A NOVEL APPROACH FOR ROOT DISTRIBUTION ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS USING ROUTH AND FULLER TABLES

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ABSTRACT

The root distribution of a given characteristic equation of a linear time-invariant system can be analyzed with the help of a Routh table using the elements of the first column in the table. In the case of unstable systems, sometimes, a zero element may appear in the third row of the first column of the Routh array. This prematurity can be suitably handled as indicated by various authors. In this paper, the given characteristic polynomial having roots in the right hand plane is multiplied by a suitable polynomial, and Routh and Fuller tables are applied for the resultant polynomial to infer the complete root distribution. Further, the column polynomials from each table are adopted to know more about root distribution, which forms the core of the proposed work. The Routh table helps in counting and locating roots in the $s$-plane, and the Fuller table helps in depicting whether the roots are distinct or complex in nature. In this regard, it is shown in this paper that the simultaneous integration of Routh and Fuller tables yields a good amount of information regarding the root distribution in the $s$-plane. The newly presented procedure is illustrated with examples.

Key Words: Routh table, Fuller table, root distribution, pseudo Routh polynomial.

I. INTRODUCTION

Though various numerical methods are available for evaluating the roots of characteristic polynomials, knowing the distribution of roots in the $s$-plane forms an interest among academic research workers. Generally, for finding the number of roots existing in the left hand plane (LHP) and right hand plane (RHP) of the $s$-plane, the well-known Routh table is applied; under special cases, namely, when the roots are on the $j\omega$-axis, the difficulty encountered in judging the nature of roots using Routh table has been indicated by many authors along with some remedial solutions [1–24]. The present work follows the same direction but involves simple procedures wherein the direct application of Routh and Fuller tables [25] are carried out simultaneously to extract the information regarding the root distribution.

II. PROPOSED SCHEME

Consider a linear time-invariant system represented by its characteristic polynomial $F(s)$:

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_3 s^3 + a_2 s^2 + a_1 s + a_0.$$  (1)
The polynomial by another polynomial is avoided. If not, checked as to whether the occurrence of zero in initial multiplied with second order continued until a suitable polynomial from first order is found to avoid prematurity occurring in the given in the Routh table, the constant term appears as the last element.

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>j-th</th>
<th>...</th>
<th>(m - 2)</th>
<th>(m - 1)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^n )</td>
<td>( a_{11} )</td>
<td>( a_{12} )</td>
<td>( a_{13} )</td>
<td>...</td>
<td>( a_{1j} )</td>
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<td>( a_{1jm-2} )</td>
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<td>( a_{1jm} )</td>
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<tr>
<td>( s^{n-1} )</td>
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<td>( a_{22} )</td>
<td>( a_{23} )</td>
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<td>( a_{2j} )</td>
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<td>( a_{2jm-2} )</td>
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<td>( a_{2jm} )</td>
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<td>( s^{n-2} )</td>
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<td>( a_{33} )</td>
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<td>( a_{3j} )</td>
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<td>( a_{43} )</td>
<td>...</td>
<td>( a_{4j} )</td>
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<td>( s^{n-4} )</td>
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<td>( a_{5j} )</td>
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<td>( s^{n-5} )</td>
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<td>( s^3 )</td>
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<tr>
<td>( s^2 )</td>
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<td>( s^1 )</td>
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<tr>
<td>( i )-th</td>
<td>( a_{i1} )</td>
<td>( a_{i2} )</td>
<td>( a_{i3} )</td>
<td>...</td>
<td>( a_{ij} )</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>( a_{ijm} )</td>
</tr>
</tbody>
</table>

Multiply the given characteristic polynomial \( F(s) \) by another polynomial \( M(s) \), which results in the transformed polynomial \( T(s) \).

\[
T(s) = F(s) \ast M(s) \tag{2}
\]

The polynomial \( M(s) \) is chosen from the given polynomial \( F(s) \) itself as shown below:

First Order \( M(s): a_1s + a_0 \)
Second Order \( M(s): a_2s^2 + a_1s + a_0 \)
Third Order \( M(s): a_3s^3 + a_2s^2 + a_1s + a_0 \)
\[\vdots\]
\((n-1)\)th Order \( M(s): a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_3s^3 + a_2s^2 + a_1s + a_0 \)

First, \( F(s) \) is multiplied with first order \( M(s) \) and is checked as to whether the occurrence of zero in initial computations of Routh Array is avoided. If not, \( F(s) \) is multiplied with second order \( M(s) \) and the process is continued until a suitable polynomial from first order \( M(s) \), second order \( M(s), \ldots, (n-1) \)th order \( M(s) \) is found to avoid prematurity occurring in the given \( F(s) \). This process of multiplication of \( F(s) \) by \( M(s) \) may avoid the occurrence of zero in the table or it may shift the zero occurrence to the bottom portion of the Routh table, during which the zero may be replaced by a negligible value \( z \to 0^+ \) and further computations may be carried out. The above operations help in reducing the computations involved in the Routh table [26].

Multiplying \( F(s) \) by \( M(s) \) does not alter the property of \( F(s) \); it is absolutely maintained along with the property of \( M(s) \).

### 2.1 Implications of Routh column polynomials

Let the transformed polynomial \( T(s) \) be of the form:

\[
T(s) = a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots
\]

containing all the terms with \( a_{ij} > 0 \).

For the polynomial in (4), the following properties are true.

**Property 1.** The number of coefficients in alternate rows is reduced by one, i.e.:

\[
a_{ij} = 0 \quad \text{for } j > [(n+3-i)/2]
\]

\[i = 1, 2, \ldots, n + 1 \tag{5}\]

**Property 2.** Because of the triangular shape of the Routh table, the constant term appears as the last element in alternate rows i.e.:

\[
a_{n+3-2k,k} = \text{constant term for } k = 1, 2, \ldots, [(n+2)/2] \tag{6}\]

Table I gives the Routh table for a general polynomial of order \( n \) (odd) where \( j_m \) is the maximum value of \( j \).
for each row and is given by:

$$j_m = [(n + 3 - i)/2]$$ (7)

Depending upon the value of ‘n’, $(\frac{n+1}{2})$ columns are formed for an odd polynomial. Thus, the maximum number of columns is given by:

$$m = \left(\frac{n + 1}{2}\right)$$ (8)

where $a_{4jm-1}$ and $a_{6jm-2}$ are equal to $a_{2jm}$ by Property 2. Thus, in an odd polynomial, the last term of the original polynomial will appear as the last term in all the columns.

From Table I, it should be noted that the computations starts from the last column and works its way towards the first column, i.e., in the $(m - 1)$-th column, $a_{3jm-1}$ is calculated first and is given by:

$$a_{3jm-1} = a_{1jm} \left[1 - \left(\frac{a_{2jm}}{a_{1jm}}\right)\times\left(\frac{a_{11}}{a_{21}}\right)\right]$$ (9)

By Property 2, the last element in the $(m - 1)$-th column is simply a shift from the last element of $m$-th column. In a similar manner, all the elements of the $(m - 2)$-th column are generated and so on. Thus, the Routh table is computed starting from the last column.

Once the column elements are generated, then the Pseudo Routh Column Polynomials can be formed as shown below. For an odd polynomial, $(\frac{n+1}{2})$ of Pseudo Routh Column Polynomials can be formed.

From Table I, the Pseudo Routh Column Polynomials can be formed as shown below:

(i) $m$-th column $\Rightarrow P_m^c(s) = a_{1jm}s^2 + a_{2jm}$ (10)

(ii) $(m - 1)$-th column $\Rightarrow P_{m-1}^c(s)$

$$= a_{1jm-1}s^3 + a_{2jm-1}s^2 + a_{3jm-1}s + a_{2jm}$$ (11)

(iii) $(m - 2)$-th column $\Rightarrow P_{m-2}^c(s)$

$$= a_{1jm-2}s^4 + a_{2jm-2}s^3 + a_{3jm-2}s^2 + a_{4jm-2}s + a_{2jm}$$ (12)

$$\cdots$$

The Routh table can be formulated for the Pseudo Routh Column Polynomials and root analysis can be done. Before applying the Routh table to Pseudo Routh Column Polynomials, we will look into the influence of the elements of the $m$-th column entering into the $(m - 1)$-th column, thereby entering into $(m - 2)$-th column and finally traversing towards the first column.

Consider the formulation of the $j$-th column element:

$$a_{3,j} = a_{1,j+1}\left[1 - \left(\frac{a_{2,j+1}}{a_{1,j+1}}\right)\times\left(\frac{a_{11}}{a_{21}}\right)\right]$$ (13)

and

$$a_{4,j} = a_{2,j+1}\left[1 - \left(\frac{a_{3,j+1}}{a_{2,j+1}}\right)\times\left(\frac{a_{21}}{a_{31}}\right)\right]$$ (14)

From (13) and (14), it is clear that the computations of the column elements are mainly based on elements from the other columns. i.e., $a_{3,j}$ is based on the $(j + 1)$-th column elements and 1-st column elements.

In general, the computation of $a_{i,j}$ is based on the elements of the columns; consider,

$$a_{i,j} = a_{i-2,j+1}\left[1 - \left(\frac{a_{i-1,j+1}}{a_{i-2,j+1}}\right)\times\left(\frac{a_{i-1,1}}{a_{i-2,1}}\right)\right]$$ (15)

By inspecting the left hand side (LHS) and right hand side (RHS) of Equation (15), evaluation of the element $a_{i,j}$ is a function of the column elements and, further, the information pertained in the last column only enters into the first column.

The construction of the Routh table for Pseudo Routh Column Polynomials, given by Equation (11), is formed as shown in Table II, where

$$M = a_{3jm-1}\left[1 - \left(\frac{a_{2jm}}{a_{3jm-1}}\right)\times\left(\frac{a_{1jm-1}}{a_{2jm}}\right)\right]$$ (16)

From the above, it is easily observed that in the computation of $M$, the column elements in the respective columns of Table II are employed. If a sign change exists in the first column of Table II, then corresponding to the sign changes, the roots of $P_{m-1}^c(s)$ lie in RHP of s-plane. This information traverses to the other columns of the Routh array in Table I and is contained in the Pseudo Routh Polynomials $P_{m-2}^c(s)$.

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>$a_{1jm-1}$</td>
<td>$a_{3jm-1}$</td>
</tr>
<tr>
<td>$s^2$</td>
<td>$a_{2jm-1}$</td>
<td>$a_{2jm}$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$M$</td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>$a_{2jm}$</td>
<td></td>
</tr>
</tbody>
</table>
given by: 

\[ P_m = P_1 \] 

The Routh table is constructed for each of these Pseudo Routh Polynomials, and the location of roots is analyzed [27]. However, if there exists a sign change in any of the columns of Table I, it is not necessary to construct a Pseudo Routh Polynomial for that particular column and, based on sign changes, root distribution is depicted.

In formulating the Routh table for \( T(s) \), if any zero occurs, then replace the zero by a factor \( s \rightarrow 0^+ \). If needed, the higher order terms in \( s \) may be dropped. Observe the elements from the last column to the first column of the Routh table to count the roots existing in LHP and RHP of the \( s \)-plane.

Thus, by constructing the Routh table and observing its column elements, the root distribution in LHP or RHP of \( s \)-plane is noted. Further, the construction of Pseudo Routh Polynomial is an added feature. The above concepts also hold when ‘\( n \)’ is even.

### 2.2 Implications of the Fuller table

This is applied for our transformed polynomial \( T(s) \). The Fuller polynomial \( N(s) \) from \( T(s) \) is given by:

\[
N(s) = T(s)|_{s = S^2} + s \left( \frac{dT(s)}{ds} \right)|_{s = S^2}
\]

(17)

The degree of \( N(s) \) is \( 2n \) and the Fuller table is formed as follows:

The transformed polynomial \( T(s) \) is of the form:

\[
T(s) = a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots
\]

(18)

The Fuller polynomial \( N(s) \) is formulated using the expression given below:

\[
N(s) = T(s) + s \left( \frac{dT(s)}{ds} \right)
\]

(19)

\[
= [a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots]
\]

\[
+ s \left( a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots \right)
\]

\[
= [a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots]
\]

\[
+ [s(na_{11}s^{n-1} + (n - 1)a_{21}s^{n-2} + (n - 2)a_{12}s^{n-3} + (n - 3)a_{22}s^{n-4} + \cdots]
\]

\[
\Rightarrow N_1(s) + N_2(s)
\]

\[
N_1(s) = [a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots]
\]

\[
+ [na_{11}s^n + (n - 1)a_{21}s^{n-1} + (n - 2)a_{12}s^{n-2} + \cdots]
\]

\[
\Rightarrow N_2(s) = [a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots]
\]

(20)

where

\[
N_1(s) = [a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \cdots],
\]

(21)

\[
N_2(s) = [na_{11}s^n + (n - 1)a_{21}s^{n-1} + (n - 2)a_{12}s^{n-2} + \cdots]
\]

(22)

The coefficients of \( s \)-terms in \( N_1(s) \) form the first row of the Fuller table and the coefficients of \( s \)-terms in \( N_2(s) \) form the second row of the Fuller table.

For simplicity, let

\[
b_{11} = na_{11}
\]

\[
b_{21} = (n - 1)a_{21}
\]

\[
b_{12} = (n - 2)a_{12}
\]

\[
b_{22} = (n - 3)a_{22}
\]

\[
\vdots
\]

in \( N_2(s) \). Based on \( N_1(s) \) and \( N_2(s) \), the Fuller table is formulated as shown below in Table III.

After the first two rows of the Fuller table are formed, the subsequent rows are computed by employing general Routh Multiplication rules. Once the entire Fuller table is formed, if the first column elements are all positive, then the roots of \( T(s) \) are distinct and negative. The sign changes in the first column indicate the presence of complex roots (or) multiple roots (proportionality in any two rows of the table indicates this situation). Thus, both the Routh and Fuller tables are applied simultaneously to the transformed polynomial and the root distribution is analyzed.

The effective utilization of the Routh and Fuller tables for Root distribution analysis is brought out in the following illustrations.

### III. ILLUSTRATIONS

#### 3.1 Illustration 1 [1]

Consider the given characteristic polynomial to be:

\[
F(s) = s^4 + s^3 + 2s^2 + 2s + 1
\]

(24)
proposed in Section II, As a result, we get the transformed polynomial which indicates the first order prematurity, multiplying the characteristic polynomial occurs in the third row first column. To eliminate this Table IV. We can notice from Table IV, that a zero from 

Table V depicts the Routh array for in Section 2.1. 

second column. While constructing the second column, only the value LHP, and this information is found to traverse to the 

while constructing the second column, it will again confirm the existence of 2 roots in the RHP. It is clearly observed

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>···</th>
<th>j-th</th>
<th>···</th>
<th>(m − 2)</th>
<th>(m − 1)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^{2n}</td>
<td>a_{11}</td>
<td>a_{21}</td>
<td>a_{12}</td>
<td>···</td>
<td>a_{1j}</td>
<td>···</td>
<td>a_{2jm}−2</td>
<td>A_{1jm}−1</td>
<td>a_{2jm}</td>
</tr>
<tr>
<td>s^{2n−1}</td>
<td>b_{11}</td>
<td>b_{21}</td>
<td>b_{12}</td>
<td>···</td>
<td>b_{1j}</td>
<td>···</td>
<td>b_{2jm}−2</td>
<td>B_{1jm}−1</td>
<td>b_{2jm}</td>
</tr>
<tr>
<td>s^{2n−2}</td>
<td>b_{31}</td>
<td>b_{32}</td>
<td>b_{33}</td>
<td>···</td>
<td>b_{3j}</td>
<td>···</td>
<td>b_{3jm}−2</td>
<td>B_{3jm}−1</td>
<td>···</td>
</tr>
<tr>
<td>s^{2n−3}</td>
<td>b_{41}</td>
<td>b_{42}</td>
<td>b_{43}</td>
<td>···</td>
<td>b_{4j}</td>
<td>···</td>
<td>b_{4jm}−2</td>
<td>B_{4jm}−1</td>
<td>···</td>
</tr>
<tr>
<td>s^{2n−4}</td>
<td>b_{51}</td>
<td>b_{52}</td>
<td>b_{53}</td>
<td>···</td>
<td>b_{5j}</td>
<td>···</td>
<td>b_{5jm}−2</td>
<td>···</td>
<td>···</td>
</tr>
<tr>
<td>s^{2n−5}</td>
<td>b_{61}</td>
<td>b_{62}</td>
<td>b_{63}</td>
<td>···</td>
<td>b_{6j}</td>
<td>···</td>
<td>b_{6jm}−2</td>
<td>···</td>
<td>···</td>
</tr>
</tbody>
</table>

The Routh array of $F(s)$ as shown in Table IV. We can notice from Table IV, that a zero occurs in the third row first column. To eliminate this prematurity, multiplying the characteristic polynomial $F(s)$ by a polynomial $M(s)$ chosen from $F(s)$ itself as proposed in Section II, i.e.:

\[
M(s) = (2s + 1) \tag{25}
\]

which indicates the first order $M(s)$ from given $F(s)$. As a result, we get the transformed polynomial $T(s)$:

\[
T(s) = (s^4 + s^3 + 2s^2 + 2s + 1)(2s + 1) = (2s^5 + 3s^4 + 5s^3 + 6s^2 + 4s + 1) \tag{26}
\]

Table V depicts the Routh array for $T(s)$ as mentioned in Section 2.1.

In Table V, if we observe the third column, we get a Pseudo Routh Polynomial:

\[
P_3(s) = (4s + 1) \tag{27}
\]

It is inferred from Equation (27), that one root is in LHP, and this information is found to traverse to the second column. While constructing the second column, only the value $a_{32}$ has to be calculated, $a_{42}$ is just a shift from $a_{24}$. After the entire second column is formed, the Pseudo Routh polynomial $P_2(s)$ can be formed and is given by:

\[
P_2(s) = (5s^3 + 6s^2 + \frac{10}{3}s + 1) \tag{28}
\]

Applying Routh test to $P_2(s)$.
that the information from the last column and the next to last column only enters the first column. Thus, all the roots are counted (2 roots in the RHP, 3 roots in the LHP).

Now, formulating the Fuller table for \( T(s) \),

\[
N(s) = T(s) + s \left[ \frac{dT(s)}{ds} \right] = (2s^5 + 3s^4 + 5s^3 + 6s^2 + 4s + 1) + s(10s^4 + 12s^3 + 15s^2 + 12s + 4)
\]

\[
= (2s^5 + 3s^4 + 5s^3 + 6s^2 + 4s + 1) + (10s^5 + 12s^4 + 15s^3 + 12s^2 + 4s)
\]

(29)

Constructing the Fuller table as shown in Table VIII.

The Fuller table shows four sign changes in the first column, indicating four complex roots of \( T(s) \) and the remaining root is distinct, existing on the negative real axis. The Fuller table can be formed either for the original polynomial \( F(s) \) or for the transformed polynomial \( T(s) \). In both cases, it will show four sign changes.

Integrating the observations from Routh array and Fuller array, the root distribution of the transformed polynomial \( T(s) \) is:

- RHP roots = 2 (both complex in nature),
- LHP roots = 3 (2 complex, 1 distinct).

The distinct root is the root of the polynomial \( M(s) \). Thus, the original polynomial \( F(s) \) will have 4 roots, with 2 roots being complex and lying in the RHP of \( s \)-plane and with 2 roots being complex and lying in the LHP of \( s \)-plane. It is also observed that the system representing this polynomial \( F(s) \) is unstable in nature.

3.2 Illustration 2 [13]

Consider,

\[
F(s) = s^{10} + s^9 + 6s^8 + 6s^7 + 19s^6 + 19s^5 + 41s^4 + 40s^3 + 20s^2 + 18s + 9
\]

(30)

The Routh array for \( F(s) \) shows prematurity in the third row. To eliminate this early zero, multiply \( F(s) \) by the polynomial \( M(s) \), chosen from \( F(s) \) itself.

(i) Selecting first order \( M(s) \) from \( F(s) \)

\[
M(s) = 18s + 9
\]

(31)

the transformed polynomial is:

\[
T(s) = F(s) \ast M(s) = (18s^{11} + 27s^{10} + 117s^9 + 162s^8 + 396s^7 + 513s^6 + 909s^5 + 1089s^4 + 720s^3 + 504s^2 + 324s + 81)
\]

(32)

From the Routh table of \( T(s) \) in Equation (32), it is observed that the early zeros are present; hence, the second order \( M(s) \) can be chosen from \( F(s) \) for further processing.
(ii) Selecting second order \(M(s)\) from \(F(s)\):

\[
M(s) = 20s^2 + 18s + 9
\]

The transformed polynomial is:

\[
T(s) = F(s) \ast M(s)
= (20s^{12} + 38s^{11} + 147s^{10} + 237s^9 + 542s^8 + 776s^7 + 1333s^6 + 1709s^5 + 1489s^4 + 1080s^3 + 684s^2 + 324s + 81)
\]

The Routh table is formed for \(T(s)\) and the Pseudo Routh Polynomials of the corresponding columns are constructed, which brings out the following observations. The Pseudo Routh Polynomial of sixth column is:

\[
P_6(s) = 684s^2 + 324s + 81
\]

The Routh test is constructed for \(P_6(s)\), which shows two roots on LHP of \(s\)-plane. This traverses towards the fifth column and the Pseudo Routh Polynomial of fifth column is:

\[
P_5(s) = 1489s^4 + 1080s^3 + \frac{9756}{19}s^2 + \frac{8730}{47}s + 81
\]

Construction of the Routh table for \(P_5(s)\) shows two sign changes, thus ensuring two roots on the LHP and two roots on the RHP. In a similar way, the Pseudo Routh Polynomials are constructed for all the other columns up to the first column. By applying the Routh test for these polynomials \(P_4(s), P_3(s), P_2(s),\) and \(P_1(s)\), the locations of roots are given as:

- \(P_4(s)\) shows 4 LHP and 2 RHP roots.
- \(P_3(s)\) indicates 6 LHP roots and 2 RHP roots.
- \(P_2(s)\) yields 6 LHP and 4 RHP roots.

Finally, the first column polynomial \(P_1(s)\) depicts eight LHP and four RHP roots.

When the Fuller table is formed for the transformed polynomial \(T(s)\), the first column of the Fuller table shows 12 sign changes indicating 12 complex roots. The root distribution of the transformed polynomial \(T(s)\) is:

- RHP roots = 4 (all four complex in nature),
- LHP roots = 8 (all eight complex in nature).

The complex LHP roots include the two complex roots of chosen second order \(M(s)\). Thus, the root distribution of original polynomial \(F(s)\) has ten roots, all of which are complex in nature. Out of the ten complex roots, four roots lie in the RHP of \(s\)-plane and six roots lie in the LHP of \(s\)-plane.

### 3.3 Illustration 3 [22]

Given a polynomial

\[
F(s) = (s^{10} + 7s^8 - s^7 + s^6 - 8s^5 - 19s^4 - 13s^3 - 18s^2 - 6s - 4),
\]

formulate a Routh table for given \(F(s)\) as shown in Table IX.

Before computation of the Routh table, on writing the elements of a given polynomial in the Routh array, it is observed that a zero occurs in the second row of the first column. Apply the proposed method to avoid this prematurity.

Considering first order \(M(s)\) from given \(F(s)\), i.e.

\[
M(s) = (-6s - 4),
\]

the transformed polynomial is:

\[
T(s) = -6s^{11} - 4s^{10} - 42s^9 - 22s^8 - 2s^7 + 44s^6 + 146s^5 + 154s^4 + 160s^3 + 108s^2 + 48s + 16
\]

A Routh array is constructed for \(T(s)\) as per the procedure in Section 2.1 and is given in Table X. In Table X, for simplicity, the entire Routh array is shown, but first, the fifth column is formed and its Pseudo Routh Polynomial is analyzed, then the fourth column is formed and its corresponding Pseudo Routh Polynomial is analyzed, and so on up to the first column.

The Pseudo Routh Polynomial of the sixth column is:

\[
P_6(s) = (48s + 16)
\]

which shows one root in the LHP of \(s\)-plane. The Pseudo Routh Polynomial of the fifth column is:

\[
P_5(s) = 160s^3 + 108s^2 + 24s + 16
\]

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s^{10})</td>
<td>10</td>
<td>7</td>
<td>1</td>
<td>-19</td>
<td>-18</td>
<td>-4</td>
</tr>
<tr>
<td>(s^9)</td>
<td>0</td>
<td>-1</td>
<td>-8</td>
<td>-13</td>
<td>-6</td>
<td></td>
</tr>
</tbody>
</table>
Hence, the original polynomial \( F(s) \) will have one distinct RHP root, five LHP roots (four complex and one distinct) and four roots on the \( j\omega \)-axis.

### 3.4 Illustration 4 [22]

Consider a polynomial,

\[
F(s) = s^4 + 2s^3 + 4s^2 - 2s - 5
\]  

(41)

Applying the proposed procedure, choose first order \( M(s) \) from \( F(s) \):

\[
M(s) = (-2s - 5)
\]  

(42)

The transformed polynomial is:

\[
T(s) = (-2s^5 - 9s^4 - 18s^3 - 16s^2 + 20s + 25)
\]  

(43)

A Routh array for \( T(s) \) is constructed, as per the proposed procedure in Section 2.1, and is given in Table XI. In Table XI, for simplicity, the entire Routh table is shown, but first, the second column is formed and its Pseudo Routh Polynomial is analyzed, then the first column is formed and its corresponding Pseudo Routh Polynomial is analyzed.

The Pseudo Routh Polynomial of the third column is:

\[
P_3(s) = 20s + 25
\]  

(44)

which shows one root in the LHP of the \( s \)-plane. The Pseudo Polynomial of the second column is:

\[
P_2(s) = -18s^3 - 16s^2 + 14.44s + 25
\]  

(45)

The first column of the Routh table of \( P_2(s) \) shows one sign change, indicating one RHP root and two LHP roots. Also, the first column of the Routh array of the Pseudo Routh Polynomial of \( P_1(s) \) shows one sign change, declaring one RHP root and four LHP roots.

To analyze the complex and distinct nature of the roots, a Fuller table is formulated for \( T(s) \). The first column of the Fuller table shows three sign changes, indicating three roots may be complex or in the RHP. From the Fuller table, it is inferred that one root is in the RHP, thus, out of three sign changes in the Fuller table, it is declared that one root is in the RHP, two roots are complex, and the remaining two roots are distinct in nature.

Thus the root distribution of \( T(s) \) is,

**RHP roots** = 1,

**LHP roots distinct** = 2,

**LHP roots complex** = 2.

---

**Table X. Complete Routh table for \( T(s) \).**

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{11} )</td>
<td>-6</td>
<td>-42</td>
<td>-2</td>
<td>146</td>
<td>160</td>
<td>48</td>
</tr>
<tr>
<td>( s^{10} )</td>
<td>-4</td>
<td>-22</td>
<td>44</td>
<td>154</td>
<td>108</td>
<td>16</td>
</tr>
<tr>
<td>( s^9 )</td>
<td>-9</td>
<td>-68</td>
<td>-85</td>
<td>-2</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>( s^8 )</td>
<td>8.22</td>
<td>81.78</td>
<td>154.89</td>
<td>97.33</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>( s^7 )</td>
<td>21.51</td>
<td>84.54</td>
<td>104.54</td>
<td>41.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^6 )</td>
<td>49.47</td>
<td>114.93</td>
<td>81.47</td>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^5 )</td>
<td>34.56</td>
<td>69.11</td>
<td>34.56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^4 )</td>
<td>16</td>
<td>32</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^2 )</td>
<td>16</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Routh table first column of \( P_3(s) \) shows no sign changes, inferring three roots in LHP of \( s \)-plane. This includes the root observed from the sixth (last) column. Constructing a Routh array for the Pseudo Routh Polynomials \( P_2(s), P_3(s), P_2(s), \) and \( P_1(s) \) of Table X shows one sign change in their first columns, indicating one root in the RHP of the \( s \)-plane. Further, in Table X, there is proportionality in the \( s^5 \)-th and \( s^4 \)-th row elements, indicating roots on the \( j\omega \)-axis. The obtained polynomials from these two rows are \( (s^4 + 2s^2 + 1)(s^2 + 1)^2 \). This results in four roots on \( j\omega \)-axis. The \( s^2 \)-th row elements also indicate the factor \( (s^2 + 1) \), which further confirms the existence of roots in \( j\omega \)-axis. Thus, it is clear that four roots lie on the \( j\omega \)-axis, one root on the RHP, and the remaining six roots are in the LHP of the \( s \)-plane.

Now, a Fuller table is formulated for \( T(s) \) to observe the complex and distinct nature of roots. The first column of the Fuller table constructed for \( T(s) \) shows nine sign changes. The Routh table first column of \( T(s) \) shows one sign change indicating one RHP root. As a result, it can be noted that nine sign changes in the first column of the Fuller table indicate one root is in RHP, eight roots are complex, and the remaining two roots are distinct in nature.

Thus, the root distribution of \( T(s) \) is as follows:

**RHP roots** = 1,

**LHP roots** = 6 (four complex roots and two distinct roots),

**Roots on** \( j\omega \)-axis

=4 (obviously, this is complex in nature)
One distinct LHP root refers to that of the polynomial $M(s)$. Thus, the original polynomial $F(s)$ will have four roots, with one root in RHP of $s$-plane, along with one distinct and two complex LHP roots in the $s$-plane.

### IV. DISCUSSION

The salient points noted in the illustration are discussed here. In cases of unstable systems, zeros might occur sometimes in the initial computation of the Routh array. This prematurity is suitably handled as indicated by various authors [1–24]. In this paper, a novel approach of multiplying the original polynomial $F(s)$ by a polynomial $M(s)$ obtained from $F(s)$ itself, removes the prematurity occurring in the initial computations of Routh array and shifts the zero occurrence to the bottom portion of the Routh table. During this process, the zero may be replaced by a negligible value $\alpha \to 0_+$ and further computations can be carried out to analyze the root distribution of $F(s)$. The Fuller table indicates the nature of roots and the Routh array yields information about the location of roots. The inferences from the Routh and Fuller tables are combined suitably, to deduce the exact root distribution.

Illustration 1 [1], the occurrence of zero elements is avoided completely, and the exact root distribution is achieved by the construction of Pseudo Routh Polynomials. These polynomials indicate that the information from the last column only traverses into the first column. Illustration 2 [13] shows the multiplication of $F(s)$ by a second order polynomial $(20s^2 + 18s + 9)$ and analyzing the nature of roots.

Illustration 3 and Illustration 4 are from [22]. In [22], the authors utilized Descartes’s rule of signs and Sturm’s test as well as Cauchy’s index for evaluating the distribution of roots. In contrast, in the proposed method, a straightforward procedure is utilized without involving any special procedures, namely, $T(s) = F(s)M(s)$, and $T(s)$ is directly dealt with in Routh’s algorithm for observing the root distribution.

Illustration 3 shows how the location of roots on $j\omega$-axis is observed, and Illustration 4 depicts how the proposed procedure is simple for analyzing the root distribution compared to [22].

### V. CONCLUSION

In the determination of the number of roots with their nature, the effective uses of the column elements of a Routh table, along with the first column of a Fuller table, are made in this paper. Further, the given $F(s)$ is suitably multiplied by an another polynomial $M(s)$ deduced from $F(s)$ itself so that the possible information regarding the number and nature of RHP, LHP and $j\omega$-axis roots are determined from the transformed polynomial $T(s)$. The proposed scheme does not employ any differentiated auxiliary polynomials when zeros occur in any of the rows of the Routh table; instead every zero element is replaced by a factor $\alpha \to 0_+$ or any other specialized procedures like Sturm’s sequence, Cauchy’s index, etc. The illustrative examples show the simplicity and applicability of the suggested scheme for the analysis of roots of characteristic polynomial.

### REFERENCES


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