\psi(t) = [\psi_1(t), \ldots, \psi_N(t)]^\top = A_t^w \cdots A_{t_0+1}^w A_{t_0}^w V_x(t_0).
\]

It follows from (36) that there exists a sufficiently small constant \( b_2 > 0 \) independent of \( t_0 \) such that, in WHCA, if \( \delta_2 < b_2, \) then \( \psi_i(t_1) \) has the same sign for each \( i \in V, \) and \( |\psi_i(t_1)| \geq |\psi_x(t_1)|/4. \) What is more, by the fact that \( A_t^w \) is a stochastic matrix for all \( t \in \mathbb{Z}_{t_0}^{+\infty}, \) we know that this property for \( \psi_i(t) \) is kept for all \( t \geq t_1. \) Thus, along the same lines as the proof of Lemma 3, we guarantee that \( x_{ij}(t) \) is uniformly bounded above zero. The rest of the proof is similar to that of Proposition 1, hence omitted.

\[\Box\]

IV. A LEADER BASED HEADING CONSENSUS ALGORITHM

Based on WHCA, in this section we discuss the consensus of the group with a leader. By leader, we mean one agent in the group who has a constant heading all the time, and is seen as an ordinary agent by the other agents. It has been mentioned in Example 2 that the heading consensus of the group may not be guaranteed if the followers update their headings according to noiseless Vicsak’s model. In the following, we propose a leader based consensus algorithm LBHCA where the heading evolutions of the followers are governed by a weighted averaging algorithm similar to that stated in the last section. We will prove that, under Assumption 1, the proposed algorithm is successful in realizing the global heading consensus of the group. In the following, without loss of generality, we assume agent \( N \) is the leader in the group.

Formally, the model of a group of agents with a leader can be put into the form: \( \forall t \in \mathbb{Z}_{t_0}^{+\infty}, \)
\[
v_t(t+1) = \frac{\sum_{j \in \mathcal{N}_t(t)} x_{ij}(t)v_j(t)}{\sum_{j \in \mathcal{N}_t(t)} x_{ij}(t)v_j(t)} v_t, \quad i \in V \setminus \{N\}
\]
\[
v_N(t+1) = v_N(t).
\]

The corresponding matrix form is
\[
V_t(t+1) = M_t(t)V_t(t),
\]
\[
V_t(t+1) = M_t(t)N_t(t)
\]
where \( M_t(t) = \) \( M_t(t) \) as in (7), and
\[
m_{ij}(t) = \begin{cases}
\frac{x_{ij}(t)}{q_{ij}(t)}, & i \in V \setminus \{N\}, \ j \in \mathcal{N}_i(t) \\
1, & i = j = N \\
0, & \text{otherwise}
\end{cases}
\]
\[
q_{ij}(t) = \left( \frac{\|\sum_{j \in \mathcal{N}_i(t)} x_{ij}(t)v_j(t)\|}{v} \right)^{1/2}
\]
\[
= \left( \sum_{k,l \in \mathcal{N}_i(t)} x_{kl}(t)x_{ij}(t)\cos(\theta_k(t) - \theta_l(t)) \right)^{1/2}.
\]

As in previous section, this model can be turned into: \( \forall t \in \mathbb{Z}_{t_0}^{+\infty}, \)
\[
V_t(t+1) = D_t^iA_t^i \cdots A_{t_0+1}^i A_{t_0}^i V_x(t_0),
\]
\[
V_t(t+1) = D_t^iA_t^i \cdots A_{t_0+1}^i A_{t_0}^i V_y(t_0),
\]
where
\[
D_t = \text{diag}(d^{1}_{cl}(t), \ldots, d^{N}_{cl}(t + 1)),
\]
\[
A_t^i(i,j) = \begin{cases}
\frac{x_{ij}(t)d_{ij}(t)}{\sum_{j \in \mathcal{N}_i(t)} x_{ij}(t)d_{ij}(t)^2}, & i \in V \setminus \{N\}, \ j \in \mathcal{N}_i(t) \\
1, & i = j = N \\
0, & \text{otherwise}
\end{cases}
\]
with the updating rule for the positive column vector
\[
d_{cl}(t) := [d^{1}_{cl}(t), \ldots, d^{N}_{cl}(t)]^\top
\]
as
\[
d_{cl}(t_0) = 1_N,
\]
\[
d_{cl}(t+1) = M_t(t)\bar{d}_{cl}(t), \quad \forall t \in \mathbb{Z}_{t_0}^{+\infty}.
\]

Here, it is not difficult to see that \( d^{N}_{cl}(t) = 1 \) for all \( t \in \mathbb{Z}_{t_0}^{+\infty}. \)

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In the rest of this section, we propose a simple leader-follower algorithm which is effective in forcing the group to reach consensus globally. Similar as in WHCA, we consider $d_{ij}(t), \forall i \in V$ in (46) as a companion state of agent $i$ at time $t \in \mathbb{Z}_0^+$. 

**Leader Based Heading Consensus Algorithm (LBHCA)**

At each time instant $t \in \mathbb{Z}_0^+, \forall i \in V$, agent $i$ takes the following steps:

1. If $t = t_0$, sets the companion state $d_{ij}(t_0) = 1$.
2. Selects the desired nonzero entries of the matrix $A^T_i$, which are denoted by $\hat{A}^T_i(i, j), j \in \mathcal{N}_i(t)$.
   
   If $i \neq N$, $\hat{A}^T_i(i, j), j \in \mathcal{N}_i(t)$ is chosen as
   
   $$\hat{A}^T_i(i, j) = \frac{1}{n_i(t)}.$$  
   
   (47)

3. Initially determines the weighting factors $\alpha_{ij}(t), j \in \mathcal{N}_i(t)$
   
   a. Sends the data $d_{ij}(t)$ to all its neighbors (receives $d_{ij}(t)$ from $j \in \mathcal{N}_i(t)$).
   
   b. If $i \neq N$, sets $\tilde{\alpha}_{ij}(t) = 1$, and according to (45), computes $\tilde{\alpha}_{ij}(t), j \in \mathcal{N}_i(t), j \neq i$ as
   
   $$\tilde{\alpha}_{ij}(t) = \frac{\hat{A}^T_i(i, j)d_{ij}(t)}{\hat{A}^T_i(i, i)d_{ij}(t)} \tilde{\alpha}_{ij}(t) = \frac{d_{ij}(t)}{d_{ij}(t)}.$$  
   
   (48)

4. Sets $\alpha_{ij}(t) = \tilde{\alpha}_{ij}(t) + \delta_x$ and $\alpha_{ij}(t) = \tilde{\alpha}_{ij}(t), j \in \mathcal{N}_i(t) \setminus \{j\}$, where $\delta_x$ is any positive number. Then recalculates $q_{ij}(t)$ (not zero for any $\delta_x > 0$) using $\alpha_{ij}(t), j \in \mathcal{N}_i(t)$.

5. Updates heading and the companion state $d_{ij}(t)$
   
   a. Updates heading according to (37) and (38).
   
   b. According to (40) and (46), updates $d_{ij}(t)$ using the rule
   
   $$d_{ij}(t + 1) = \begin{cases} \sum_{j \in \mathcal{N}_i(t)} \frac{\alpha_{ij}(t)}{q_{ij}(t)}d_{ij}(t), & \text{if } i \neq N \quad (49) \\ 1, & \text{otherwise.} \end{cases}$$

Now we show that, under Assumption 1, LBHCA guarantees the global heading consensus of the group. For the same reason stated before Proposition 3, we only need to prove this result for small $\delta_x$.

Lemma 4. ([13]) Suppose $\{C(t)\}_{t_0}^\infty$, $t_0 \in \mathbb{Z}$, is a sequence of stochastic matrices in $\mathbb{R}^{|N| \times |N|}$ whose diagonal entries are positive, and all the nonzero entries are uniformly lower bounded by a positive real number. If, further, there exists $T \in \mathbb{Z}^+$ such that for all $t \in \mathbb{Z}_0^+$, the graph $U_{k=T}^T \mathcal{G}(k)$ has a spanning tree, then $\lim_{t \to \infty} C(t) \cdots C(t_0 + 1)C(t_0) = I_Nc_0$, where $c_0 \in \mathbb{R}^{|N| \times |N|}$. Moreover, $\forall t_0 \in \mathbb{Z}$ and $\forall \varepsilon > 0$, $\exists T_\varepsilon$ such that $\forall t > t_0 + T_\varepsilon, \|C(t) \cdots C(t_0 + 1)C(t_0) - I_Nc_0\| \leq \varepsilon$.

Proposition 4. Suppose Assumption 1 holds, then by LBHCA, all the headings of the group converge to a common value. In addition, $\forall i \in V \setminus \{N\}, \forall j \in \mathcal{N}_i(t)$, $\lim_{t \to \infty} d_{ij}(t) = 1$, $\lim_{t \to \infty} \alpha_{ij}(t) = 1$.

**Proof.** Let $\delta_x < 1/|N|$. First, it can be shown that, at each time $t \in \mathbb{Z}_0^+$, LBHCA makes $A_i^T$ a stochastic matrix whose nonzero entries are uniformly lower bounded
Since \( \lim_{t \to +\infty} q^l_k(t) \neq 0, \forall i \in V, \forall t \in \mathbb{Z}_{t_0}^{+\infty} \) in Step 3c of LBHCA, the nonzero entries of \( A^l_k \) are uniformly lower bounded by \( 1/N \); otherwise, by similar analysis as in the proof of Proposition 3, a lower bound for the nonzero entries of \( A^l_k \) is \( 1/N - \delta_z \). Second, by Assumption 1, it is not difficult to see that for all \( t \in \mathbb{Z}_{t_0}^{+\infty} \), the graph \( \{d_{i,j}^T \} \) has a spanning tree with a root at agent \( N \). Thus, by Lemma 4, it follows that \( \lim_{t \to +\infty} A^l_0 \cdots A^l_{t_0} + A^l_{t_0} = 1/c \), where \( c \) is a constant row vector. Further, it can be seen that \( c = [0, \ldots, 0, 1] \) since the last row of \( A^l_k \) is invariant over time. Hence, we have

\[
\lim_{t \to +\infty} A^l_0 \cdots A^l_{t_0} + A^l_{t_0} V_x(t_0) = 1/c V_x(t_0),
\]

\[
\lim_{t \to +\infty} A^l_0 \cdots A^l_{t_0} + A^l_{t_0} V_y(t_0) = 1/c V_y(t_0). \tag{50}
\]

Since \( v_x^N(t_0) \) and \( v_y^N(t_0) \) can not both be zero, by the similar reasoning in the proof of Proposition 1, we see that there exist \( t_1 \in \mathbb{Z}_{t_0}^{+\infty} \) and an angle \( \beta \) such that \( \forall t \in \mathbb{Z}_{t_0}^{+\infty}, \forall i \in V, \theta_i(t) + \beta \in (-\pi/2, \pi/2) \).

Now, let \( \tilde{\theta}_i(t) := \theta_i(t) + \beta \), and turn the attention to the heading updating model: \( \forall t \in \mathbb{Z}_{t_0}^{+\infty}, \)

\[
\tan(\tilde{\theta}_i(t + 1)) = \frac{\sum_{j \in N_i(t)} x_{ij}(t) \cos(\tilde{\theta}_j(t))}{\sum_{j \in N_i(t)} x_{ij}(t) \sin(\tilde{\theta}_j(t))}, \quad i \in V \setminus \{N\},
\]

\[
\tan(\theta_N(t + 1)) = \tan(\tilde{\theta}_N(t)).
\]

By the same arguments in the proof of Proposition 1, we have that \( \forall t \in \mathbb{Z}_{t_0}^{+\infty}, \forall i \in V, \cos(\tilde{\theta}_i(t)) \) is uniformly lower bounded by \( \min_{i=1}^N \cos(\tilde{\theta}_i(t_1)) \). In addition, note that the uniform boundedness of \( x_{ij}(t) \) is ensured by
the same reasoning in the proof of Lemma 3. Hence, by Lemma 4, it follows \( \forall i \in V, \lim_{t \to +\infty} \tan(\theta_i(t)) = c \), where \( c \) is a constant. This is equivalent to saying that all the headings of the group converge to a common value, which is, obviously, the constant heading of the leader.

Without loss of generality, let us assume \( v^N_t(t_0) \neq 0 \). Then from (42) and (50), it is straightforward to see that for any \( i \in V \setminus \{N\}, \lim_{t \to +\infty} d_{ij}(t) = 1 \). Further, by (48), \( \lim_{t \to +\infty} \alpha_{ij}(t) = 1, \ j \in N(t) \).

\( \square \)

V. SIMULATION RESULTS

In this section, we test our proposed heading consensus algorithms: WHCA and LBHCA. First, we simulate the WHCA and LBHCA for randomly chosen initial conditions. In Figs. 2 and 3, the positions and headings of 30 agents are randomly chosen in the interval \([0, 2] \times [0, 2]\) and \([-\pi, \pi]\) respectively; and \( r = 0.8, v = 0.2 \). To keep the joint connectivity of the group, a periodic boundary condition is imposed. We can see that the headings of the group converge to the same value in both cases, and, in leader-follower case, all the headings converge to the heading of the leader.

Next, we simulate WHCA and LBHCA for some special initial conditions to test these two algorithms with (nearly) extreme cases. In Fig. 4, the initial headings in Example 1 are slightly modified: the heading of the agent at the top changes to \( \pi/2 - \pi/200 \) such that the initial headings are nearly balanced. Since the critical issue in the heading consensus of the group is the connectivity, we fix the positions of the group at all times, and let the sensing range \( r \) still be 0.8. In this way, it is easy to check that the connectivity condition in Fig. 4 is the same as that of Example 1. Our simulations show the WHCA makes a fast convergence of all the headings to a common value, but the original Vicsek’s model fails to drive the headings to reach consensus (Fig. 5). In Fig. 6, we choose the initial headings of the group to be the same as those in Example 1 (i.e. balanced initial headings), and also fix the positions of the group at all times to ensure the same connectivity condition as in that example. Simulation results demonstrate that the proposed LBHCA guarantees the fast consensus of the headings.

VI. CONCLUDING REMARKS

In this paper, to overcome the drawbacks of Vicsek’s model such as singularity behavior and non-
global attractivity, we propose two modified decentralized heading consensus algorithms based on Vicsek’s model: WHCA and LBHCA. We prove that, under a well-known connectivity assumption, the WHCA ensures almost global consensus of the headings, except when they are initially balanced. However, for the LBHCA, the global heading consensus is guaranteed under the same connectivity assumption. Our future work will be directed at the sensitivity analysis of the proposed algorithms, and at other interesting issues such as robust convergence and asynchronous consensus.

REFERENCES


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