SINGULARITY COMPUTATION FOR ITERATIVE CONTROL OF NONLINEAR AFFINE SYSTEMS

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ABSTRACT

This paper considers a gradient type of iterative algorithm applied to the open loop control for nonlinear affine systems. The convergence of the algorithm relies on the control signal in each iteration being nonsingular. We present an algorithm for computing the singular control for a general class of nonlinear affine systems. Various nonlinear mechanical systems, including nonholonomic systems, are included as examples.

KeyWords: Affine system, nonlinear control, nonholonomic system, path planning, gradient algorithm, singular control.

I. INTRODUCTION

This paper considers the open loop control of nonlinear affine systems with specified initial and final states:

\[ \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i; \quad x(0) = x_0; \quad x(T) = x_d. \]  

(1)

Such problems can occur, for example, in the point-to-point motion planning of nonlinear mechanical systems such as satellites, robots, and vehicles. A special subclass is the nonholonomic systems [18,12] which contain non-integrable generalized velocity constraints (e.g., wheel no-slip constraint). Since velocity is considered to be the input in this case, the system is usually driftless (i.e., \( f_0 = 0 \)). Other commonly considered examples include underactuated multi-body systems such as serial open-chain robots with passive joints, and satellites with one or more unactuated degrees of freedom. Since dynamics need to be considered for such systems, the drift term, \( f_0 \), is typically non-zero.

The various approaches to this problem can be roughly classified as follows:

• Search-based algorithms. These algorithms can effectively handle obstacles for relatively simple systems, e.g., a tractor with one trailer. Heuristics are sometime incorporated.
• Control-theoretic algorithms (including optimal control, steering sinusoids, and piecewise constant controls).

These planners are usually constructive in nature, and are derived based on nonlinear geometric, and optimal control theory. Inequality constraints that arise in joint stops, collision avoidance, etc. are typically not considered.

• Differential Flatness approach. This method is based on the flatness property of certain nonlinear systems [11], and involves lifting of a trajectory from a smaller, but constraint-free, space of the flat outputs, into the original configuration space. Inequality constraints are typically not considered.
• Path-space iteration with penalty functions. They include the continuation method, Newton-type and gradient-type algorithms. The methods can incorporate obstacle avoidance using penalty functions, but may have numerical convergence problems due to singular controls.

This paper focuses on path-space iteration algorithms, proposed in [7,33,27]. The convergence of these algorithms is guaranteed if the control in each iteration is “non-singular” (in the sense that the corresponding time-varying linearized system is controllable), and if the control is uniformly bounded for all iterations (which is equivalent to the lack of singularity at infinity).

For affine nonlinear systems, singular controls are ill-understood. In [27], it was shown that singular controls are “non-generic” and, in the case of drift-free systems, can be avoided by using a generic loop. In [33], a strong bracket generating condition (SBGC) is introduced for driftless systems which guarantees that all non-zero controls are non-singular. However, the condition is too strong for most nonholonomic systems, even for the classic car-trailer example. In the case of the front wheel driven car, the singular controls are identified, and the path-lifting algorithm in [33] is proven to work globally.
The uniform boundedness of the control signals for unicycle and car is also shown in [33].

The goal of this paper is to present different approaches to characterize singular controls for nonlinear systems. In particular, we present a general algorithm for identifying finite singular controls in nonlinear affine systems whose Control Lie Algebra is finitely generated. We also demonstrated the usage of the algorithm by applying it to a number of common nonlinear systems, with and without the drift term. In the case of the multi-chain system, we also show that there is no infinite singular controls (i.e., the control function is guaranteed to be uniformly bounded in all iterations). This generalizes the corresponding result for one-chain [22]. Finally, using the singularity results, we can extend the iterative path planner for systems evolving on manifolds.

Once singular controls are found for a given system, methods for avoiding them still need to be developed. For driftless systems, generic loop has been proposed in [27]. Methods for avoiding them still need to be developed. For driftless systems, generic loop has been proposed in [27]. However, a general method is still lacking at this point.

II. ITERATIVE CONTROL ALGORITHM

2.1 Gradient Based Iteration

Consider the point-to-point open loop control of the nonlinear affine system in (1) with \( x \in M \), an \( n \)-dimensional smooth and connected manifold, immersed in \( R^m \). The controls \( u_i \) are assumed to lie in \( H = C^\infty([0, T], R^n) \), while the vector fields \( f_i(x) \), \( i \geq 0 \) are analytic. We also assume that the vector fields \( f \) satisfy the Lie Algebra Rank Condition, LARC (driftless case) or the Accessibility Rank Condition, ARC (drift case). Driftless systems satisfying LARC are completely controllable [5], while systems with drift and ARC are strongly accessible [14,29].

Given \( \bar{u} = \{u(t) : t \in [0, T]\} \), denote the end-point map of (1) by \( \phi_T(x_0, \bar{u}) \). Define the final state error as

\[ e(x_o, \bar{u}) = \phi_T(x_o, \bar{u}) - x_d. \]

Starting with an initial guess, \( \bar{u}_0 \), standard iterative methods can be applied to update \( \bar{u} \) to reduce \( |e| \) to 0. For example, with Newton or gradient updates the path-planner starts with an initial guess of control, and successively iterate feasible paths in order to reach the final configuration, as shown in Fig. 1. The update requires the computation of the gradient operator

\[ D(\bar{u}) = \nabla_{\bar{u}} \phi_T(x_o, \bar{u}). \]

If \( \tau \) is the iteration variable, the control along the entire path is updated according to:

\[ \frac{d\bar{u}}{d\tau} = -D(\bar{u})^\dagger e(x_o, \bar{u}). \]  

(2)

where \# = + (pseudo-inverse) for Newton’s method, and \# = * (adjoint) for gradient’s method [9,27,19]. In the first case, it is easily shown that the end-point map error shrinks exponentially to zero:

\[ \frac{de}{d\tau} = \nabla_{\bar{u}} \phi_T \frac{d\bar{u}}{d\tau} = D(\bar{u}) \frac{d\bar{u}}{d\tau} = -\alpha D(\bar{u}) e = -\alpha e. \]

(3)

Proposition 2.1. For the affine system (1), if the path-space iteration (2) converges to a control \( \bar{u}_0 \), then \( \bar{u}_0 \) steers the system from \( x_0 \) to the desired state \( x_d \).

The iterative algorithm will fail to produce a feasible path in two cases, namely, if \( D(\bar{u}) \) cannot be inverted, and also if the iterated controls become unbounded.

Definition 2.2. (Singular Control) A given control \( \bar{u} \), is called singular, if the operator \( \nabla_{\bar{u}} \phi_T = D(\bar{u}) \) loses rank. We denote the set of all singular controls by \( \mathcal{S} \in H \).

Definition 2.3. (Non-Explosion or Wellposedness Condition) The control update (2) satisfies the “non-explosion condition”, if \( \bar{u}(\tau) \) is defined \( (\forall) \tau > 0 \).

In order to take advantage of the exponential convergence property of the end-point map to zero, we must guarantee that singular controls are not encountered during iteration, and that the differential equation updating the control law (2) is well posed.

The pseudo-inverse update (2, with \# = +) can be further augmented to include terms in the null-space of \( D(u) \)

\[ \frac{d\bar{u}}{d\tau} = -\alpha D(\bar{u}) e(x_o, \bar{u}) + \bar{D}(\bar{u}) \beta, \]

(4)

in which \( \bar{D} = (I - D'D) \) is the annihilator operator of \( D(u) \), i.e., \( \bar{D}D = 0 \), and \( \beta \) is arbitrary. The advantage of using the null-space of the gradient operator lies in the fact that we can impose additional constraints on the feasible controls [20].

Closely related to the above path-planning algorithms is the “continuation method” (CM), first introduced by H.J. Sussmann in [33]. CM is the application of
Poincare’s continuation method to driftless nonholonomic systems over connected Riemannian manifolds. The idea behind the method is to connect the end-point of the initial control guess \( \tilde{u}_0 \) and \( x_0 \) with a \( C^r \) curve contained in \( M - M_s \), where \( M_s \) is the manifold generated by end-points corresponding to singular controls. Then, the control is iteratively updated such the end-point map to follow the given curve by using a procedure known as “Path-Lifting”:

- Pick \( \tilde{u}_o \), a nonsingular control for which \( \phi_t(x_0, \tilde{u}_o) \in M - M_s \).
- Select a \( C^r \) curve contained in \( M - M_s \), \( \pi(t) : [0, 1] \to M \), uniformly away from the singular control manifold \( M_s \), and such that \( \pi(0) = \phi_t(x_0, \tilde{u}_o) \), \( \pi(1) = x_s \). Such a curve could always be found provided that \( M - M_s \) is connex.
- Construct the path Lifting Equation (PLE), by imposing

\[
\pi(t) = \phi_t(x_0, \tilde{u}(t)), \quad \tilde{u}(0) = \tilde{u}_o
\]

and differentiate, to get

\[
\pi'(\tau) = \nabla_{\tilde{u}} \phi_t(x_0, \tilde{u}(\tau)) \frac{d\tilde{u}}{d\tau},
\]

which is solved by using the pseudo-inverse operator:

\[
\frac{d\tilde{u}}{d\tau} = D(\tilde{u}(\tau))^\dagger \pi(\tau).
\] (5)

By solving equation (5), one finds the control \( \tilde{u} = \tilde{u}(1) \) which steers the system from \( x_0 \) to \( x_s \).

In fact, one can see that Newton’s update (2) is a particular case of (5), with \( \pi(\tau) = e^{\pi \tau} e(x_0, \tilde{u}_o) \), and infinite lift time. Since the curve \( \pi(s) \) stays away from the singular set \( M_s \), as long as the PLE has a solution, the computed controls \( \tilde{u}(\tau) \) will be non-singular, therefore, the only condition we need to ensure is the finite time “non-explosion condition” for the PLE.

Guaranteeing convergence for all the iterative methods depends on the knowledge we have about the singular set of controls, as well as on the control law update being wellposed.

2.2 Computation of the Gradient

It is straightforward to show that \( \nabla_x \phi_t(x_0, \tilde{u}) = D(\tilde{u}) \) corresponds to the solution of the variational equation

\[
\dot{z} = A(t)z + B(t)u
\] (6)

in which \( A(t) = \nabla_x f(x) + \nabla_y f(x)u \), and \( B(t) = \nabla_y f(x)u \) [27]. Hence

\[
\nabla_x \phi_t(x_0, \tilde{u}) = \int_0^T \Phi(T, t) B(t) u(t) dt
\] (7)

where \( \Phi \) is the fundamental matrix associated to the variational system. In general, \( \Phi \) cannot be written as an exponential operator, since \( A(t_1) \) and \( A(t_2) \) may not commute. A given control \( \tilde{u} \) is singular for if the system (6) is uncontrollable, i.e., its controllability Grammian

\[
Q(T) = \int_0^T \Phi(T, t) B(t) B(t)^\dagger \Phi(T, t)^\dagger dt
\] (8)

is singular. If a control is singular, then there exists a non-zero vector \( z \in R^r \) such that

\[
z^\dagger Q(T) z = 0, \text{ or } z^\dagger \Phi(T, t) B(t) = 0, \forall t \in [0, T].
\]

A sufficient condition for the convergence of the end point map error \( \epsilon(x) \) to zero as \( \tau \to \infty \) is that \( D(\tilde{u}) \) is full rank for all \( \tau \). It has been shown in [27] that for strongly accessible systems the non-singularity condition is generically true for \( u \in C^r ([0, T], R^n) \). Moreover, “genericity” can be interpreted in a larger, probabilistic sense. The wellposedness condition has also been investigated in [34,4], and proven to work under restrictive assumptions on the vector fields. It was pointed out that singular controls are also the abnormal extremals, or singular arcs, of a time-optimal control problem with a trivial index. They could be defined in terms of the “switching functions” defined by

\[
g_i(t) = \lambda^T \Phi(T, t) \dot{y}_i(x(t)), \quad \lambda = 1, \quad i = 1...m.
\] (9)

If a control is singular, then all the switching functions are identically zero and, equivalently, the controllability Grammian is singular.

For computation reasons, (2) needs to be discretized and approximated by a finite dimensional system. This could be achieved by either using a basis in \( L^2_{2}(0, T) \) (such as the Fourier or Wavelet bases), or directly through the piecewise constant approximation. For this case, let \( N \) be the number of sampling points along the path, i.e., of the \( [0, T] \) time interval. The discretized operator \( D(\tilde{u}) \) is, then, a “fat” matrix in \( R^r \times mN \), and \( \tilde{u} \) is an \( mN \) dimensional vector constructed by stacking the \( m \) discretized controls. A computational procedure for calculating the discrete version of \( D \) involves the use of the variational system derived from (1) [9].

Consider \( (\Psi(t)), 1 \leq i \leq \infty \) a basis of functions with a dense span in the control space \( H \). Then, every control vector \( u(t) \) could be approximated arbitrarily well by a linear combination of the basis functions

\[
u(t) = \Psi(t) \lambda,
\]
where $\lambda$ is represented by a $k$-th dimensional constant vector, and $\Psi$ is a matrix function dependent on the choice of the basis function.

Now discretize the path into $N$ equal time frames in $[0, T]$, and approximate the dynamics of the variational system (6) as a discrete time varying linear system

$$\delta x_{i+1} = A_i \delta x_i + B_i \delta \lambda_i, \delta x_0 = 0, 0 \leq i \leq N,$$

in which all the $N$ iterations are linear approximations of the corresponding systems between time $0 \leq t_i \leq T$. In particular,

$$A_i = e^{\frac{1}{T} A_{\lambda}^{(i)}} B_i = \int_0^1 e^{\frac{1}{T} A_{\lambda}^{(i)} d\tau B_{\lambda}(\tau)} \Psi(\tau) \, d\tau$$

Hence, if we define $D$, as the mapping $D_i \delta x = D_i \delta \lambda$, we obtain the following iteration:

$$D_{i+1} = A_i D_i + B_i, 0 \leq i \leq N - 1,$$

and thus we can approximate $D(\delta u) = D_N$. The quality of the approximation will depend on the choice of the basis functions, and the discretization constants $k$ and $N$. Note that for those systems for which the steering with sinusoids method could apply (such as one-chain), it makes much sense to implement the algorithm using the Fourier basis.

Let $D_\lambda$, $k \in \{0, 1, ..., N\}$ be the discrete gradient corresponding to the sample at times $t_k = k \frac{T}{N}$. Then the discrete iterations step for $u^{(0)}$ will be

$$u^{(i+1)} = u^{(0)} - \hbar \alpha D_\lambda^{(i)} \Psi(T/T) e^{(i)}$$

in which $\hbar$ is the discrete $d\tau$ step. To improve the convergence rate of the iteration, we may combine it with line searches, obtaining a best-step Newton-Raphson algorithm [8].

Alternatively, one might not need to implement the discretization of the iterative path-planning algorithm, but might want to compute the gradient operator directly. The gradient operator relates small changes in control to small changes in the end-point map by

$$D(\delta u) = \delta x(T) = \int_0^T \Phi(T, t) B(t) \delta u(t) dt.$$  

$\Phi(T, t)$ can be computed explicitly in the special case of $A$ being nilpotent. The fundamental matrix $\Phi(t, T)$ can be expressed in terms of Peano-Baker series (equivalent to the Chen-Fliess series for the linearized system)

$$\Phi(T, t) = I_K + \sum_{i=1}^{\infty} \int_t^T A(\tau) \cdots \int_{t_{i-1}}^T A(\tau_i) d\tau_i \cdots d\tau_1.$$  

If $A$ is nilpotent, the series is finite, therefore one can explicitly find the operators

$$D'(\delta x(T))(t) = B(t) \Psi(T, t) \delta x(T)$$

and

$$D'(\delta x(T))(t) = B(t) \Psi(T, t) \left[ \int_0^T \Phi(T, t) B(t) \Phi(T, t) dt \right]^{-1} \delta x(T).$$

Hence, in the second case, the update law could be discretized directly from

$$\frac{du}{d\tau} = -\alpha B(t) \Psi(T, t) \left[ \int_0^T \Phi(T, t) B(t) \Phi(T, t) dt \right]^{-1} y(x_0, u).$$

### 2.3 Illustrative Examples

We have applied the path-space iteration to many nonlinear systems, including systems with drift. As an illustration, consider the Acrobot [30,31], shown in Fig. 2. The Acrobot is a two-link, under-actuated manipulator (the attached link is not actuated). The control of the Acrobot is inherently challenging because the upper-arm can only be affected through Coriolis torque. Initially the system is at its stable equilibrium, and it is swung up to its unstable equilibrium.

In the second example, consider a satellite with two torques on $y$ and $z$ axes reorienting by 90 degrees around the $x$ axis (Fig. 3). In this case, we consider the kinematic control problem, i.e., the angular velocity corresponding to each torque is assumed to be the control variable. The model of the satellite was formulated over $R^3$ using the Rodriguez parameters $\lambda$:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_3 - \lambda_2 \lambda_3 \\ 1 - \lambda_2^2 - \lambda_3^2 \\ -\lambda_1 - \lambda_2 \lambda_3 \end{bmatrix} u_1 + \begin{bmatrix} \lambda_1 \lambda_3 - \lambda_2^2 \\ \lambda_3 + \lambda_2 \lambda_3 \\ 1 - \lambda_2^2 \lambda_3^2 \end{bmatrix} u_2.$$  

It is possible to reorient the satellite using the quaternion notation using an extension of the path-planner to handle systems evolving over general manifolds (see Section 4).

### 3. Characterization of Singular Controls

The focus of this paper is to characterize singular controls for a given nonlinear system. We will start by
considering a special class of driftless systems, the so-called multi-chain systems, whose gradient is nilpotent. We will then present a more general approach for a broad class of nonlinear systems – those with finitely generated control Lie Algebra.

3.1 Multi-Chain Systems

Consider the single generator multi-chain system with at least one chain of length \( n \geq 4 \), and smooth inputs, the only singular controls are given by \( u(\cdot ) = 0, \forall t \in [0, T] \).

Theorem 3.1. For the multi-chained system (20), with at least one chain of length \( n \geq 4 \), and smooth inputs, the only singular controls are given by \( u(\cdot ) = 0, \forall t \in [0, T] \).
Proof. For our system, \( A(t) \) is nilpotent, and has a simple block-matrix diagonal form given by
\[
A(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & L_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & L_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & L_m
\end{pmatrix},
\]
in which the \((n_i-1) \times (n_i-1)\) sub-block \( L_i \) is given by
\[
L_i = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & u_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & u_i & 0
\end{pmatrix}.
\]
Due to the nilpotency of \( A(t) \), the Peano-Baker series (15) is finite, and has the same block-diagonal structure:
\[
\Phi(T, t) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & T_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & T_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & T_m
\end{pmatrix},
\]
where \( T_i \) are given by
\[
T_i = \begin{pmatrix}
\alpha_1 & 1 & \cdots & 0 & 0 \\
\alpha_2 & \alpha_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{n_j-2} & \alpha_{n_j-3} & \cdots & \alpha_1 & 1
\end{pmatrix}
\]
\[
\alpha_i(t) = \int_{t_1}^{t_2} u_i(\tau_1) \int_{t_1}^{\tau_2} u_i(\tau_2) \cdots \int_{\tau_{n_j-1}}^{\tau_{n_j-2}} u_i(\tau_{n_j-2}) d\tau_n \cdots d\tau_1,
\]
k = 1, 2, \ldots, \( n_i - 2 \). Now, suppose \( u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \) is a singular input. Then, there exists a non-zero constant vector \( z \in \mathbb{R}^n \) such that
\[
z^T \Phi f_1 f_2 \cdots f_m = 0, \quad \forall t \in [0, T].
\]

The equation corresponding to column \( j, m \leq j \leq 2 \) can be written as
\[
y_j + y_j \Phi f_1 f_2 \cdots f_m = 0, \quad t \in [0, T],
\]
where \( \{y_1, y_2, \ldots, y_m\} \) are the entries of \( z \) corresponding to the states in chain \( j \). Suppose \( u_i \) is not identically zero, and let \( k \) be the largest index value such that \( y_k \neq 0 \), and suppose that \( k \neq 1 \). By taking \( k-2 \) successive derivatives of (21) and simplifying by \( u_i(t) \) at each step, we arrive at
\[
y_i u_i(t) = 0, \quad \forall t \in [0, T], \quad \forall k \geq 2
\]
i.e., \( y_i = 0 \) or \( u_i = 0 \), contradicting the assumptions. Therefore, either \( u_i \) is identically zero on \([0, T]\), or \( k = 1 \). However, if \( k = 1 \), then \( y_1 = 1 \). Thus \( y_i Q(T) y = T \neq 0 \), and hence the only possible alternative is to have \( u_i(t) = 0 \).

So far we have proved that if \( u(t) \) is singular, then \( u_i = 0 \). To characterize the singular controls exactly, we first note that for \( u_i(t) = 0 \), \( \Phi(T, t) = I_n \). Let \( i \) be the index for which \( n_i \geq 4 \), let \( u_1 = 0 \) and \( z \in \mathbb{R}^{3 \times n} \) be a vector such that all entries of \( z \) are zero except those corresponding to chain \( i \), \( y_i \), which satisfy the condition
\[
y_1 + y_3 x_2 (0) + \cdots + y_n x_{n_i - 1}, (0) = 0.
\]
Then, \( z^T \Phi B(t) = 0 \) on \([0, T]\) because \( x_2, \ldots, x_{n_i - 1} \) are all constants. This shows that \( u_i = 0 \) is indeed a singular control.

It can be easily shown that the above result also holds for regular controls \( u \in H^1[0, T] \), in which case the singularity inputs are the set \( S = \{ u | u_i(t) = 0 \text{ a.e. in } [0, T] \} \).

Using the singularity results for one-chain systems, we can identify the set of singular controls for any nonholonomic system reducible to the one-chain form. A particular example of such system is the \( N \)-trailer system
\[
\dot{x} = f_1(x) u_1 + f_2(x) u_2
\]
in which
\[
f_1 = \begin{pmatrix}
\cos(\theta_n) \\
\sin(\theta_n) \\
\vdots \\
1 \prod_{j=2}^{n} \sec(\theta_{j-1})\tan(\theta_j - \theta_{j-1}) \\
0
\end{pmatrix}
\]
\[
f_2 = (0, 0, \ldots, 0, 1)^T, \quad \text{and } x = (x_0, y_n, \theta_n, \ldots, \theta_3)^T.
\]
The chain system is singular if \( u_i(t) = 0 \). If \( u_i \) is not identically zero, i.e., the tractor is not moving, the trailers cannot not stay jackknifed, the singularity occurs either because \( u_i(t) = 0 \), i.e. the tractor is not moving, or the steering angle is ±90 degrees. The expression in (25) was obtained in [23] using the Goursat normal form.

The property which we exploited in order to identify singularities is nilpotency. Other nonholonomic systems are nilpotent, although not reducible to a one-chain. Consider for example the ball rolling on a plate problem [3], reducible through transformations to the following five dimensional differential equation:

\[
\dot{x} = f_1 u_1 + f_2 u_2 = f(x(t)) u(t)
\]  

where \( f_1 = (1, 0, -x_2, x_1, 0)^T \), and \( f_2 = (0, 1, x_1, 0, x_3)^T \). The system is nonholonomic, nilpotent, and globally controllable. Using a similar approach to the one-chain case [21], we can show

**Lemma 1.** For the linearized “ball one a plate” system (26), a control is singular if and only if \( u_i(t) = 0 \), \( \forall t \in [0, T] \), or \( u_i(t) = 0 \), \( \forall t \in [0, T] \), or there exists a real constant \( c \) such that \( u_i(t) = cu_i(t), \forall t \in [0, T] \).

The proof is omitted here since we will show the same result using a general method to be presented in the next section. The key observation is that \( A(t) \) of this system is order three nilpotent, and the Peano-Baker series is finite:

\[
A(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
u_2 & u_1 & 0 & 0 & 0 \\
0 & 0 & u_1 & 0 & 0 \\
0 & 0 & u_2 & 0 & 0
\end{pmatrix}
\]

\[
\Phi(T, t) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\alpha_5 - \alpha_1 & 1 & 0 & 0 & 0 \\
\beta_{12} - \beta_{11} & \alpha_1 & 0 & 0 & 0 \\
\beta_{22} - \beta_{21} & \alpha_2 & 0 & 1 & 0
\end{pmatrix}
\]

3.2 A General Method of Identifying Singular Controls

We would like to generalize the above method to finding singular controls for any nonholonomic system, using the nilpotency property of the linearized system. Unfortunately, in a general \( n \)-dimensional case this proved intractable. We have therefore adopted a different approach using a classic LTV controllability criterion [6,29]:

**Lemma 1.** The linear, time varying analytic system (6) is completely state-controllable on the interval \([0, T]\), if and only if any of the following two holds:

- The controllability Grammian

\[
Q(T) = \int_0^T \Phi(T, t) B(T) B(t)^T \Phi(T, t)^T dt
\]

is nonsingular.

- The controllability matrix:

\[
Q_i(t) = [B(t), \Delta B(t), \ldots, \Delta^{n-1} B(t)]
\]

is full rank at lest on a sub-interval of \([0, T]\), where the operator \( \Delta \) is given by:

\[
\Delta = -A(t) + \frac{d}{dt}.
\]

Note that using the Grammian is only possible if the Peano-Baker series of the fundamental matrix \( \Phi \) is nilpotent. However, using the controllability matrix \( Q_i(t) \) seems more attractive since only a finite number of derivatives need to be considered. If we try to compute \( Q_i(t) \) directly, for a driftless nonholonomic system with two inputs we obtain the following result:

**Lemma 2.** For the system (1) with \( m = 2 \) and \( f_i(x) = 0 \) the columns of \( Q_i(t) \) are given by:

- \( c_1 = f_1, c_2 = f_2 \).
- \( c_3 = u_2(t)[f_1, f_2] \), \( c_4 = -u_1(t)[f_1, f_2] \)
- \( c_{2+1} \), and \( c_{2+2} \) are linear combinations of all brackets of degree smaller than or equal to \( i \), with scalars given by polynomials of \( u_1, u_2 \) and at most \( i-2 \) of their derivatives.

**Proof.** Let us first compute \( c_3 \) and \( c_5 \),

\[
c_3 = \Delta f_1 = -A(t) f_1 + \frac{df_1(x(t))}{dt} \tag{27}
\]

If we denote the Jacobians

\[
J_i = \frac{\partial f_i(x)}{\partial x_i}.
\]

\( i = 1, 2 \), then

\[
A(t) = J_1 u_1 + J_2 u_2;
\]

hence (27) becomes

\[
c_3 = -(J_1 u_1 + J_2 u_2) f_1 + J_1 \dot{x}
\]

\[
= -(J_1 u_1 + J_2 u_2) f_1 + J_1 (f_1 u_1 + f_2 u_2)
\]

\[
= u_2 (J_1 f_2 - J_2 f_1) = u_2 [f_1, f_2].
\]
Now let \( f_k = [f_1, f_2] \), and write
\[
c_5 = \Delta(u_2 f_3) \\
= u_2 f_3 + u_2(-(J_1 u_1 + J_2 u_2) f_3 + J_3(u_3 u_1 + f_2 u_3)) \tag{28}
\]
or
\[
c_5 = \dot{u}_2 f_3 + u_2(u_1 f_3, f_1) + u_2(f_3, f_2)) \tag{29}
\]

When we compute the rest of the columns of \( Q(t) \), we will obtain higher and higher order brackets of the control Lie algebra. Suppose that for some \( i \geq 1 \) we have already established the result for columns \( c_{2i+1} \) and \( c_{2i+2} \), i.e., we can write
\[
c_{2i+j} = \sum_{d \in \{1, 2 \}} a_{j} \hat{e}_{j} f_{j}, \quad j = 1, 2, \ldots
\]
as linear combinations of brackets with degree smaller than \( i \), where \( a_{j} \) are polynomials of \( u_1, u_2 \) and their derivatives. The next two columns of the controllability operator can be computed as \( c_{2i+1+j} = \Delta(c_{2i+j}), j = 1, 2, \) or
\[
c_{2i+1+j} = -J_1 u_1 c_{2i+j} - J_2 u_2 c_{2i+j} + \frac{dc_{2i+j}}{dt} \tag{30}
\]
\[
= -\sum u_1 J_1 a_{j} \hat{e}_{j} f_{j} - \sum u_2 J_2 a_{j} \hat{e}_{j} f_{j} + \sum a_{j} J_1 \hat{e}_{j} f_{j} + a_{j} J_2 \hat{e}_{j} f_{j} \tag{31}
\]
\[
= \frac{\dot{J}_1 u_1}{\sum_{p=1}^{\infty} \sum_{j=1}^{m} a_{j} \hat{e}_{j} (f_1 f_2) J_2 \hat{e}_{j} f_{j}) + \sum a_{j} f_{j} \tag{32}
\]
\[
= \frac{\dot{J}_2 u_2}{\sum_{p=1}^{\infty} \sum_{j=1}^{m} a_{j} \hat{e}_{j} (f_1 f_2) J_1 \hat{e}_{j} f_{j}) + \sum a_{j} f_{j}. \tag{33}
\]

Since the Lie-brackets of degree smaller than \( i + 1 \) are of the form \( [f_1, f_3] \) or \( [f_2, f_3] \), with \( \text{deg}(f_3) \leq i \), the new scalar coefficients of the linear combination of brackets for the column \( c_{2i+1+j} \) will be given by
\[
a_{j}^{(i+1)} j = u_1 a_{j}^{(i)} \hat{e}_{j} f_{j} + a_{j}^{(i)} \hat{e}_{j} f_{j}, \quad \text{if} \quad r = \text{deg}(f_3) \leq i \tag{34}
\]
\[
a_{j}^{(i+1)} j = u_2 a_{j}^{(i)} \hat{e}_{j} f_{j} + a_{j}^{(i)} \hat{e}_{j} f_{j}, \quad \text{if} \quad \text{deg}(f_3) = i + 1 \tag{35}
\]
in which \( j = 1, 2, p = 1 \) or \( p = 2 \), is such that \( f_i = [f_1, f_2] \). Equation (34) defines a recursive formula to compute the new linear combination of brackets in column \( 2(i+1) \), based on the brackets in columns \( 2i+j \). If the latter are polynomials of \( u_1, u_2 \) and at most \( i - 2 \) of their derivatives, then the former are also polynomials of \( u_1, u_2 \) and at most \( i - 1 \) of their derivatives, proving the result by induction.

We now need to use the fact that analytical distributions are “finitely generated” [14,36]:

**Definition 3.2.** (Locally Finitely Generated Distributions)

A distribution \( \Delta = \text{span}(f_j) \) is locally finitely generated if \( \forall x \in M \) there exists a neighborhood \( U \) of \( x \) and a finite set of vector fields in \( \Delta, f_j, 1 \leq i \leq r \), such that \( \forall f \in \Delta \), can be written as a linear combination of the form
\[
f(x) = \sum_{i=1}^{n} c_i f_i(x), \tag{36}
\]
where \( c_i \) are real-valued smooth functions on \( U \).

Now assume that the \( m \) input system (1) is globally controllable, driftless, and finitely generated. Pick a size \( n \) basis for its control Lie algebra \( B = \{f_1, f_2, \ldots, f_m\} \). Using the same argument as in Lemma 2 we can show that

**Theorem 3.3.** The controllability matrix \( Q(t) \) has the same rank as the time varying \( n \times mn \) matrix
\[
C(t) = [a_{11}(x), t \{f_{11}, \ldots, a_{1m}(x), t \{f_{1m}, \ldots, a_{n1}(x), t \{f_{nm} \}
\]
\[
\ldots a_{nm} \{x, t \{f_{nm} \}
\]
\[
C_{i}(t) = \mathcal{J}(t) \mathcal{A}(x(t), t) \tag{37}
\]
\[
\text{rank}(C_{i}(t) = \text{rank}(\mathcal{A}(x)) \quad \forall t \in [0, T], \tag{38}
\]
in which \( a_{ij}(x, t) \) are time varying scalar \( C \) functions depending on \( u(t) \) and the state vector \( x(t) \) through the basis function coefficients \( a_{i} \), \( i = 1 \ldots n, j = 1 \ldots m \). The scalar coefficients \( a_{ij}(t) \) are determined through column reduction operations applied on the matrix \( Q(t) \). \( J \) is a \( n \times n \) matrix containing the basis brackets \( B \), and \( A \) is a \( n \times mn \) matrix containing coefficients dependent on the controls and at most \( n - 2 \) of their derivatives and \( a_{ij}(t) \).

This result is a generalization of Lemma 2 to \( m \) inputs. The existence of the basis \( B \) uses the finite-generated assumption. The rest of the brackets present in each of the columns of \( Q(t) \) could be written as a linear (time-varying) combination of this basis. The coefficients will depend on \( u(t) \) and their derivatives, and also (possibly) on the state vector \( x(t) \) through \( \mathcal{A}(t) \)’s, which, in turn, depends uniquely on \( u(t) \). As a result, the scalar time varying coefficients resulting through column operations applied to \( Q(t) \), \( a_{ij}(t) \) will depend solely on the \( m \) controls \( u(t) \), and will be uniquely determined.

Because of Theorem 3.3, the singular controls of the system can be determined as follows:

**Theorem 3.4.** Given finitely generated, driftless, controllable system (1), with \( n \geq 3 \), the only singular controls are those for which for all time sub-intervals \( [t_1, t_2] \) of \( [0, T] \), there exists an index \( 1 \leq i \leq n \) such that all scalar functions \( a_{ij}(t), 1 \leq j \leq m \), defined in Theorem 3.3 are identically zero. Alternatively, the singular controls are exactly those controls for which \( \mathcal{A}(t) \) loses rank for all \( t \).
Proof. Since \( \mathcal{G} \) is always full rank, the system is singular if and only if the time-varying controllability matrix \( Q_c(t) \), and consequently \( C_c(t) \) and \( A(t) \), are singular for all \( t \in [0, T] \). Because the system has analytic vector fields, and smooth controls, we conclude that \( Q_c(t) \) loses its rank at \( t = t_o \in [0, T] \) if and only if one of the Lie brackets from the minimal basis \( \mathcal{B} \) is absent, i.e., its corresponding scalar time-varying functions \( a_{ij}(t) \) are all identically zero on a sub-interval containing \( t_o \). However, these equations (it could happen that they are dependent) are differential equations in \( u_i \), which either admit no solution or a non-dense subset of \( C^\infty[0, T] \) as solutions. Note that a similar result holds for analytic systems which are not finitely-generated, only the Lie basis \( \mathcal{B} \) can change along the system’s trajectory, and the above results hold locally.

Comment 1. From Theorem 3.4 we see that singular controls are isolated points in the control space \( \mathcal{H} \), because they need to satisfy differential or algebraic equations of the form \( a_{ij}(u(t), x(t)) = 0 \). This confirms Sontag’s result that they are “non-generic” in \( C^\infty([0, T]) \).

Comment 2. Singular controls are also related to other “singularity” concepts, such as the singular locus of the nonholonomic system, which is defined as the minimum level of bracketing needed to obtain a basis for the Control Lie Algebra of the system. It was shown that for the \( N \)-trailer system, the singular locus consists of all con-figuration where at least one trailer is jackknifed [15]. From Theorem 3.4, we see that for a general nonholonomic system if a trajectory contains only points in the singular locus, then it is necessarily generated by a singular control. Indeed, if a point belongs to the singular locus, then the controllability matrix \( Q_c(t) \) is not full rank because it depends only on brackets of degree smaller than the dimension of the system. However, \( Q_c \) being singular implies that the control is singular.

Comment 3. An immediate consequence of Theorem 3.4, for an analytic system with two controls and three states such that \( \{ f_1, f_2, [f_1, f_2] \} \) is a basis for the control Lie Algebra, then the only singular controls \( u_1(t) \), \( u_2(t) \) are both identically zero on \([0, T] \).

Theorems 3.4, 2 and Lemma 3.3 are the basis of a procedure to find the singular controls. Finding the singularities involves first finding a generating basis for the control Lie-Algebra, and their corresponding scalar coefficients \( a_{ij}(t) \). For simple systems, especially polynomial systems, this can be accomplished using any symbolic toolbox such as MAPLE. However, there are cases where the symbolic computations become intractable, or lead to very complicated differential equations in \( u(t) \) which may not have straightforward solutions. Numerical approximations must be used in such cases.

3.2.1 Example: Linearized “Ball on a Plate”

This system (26) is bilinear and known to be non reducible to the chain form [3]. A basis of the control Lie algebra is given by \( f_1, f_2, f_3 = [f_1, f_2], f_4 = [f_1, f_1], f_5 = [f_1, f_2] \). Moreover, the system is order 4 nilpotent, i.e., \( f_i = 0, \) if \( i \geq 6 \). We have two cases:

- \( u_1(t) \) and \( u_2(t) \) are both nonzero on a subinterval of \([0, T] \). In this case, the only candidate columns which can render \( Q_c(t) \) nonsingular are \( c_1, c_2, c_3, c_5 \). Since

\[
\begin{align*}
  c_5 &= \bar{u}_1(t)f_5 + \bar{u}_2(t)f_4 + \bar{u}_3 f_5, \\
  c_7 &= \bar{u}_1(t)f_7 + \bar{u}_2(t)f_6 + \bar{u}_3 f_7 + \frac{\bar{d}(\bar{u}_1, \bar{u}_3)}{dt} f_6 + 2\bar{u}_2 \bar{u}_3 f_5.
\end{align*}
\]

These could be simplified by column operations to

\[
c_5 = \bar{u}_1(t)f_4 + \bar{u}_2 f_5,
\]

In order for the controllability matrix to be singular, we must have \( c_5 \) and \( c_7 \) proportional to each other, i.e.,

\[
\bar{u}_1 = \frac{c(t)}{\bar{u}_3}, \quad \bar{u}_2 = c(t)2\bar{u}_2
\]

By dividing the last two relations, we obtain

\[
\frac{\bar{u}_1}{\bar{u}_2} = \frac{c(t)}{\bar{u}_3},
\]

which means that \( u_1(t) \) and \( u_2(t) \) must be proportional at all times.

- Either one of \( u_i \) and \( u_3 \) are identically zero on \([0, T] \). In this case, it is very easy to verify that \( Q_c(t) \) is always singular.

Hence, we have shown that for the linearized ball and plate system (26), a control is singular if and only if \( u_i(t) = 0, \forall t \in [0, T] \), or \( u_3(t) = 0, \forall t \in [0, T] \), or there exists a real constant \( c \) such that \( u_i(t) = cu_3(t), \forall t \in [0, T] \). The same result was obtained is Section 3 using the nilpotence property of \( A(t) \) for this system.

3.2.2 Example; Original “Ball on a Plate”

This example is particularly important because the system is neither differentially flat, not is it reducible to a chain-from.
\[ f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sin(x) & 1 \\ \cos(x) \cos(x) & -\sin(x) \\ -\cos(x) \sin(x) & \cos(x) \end{pmatrix} \]  

(40)

A basis for the control Lie algebra is given by \( f_1, f_2, f_3, f_4, f_5 = [f_1, f_2] \). After computing the columns of \( Q \), we notice that if \( u_i \) and \( u_j \) are not identically zero, the columns \( c_1, c_2, c_3, c_5, c_7 \) are always linearly independent. For the remaining cases, setting \( u_i(t) = 0 \) we obtain a singularity, while \( u_2(t) = 0 \) and \( u_5(t) \) not identically zero is not a singularity.

### 3.2.3 Example: Systems in Power Form

Nonholonomic systems in power form with \( n \geq 4 \) states and \( m = 2 \) controls are defined by the vector fields [24]

\[ f = [f_1, f_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & \frac{1}{2}x_i^2 \\ \vdots \\ 0 & \frac{1}{(n-2)!}x_i^{n-2} \end{pmatrix} \]  

(41)

The control Lie-Algebra is spanned by \( \{f_1, f_2\} \) and \( f_i = ad^i_{f_1} f_2, \ i \geq 3 \), and all the other brackets are zero. As a result, the controllability matrix \( Q_i(t) \) has columns given by

\[ Q_i(t) = \langle f_1, f_2, u_1 f_3 - u_2 f_2, u_1 u_2 f_4 + \dot{u}_2 f_3, \ldots \rangle. \]

If \( u_i(t) \) is not identically zero on some sub-interval \([t_1, t_2]\) of \([0, T]\), then by odd-column reduction the rank of \( Q_i(t) \) in \([t_1, t_2]\) is the same as the matrix

\[ C_i(t) = \langle f_1, f_2, u_1 f_3, u_2 f_2, \ldots, u_{n+2-i} f_n, u_{n+3-i} f_n, \ldots \rangle, \]

which is \( n \) because the first two columns and the rest of the odd-numbered ones have non-identically zero scalar coefficient functions. Alternatively, if \( u_i(t) \) is identically zero in \([0, T]\), then the controllability matrix has rank \( n \geq 4 \) and is, therefore, singular.

### 3.3 Extension to Systems with Drift

For analytic systems with drift, they are of the form

\[ \dot{x} = f(x) + \sum_{i=1}^{m} f_i(x) u_i \]

CLA is the Lie algebra of all the iterated Lie brackets \( CLA = \mathcal{L}(ad_{f_1}) \), and \( LARC \) is equivalent to strong accessibility [27,16]. The system may fail to be small-time locally controllable even if it is analytic and strongly-accessible. Nevertheless, the path-planning algorithm can be applied to steer the system towards any reachable final point using the iterative algorithm. It is possible to extend the same singularity results for systems with drift. Consider first systems with drift with only one control variable, i.e., systems of the form

\[ \dot{x} = f(x) + g(x) u, \]  

(42)

analytical, finitely generated and globally controllable on \( M = \mathbb{R}^n \). For this system we can prove the following statement.

**Lemma 3.** For the system (42), the \( i \)-th column of the controllability matrix \( Q_i(t) \) is equal to

\[ c_1 = g \]

(43)

\[ c_2 = ad g \]

(44)

\[ c_i = \sum_{j<i} a_{ij} f_j, \ i \geq 3, \]

(45)

where \( f_i \) is an iterated Lie bracket of \( f \) and \( g \) with degree smaller than \( i \), and \( a_{ij} \) are nonzero scalar functions of \( u \) and at most \( i - 3 \) of its derivatives.

**Proof.** Trivially, \( c_1 = g \), since for this system,

\[ A(t) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial u}, \ B(t) = g. \]

After applying the operator \( \Delta \) to \( g \), we obtain

\[ c_2 = (-A + \frac{d}{dt})g = -\frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} gu + \frac{\partial g}{\partial x} (f + gu) \]

\[ = -\frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f = [f_1, f_2]. \]

Now suppose that the \( i \)-th column of \( Q_i(t) \) is in the form required by the theorem, i.e. \( c_i = \Sigma_{j<i} a_{ij} f_j \). Then, the \( (i+1) \)-th column will be given by

\[ c_{i+1} = (-A + \frac{d}{dt}) \Sigma_{j<i} a_{ij} f_j \]  

(46)

However, since
\(-A + \frac{d}{dt} a_{\mu} f_j = -\frac{\partial f}{\partial x} a_{\mu} f_j - \frac{\partial g}{\partial x} u a_{\mu} f_j + \frac{\partial f_j}{\partial x} a_{\eta}(f + gu), \)

\(-A + \frac{d}{dt} a_{\mu} f_j = \dot{a}_{\mu} f_j + a_{\mu} [f, f_j] + a_{\mu} u [g, f_j].\)

Hence, if we put
\[ a_{\mu j+1} = \dot{a}_{\mu}, \text{ if } \deg(f_j) \leq i, \]
and
\[ a_{\mu i+1} = a_{\mu}, \text{ or } u a_{\mu}, \text{ if } \deg(f_j) > i \]
the \((i + 1)\)-th column will list all the brackets of length less than or equal to \((i + 1)\), and the lemma follows by induction. ■

As a result, since all the Lie brackets appearing in the columns of \(Q\) are all the brackets of the CLA of the system, a similar singularity theorem as Theorem 3.4 will hold for finitely generated systems with drift as well. Finally, because of the structure of the first two columns of \(Q\), any controllable system with drift, one control and two states, will have no non-trivial singularities.

3.3.1 Example: Dubin’s Car

Consider the following system with drift:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{pmatrix} =
\begin{pmatrix}
y \cos(\theta) \\
y \sin(\theta) \\
-\cos(\theta)
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} u . \tag{47}
\]

This system represents a modified version of Dubin’s car problem [4]. It was proven that the system is controllable, and, moreover, it is controllable with bounded input \(-\epsilon \leq u \leq \epsilon\), if and only if \(\epsilon \geq 1\). The first few Lie brackets in CLA are

\[
[f, g] = \begin{pmatrix} y \sin(\theta) \\ -y \cos(\theta) \\ -\cos(\theta) \end{pmatrix}, \text{ if } [f, g] = g, \text{ or } [f, g] = f .
\]

Therefore, from Lemma (3) the columns of the controllability matrix are

\[
c_1 = g, c_2 = [f, g], c_3 = g + uf.
\]

If \(u \neq 0\), the matrix has full rank, and such controls are not singular. If \(u\) is identically zero on \([0, T]\), the controllability matrix loses rank.

3.3.2 Example: Sussmann’s STLC System

The system with drift, \(n = 3\) and one input given by

\[
\begin{align*}
\dot{x}_1 &= u \tag{48} \\
\dot{x}_2 &= x_1 \tag{49} \\
\dot{x}_3 &= x_2^2 + x_1^3 \tag{50} \\
x(0) &= 0 \tag{51}
\end{align*}
\]

is a classic example of a small-time locally controllable system with drift satisfying Sussmann’s sufficient conditions [16]. A basis \(B\) for the Lie-Algebra is \(\{g, [g, f]= [0, 1, 3x_1^2]\}, g_3 = ad_f^2 g = [0, 0, 2]^T\), and the controllability matrix is

\[
Q_c(t) = [g, -[g, f], \left(\frac{1}{3} x_2 - 2u(t)x_1\right)g_3],
\]

and, therefore, the singular controls are defined by \(x_2(t) = 3u(t)x_1(t) = 0, \forall t \in [0, T]\). A simple computation leads to \(u(t) = \frac{\dot{x}_1}{\dot{x}_2}\) which is the only singular control for this system.

3.3.3 Example: Pushing and Steering a Knife Edge

Consider the control of a knife edge in point contact with a plane surface [1]. A reduced set of equations for this system is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{pmatrix} =
\begin{pmatrix}
f(x) + u_1 g_2(x) + u_2 g_3(x) \\
f(x) + u_3 g_2(x) + u_4 g_3(x) \\
-x_4 x_5 \\
x_4 x_5 \\
-3 x_2^2 + u_3 g_2(x) + u_4 g_3(x)
\end{pmatrix} . \tag{52}
\]

A basis for the control Lie-Algebra is \(B = \{g, g_2, g_3, g_4 = [f, g_3], g_5 = [g_3, [f, g_2]]\}\), and the controllability matrix becomes

\[
Q_c = \begin{pmatrix}
g_1 & g_2 & g_3 & g_4 & x_4 g_3 & (x_4 x_2 - x_3 g_3) \\
-x_2 g_3 & -x_3 g_3 & x_4 x_2 - x_3 g_3 + u_3 g_3 & \ldots
\end{pmatrix}.
\]

\(Q_c(t)\) is rank-reducible to \(C_c = \{g_1, g_2, g_3, g_4, u_5, g_5, u_6, g_6, \ldots\}\). If neither \(u_1(t)\) nor \(u_2(t)\) are identically zero in \([0, T]\), then \(Q_c(t)\) will be nonsingular on at least a sub-interval of \([0, T]\). If both \(u_1\) and \(u_2\) are identically zero, then \(x_4(t) = 0\), identically, is the only non singularity for this system.
3.3.4 Example: Inverted Pendulum

This example shows that in some cases, Theorem 3.4 does not give any clear insight about the singularities of the system. A set of reduced dynamical equations of the inverted pendulum obtained by partial feedback linearization [30] is given by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{g}{T} \sin(x_1) - \frac{1}{T} \cos(x_1) u \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= u,
\end{align*}
\]

in which, \(x_1 = \theta, x_2 = \dot{\theta}, x_3 = x, x_4 = \dot{x}\), and \(F = (M + m \sin^2(x_1))u - mlx_2^2 \sin(x_1) + mg \sin(x_1) \cos(x_1)\) is the horizontal force applied to the cart.

An exact characterization of singular controls proves intractable. The system evolves over \(\mathbb{R}^4\) and the controllability matrix has the form

\[
Q(t) = \begin{bmatrix} f_1, f_2, f_3 + uf_4, f_4 + uf_4 + uf_3 \end{bmatrix},
\]

where \(f_3 = [f_\theta, f_\dot{\theta}], f_4 = [f_x, f_\dot{x}], f_5 = [f_x, f_\dot{x}], f_6 = [f_\theta, f_\dot{\theta}]\) and \(f_1 = [f_\theta, f_\dot{\theta}]\). Computing the determinant of \(Q\), we obtain

\[
det(Q) = -\frac{11}{2} \cos(x_1)x_2^2 + \frac{3}{2} x_2^2 \cos(3x_1) + \frac{5}{2} x_2^4 + \frac{1}{2} x_2 \cos(2x_1) + \frac{3}{2} \mu \sin(3x_1) - \frac{13}{2} x_2 \mu \sin(x_1) + \cos(4x_1) + 1 + 2 \mu \sin(4x_1) - \cos(2x_1) - \mu \sin(2x_1) + u^2 - u^2 \cos(4x_1) + \mu \dot{x}_1 \mu \cos(x_1) - \mu \dot{x}_1 \mu \cos(3x_1),
\]

which may or may not be identically zero. Thus, in this case all we obtained was an algebraic test for the singularity of any control \(u\).

However, one can integrate the original affine system together with the additional \(\det(Q(t)) = 0\) to obtain a numerical approximation of the singular control. For \(x_0 = [1, 1, 0, 0]^T\) and \(u(0) = 1\), we obtain the non-trivial singular control shown in Fig. 4. However, from the rest position (pendulum down) \(x_0 = 0\), since \(\det(Q(0)) = 1\), it follows that no control can be singular.

IV. EXTENSION TO SYSTEMS OVER MANIFOLDS

In this section we present the path-planning algorithm as it applies to systems restricted over an analytic, connected manifold \(M \in \mathbb{R}^n\). This extension is important as many mechanical systems evolve on a curved manifold instead of in the flat Cartesian space, for example, orientation evolves on \(SO(3)\), and robots with revolute joints evolve on \(S(N)\).

Assume that the affine system (1) has analytic vector fields and, as a result, the maximal integral manifolds property holds. Denote \(M\) to be the maximal integral manifold onto which the state space of the system is restricted. Now suppose that the rank of the Accessibility Lie-Algebra (ALA) is equal to \(p\) at \(x_0\). Then, a classical result [14] guarantees that the reachable set in time \(T, R^*({x_0}) \subseteq M\), from \(x_0\) contains at least a manifold of dimension \(p, N \subseteq M\). Similar to the case \(N = \mathbb{R}^n\), we can show that the gradient operator \(D(\bar{u})\) has rank \(p\) as well, except for case when \(\bar{u}\) is a “singular control”, i.e., \(rank(D(\bar{u})) < p\).

Theorem 4.1. Consider the affine system with drift (1), which satisfies \(dim(ALA) = dim(N) = p\) and is analytic (and therefore locally finitely generated). Then, the set of controls \(\bar{u}\) for which \(rank(Q(t)) = p\) at least on some subinterval of \([0, T]\) is generic in \(C^\infty([0, T])\). For such controls \(rank(D(\bar{u})) = p\).

Proof. The Accessibility Lie-Algebra (ARC) for this system has rank \(p\), and the fact that the set of reachable points from \(x_0\) in time \(T\) is a submanifold \(N\) of the same dimension is a consequence of classical results for analytic affine systems [14]. The ALA is also locally finitely generated, and therefore \(Q(t)\) on sufficiently small intervals of \([0, T]\) can be column reduced to the \(p\) local generating brackets of the ALA, each multiplied by a scalar function depending on \(x\) and \(u\) and at most \(n\) of its derivatives (this is a straightforward generalization of Lemma 3 and Theorem 3.4). The set of singular controls is defined by setting some of the scalar coefficients to zero, which in turn implies the genericity result. The

![Fig. 4. The singular control of the inverted pendulum.](image-url)
rank of \( D(u) \) is exactly the dimension of the reachable manifold in time \( T \) for the time varying system (6). If the time \( t \) is chosen to be an \((n+1)\)-th independent coordinate, we obtain a following nonlinear, time-invariant system over \( \mathbb{R}^{n+1} \):

\[
\dot{z} = A(t)\dot{x} + B(t)u \\
i = 1 
\]

This system is an affine system with drift \( s = g_o(s) + \sum_{i=1}^n g_i(x)u_i \), with the vector fields given by

\[
s = \begin{pmatrix} \dot{z} \\ t \end{pmatrix}, s(0) = 0, g_o(x) = \begin{pmatrix} A(t)\dot{x} \\ 1 \end{pmatrix}, g_i(s) = \begin{pmatrix} f_i(x(t)) \\ 0 \end{pmatrix}.
\]

(54)

Computing the first lie brackets in the CLA for the system we obtain

\[
[g, g] = \begin{pmatrix} A(t) & \dot{A}(t)\dot{x} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(x(t)) \\ 0 \end{pmatrix} = \begin{pmatrix} -\Delta f_1 \\ 0 \end{pmatrix},
\]

where \( \Delta \) is the operator defined in Lemma (1). The higher order brackets in the CLA will be exactly the columns of the controllability matrix \( Q(t) \). If the control \( u \) is not singular, the rank of \( Q(t) \) will be \( p \). In addition the CLA contains the vector field \( g_o \) and, thus, it has rank \( p + 1 \). According to the well-known reachability criterion [14], the reachable set from 0 of this system contains an manifold of dimension \( p + 1 \). Hence, a slice through this manifold with \( t = T \) must have dimension \( p \) and hence, rank \( (D(\tilde{u})) = p \).

As a result of Theorem 4.1 we can reformulate the Newton iteration by reducing the system evolution to \( \mathbb{R}^p \) through local parametrizations. If

- \( N \) is a connected manifold, and the desired final point \( x_f \) belongs to \( N \).
- \( N \) can be described in a neighborhood \( V(x_f) \) around \( x_f \) by the explicit analytic equation \( x = h(x) \), where \( x = [x_1, x_2]^T \in V(x_f), \text{dim}(x_2) = n - p, \text{dim}(x_1) = p \).
- \( \tilde{u}_0 \) is an arbitrary initial control, such that its end-point map belongs to \( V(x_f) \).

The following path-space iteration scheme will converge of the end-point map to \( x_f \), provided that singularities are not encountered, and that the wellposedness condition holds:

\[
\frac{d\tilde{u}}{d\tau} = -AD(\tilde{u})^{++}e(x_f, \tilde{u}),
\]

(56)

in which \( D(\tilde{u})^{++} \) is an operator such that

\[
D(\tilde{u})D(\tilde{u})^{++} = \begin{pmatrix} I_p & 0 \\ \frac{\partial h}{\partial x_1} & 0 \end{pmatrix}
\]

Indeed, \( \frac{de}{d\tau} = -\alpha e, \) and \( \frac{de}{d\tau} = \frac{\partial h}{\partial x_1} \frac{de}{d\tau} \), which guarantees \( e(\tau) \rightarrow 0 \).

Not here that the above path-planner works only “locally”, since we require that the initial end-point map be in a vicinity of \( x_f \). If \( N \) is a connected manifold, then it will still be possible to steer the system to \( x_f \) by locally steering the end-point map through a finite number of neighborhoods similar to \( V(x_f) \) which approach \( x_f \). Alternatively, without using local manifold patches, we can always choose an operator \( D(\tilde{u})^{++} \) such that

\[
\Pi(p)DD^{++} = -\alpha \Pi(p),
\]

where \( \Pi(p) \) is the projection operator onto a \( p \)-dimensional subspace of \( \mathbb{R}^p \). In turn, this guarantees that

\[
\frac{d\Pi(p)e}{d\tau} = \Pi(p)D(\tilde{u})D(\tilde{u})^{++}e = -\alpha \Pi(p)e,
\]

(57)

i.e., \( e_1 = \Pi(p)e \rightarrow 0 \), while \( e_2 \) may of may not converge to zero. Take the example of the underactuated satellite in Fig. 3, written in quaternion form:

\[
\begin{align*}
\dot{q}_0 &= -q_1\dot{q}_1 - q_2\dot{q}_2 \\
\dot{q}_1 &= q_0\dot{q}_1 - q_3\dot{q}_2 \\
\dot{q}_2 &= q_0\dot{q}_2 + q_1\dot{q}_3 \\
\dot{q}_3 &= -q_1\dot{q}_1 + q_0\dot{q}_2 \\
1 &= q_0^2 + q_1^2 + q_2^2 + q_3^2.
\end{align*}
\]

(58)

The system’s CLA is generated by \( f_1 = (-q_1, q_0, q_3, -q_2)^T, f_2 = (-q_2, q_3, q_0, q_1)^T, \) and \( f_3 = [f_1, f_2] = 2(q_2, q_3, q_1, -q_0)^T, \) and has dimension three. The quaternion formulations is the natural formulation of the satellite attitude problem since parametrizations with three parameters are made up of patches. The reorientation around \( x \) using \( y \) and \( z \) torques could be reformulated on the unit sphere in \( \mathbb{R}^2 \) between \( q_{init} = (1, 0, 0, 0)^T \), and \( q_{final} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T \). Numerically, the operator \( D^{++} \) is then computed using the QR decomposition of \( D(\tilde{u})^{++} \), by inverting only the nonsingular \( 3 \times 3 \) portion of the upper triangular matrix \( R \).
The iterated vector quaternion paths and the control efforts are shown in Fig. 5.

V. WELLPOSEDNESS CONSIDERATION

To guarantee convergence of the iterative method described earlier, we need both singularity avoidance and wellposedness. The wellposedness condition requires that the differential equation over \( H \) given by (2) have a solution \( u(t, \tau) \) defined for all \( \tau \in [0, \infty) \). This is similar to the non-explosion condition in the continuation method, where the non-explosion refers to the control law update not blowing up in finite time.

The following example shows how the violation of the wellposedness condition can contribute to the lack of convergence of the iterative algorithm. Consider the following nonlinear system with drift given by

\[
\begin{align*}
\dot{x}_1 &= u \\
\dot{x}_2 &= x_1^m
\end{align*}
\] (59)

with \( m = 2 \). Suppose that \( x_0 \) is the origin and \( x_d = (0.5, -1) \) is the desired final state. We know a priori that \( x_d \) is not reachable from \( x_0 \) since \( x_2(t) \) cannot be negative.

By using the singularity computation algorithm presented earlier, it is straightforward to establish that \( u(t) = 0, \forall t \in [0, T = 1] \) is the only singular control. When the iterative algorithm is applied to this system, the state trajectory iteration converges to a non-singular path, while the control trajectory becomes unbounded. Figure 6 shows the path of the system after several iterations. In this example, the lack of convergence of the algorithm is not attributed to the singular control, but rather to the fact that \( u(t, \tau) \) blows up during the iteration (i.e., escapes for finite \( \tau \)).

In general, a sufficient condition for wellposedness is [34,35,4]:

\[
\left| D(\tilde{u})^+ \right| \leq \beta \| \tilde{u} \| + \beta_o . \tag{60}
\]

If a singular control is not encountered, then an explicit uniform bound on \( |\tilde{u}| \) can be derived from (2) and (3). Noting that \( D^* = D(DD^*)^{-1} \) when \( \tilde{u} \) is not a singular control, a sufficient condition for (60) that is easier to check is the uniform positive definiteness and uniform boundedness of \( DD^* \):

\[
0 < c_1 \leq D(\tilde{u})D(\tilde{u})^* \leq c_2 < \infty . \tag{61}
\]

To show that (60) and non-singularity implies wellposedness, consider (2).

Condition (60) is difficult to check for general nonlinear systems. In [34,4], it was shown that systems satisfying the Strong Bracket Generating Condition are well posed. However, for \( m = 2 \) controls, this condition reduces to the CLA being spanned by \( \{f_1, f_2, [f_1, f_2]\} \), i.e., the system must necessarily have three states. It was also shown in [34], that the front-wheel driven car (\( n = 4 \)) is wellposed. However, there has been no other general result so far. In the following section, we will demonstrate wellposedness for the general one-chain system (which includes unicycle and front-wheel driven car).
5.1 Wellposedness for One-Chain Systems

For the front wheel-driven car, which is reducible to the one-chain form with \( n = 4 \), the non-explosion condition is formulated as [34]:

Given a compact set \( K \) such that \( \pi([0, 1]) \subset K \) and \( \text{dist}(M_s, \ K) \alpha > 0 \), show that there exists a positive constant \( C_k \) such that for every control \( \tilde{u} \) initiating a trajectory with the end-point in \( K \)

\[
\left| P(\tilde{u}) \right| \leq C_k \left| \tilde{u} \right| .
\]  
(62)

where the norms in (62) are taken in the appropriate spaces, and \( P(\tilde{u}) = (D'D)^{-1}D^\tau \). This condition ensures both singularity avoidance (through the choice of \( K \)) and uniform boundedness of \( \tilde{u} \). The condition can be written in terms of the norm of \( (D'D) \) as

\[
\left\| (D'D) \tilde{u}(\tau) \right\| \geq \frac{1}{C_k} .
\]

Consider now the case of the one-chain system (20); with \( n \geq 4 \):

\[
\dot{x} = f_1 u_1 + f_2 u_2 ,
\]  
(63)

where

\[
\begin{bmatrix}
1 \\
0 \\
x_2 \\
\vdots \\
x_{n-1}
\end{bmatrix} f_1 = 
\begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} f_2 = 
\begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix} .
\]

We have already shown that singular control are given by \( u_1 \equiv 0 \). Given \( x_{\infty} \), the corresponding final state is therefore a line \( M_s = \{ x_0 + f_2 \ell : \ell \in \mathbb{R} \} \). If \( M_s - M_s \) will also be connected. Therefore, we can always find a path \( \pi(\ell) \) between any two points \( p \) and \( q \), which stays away from the singular set. This means that we can construct the requisite set \( K \) that stays uniformly away from the singular configuration set \( M_s \).

We next focus on the the non-explosion condition which may be written as

\[
\left\| Q(T) \tilde{u}(\tau) \right\| > \frac{1}{C_k} .
\]  
(64)

with \( Q(T) \) being the controllability Grammian of the variational system (6). By expanding the Grammian, and using the fact that \( \left| Q \right| = \inf_{\left| L \right| = 1} (z^T Q z) \), condition (62) becomes:

Let \( H_K \) be the set of all controls \( u \) which bring the system into the compact \( K \). The following condition holds:

\[
C = \inf_{u \in H_K} \left\{ \left( \int_0^T (u_1^2 + u_2^2) dt \right)^{1/2} \right\} > 0
\]

For the one-chain system, we have the close form expression for \( \Phi(T, t) \), so \( C \) can be written as

\[
C = \inf_{u \in H_K} \left\{ \left( \int_0^T (u_1^2 + u_2^2) dt \right)^{1/2} \right\}
\]

wher, for all \( 1 \leq m \leq n - 2 \),

\[
\alpha_m(t) = \frac{\int_0^T u_1(\tau)d\tau^m}{m!}, \quad \beta_m(t) = x_m + t \alpha_1 + \cdots + x_m t \alpha_{m-1} .
\]

We have the following sufficient condition for \( C \) to be strictly positive:

**Theorem 5.1.** The constant \( C \) in (66) is positive if the compact set \( K \) does not contain any points in the hyper plane \( x_1 = x_{\infty} \).

**Proof.** Under the stated condition, there exists a positive constant \( C_1 \) such that

\[
\left| \int_0^T u_1(\tau)d\tau \right| > c_1
\]

(67)

for all \( u \in H_K \).

Assume without loss of generality that \( T = 1 \).

Applying the Cauchy-Schwartz inequality

\[
\int_0^1 (f_1^2 + f_2^2) \geq \int_0^1 f_1 \int_0^1 f_2 \geq (\int_0^1 f_1 g_2) \geq (\int_0^1 f_1 g_2)^2 ,
\]

to equation (66), we have

\[
C \geq \inf_{u \in H_K} \left\{ \left( \int_0^1 (g_1 + \sum_{i=3}^n g_i) u_i \right)^2 \right\} .
\]  
(68)

The infimum \( C \) from (66) is a limit achieved by a sequence of controls. Let \( (u_{1\infty}, u_{2m}) \in H_K \) be a sequence of
controls and \( z^{(m)} \) be a sequence of unit vectors so that
\[
\left( \sum_{i=1}^{n} (z_{n}^{(m)} + \sum_{i=3}^{n} z_{i}^{(m)} a_{i} u_{i})^{2} \right)
\]
converges to \( C \). Since the unit ball in \( \mathbb{R}^{n} \) is compact, the sequence \( (z^{(m)}) \) contains a convergent subsequence. Let its limit be \( z^* \). Consider the two possibilities below:

- \( z^*_1 = 1 \). In this case, from (66) and (67) it immediately follows that
\[
C > \inf_{u \in H_{K,N}} \int_{0}^{1} (u_{1}^{2} + u_{2}^{2}) \geq \inf_{u \in H_{K,N}} \int_{0}^{1} u_{1}^{2} \\
\geq \inf_{u \in H_{K,N}} (\sum_{i=3}^{n} z_{i}^{*} a_{i} u_{i})^{2} > 0.
\]

- \( z^*_1 < 1 \). In this case, from (68) and (66) we can write
\[
C \geq \sup_{t \in [0,1]} \inf_{u \in H_{K,N}} \int_{0}^{t} \left( z_{2}^{2} + \sum_{i=3}^{n} z_{i}^{*} a_{i} u_{i} \right)^{2}.
\]

By a simple change of variable, we obtain
\[
C \geq \sup_{t \in [0,1]} \inf_{u \in H_{K,N}} \int_{0}^{t} \left( z_{2}^{2} + \sum_{i=3}^{n} z_{i}^{*} u_{i} \right)^{2}.
\]

in which \( \nu = \nu(t) = \int_{0}^{t} u_{1} \). Note that \( |\nu(0)| = 0 \), and \( |\nu(1)| > c_{1} \). Since not all \( z_{2}, z_{3}, \ldots, z_{n} \) are zero, the polynomial
\[
h(\nu) = \left( z_{1}^{*} \nu + z_{2}^{*} \nu^{2} + \ldots + z_{n}^{*} \frac{\nu^{n-1}}{(n-1)!} \right)^{2},
\]
defined over a real domain for \( \nu \) which includes at least one of the subintervals \([0, c_{1}]\) or \([-c_{1}, 0]\) will attain a nonzero maximum. Therefore, there will exist \( \nu_{0} \) on either of these subintervals, such that \( h(\nu_{0}) > 0 \). Since \( |\nu(1)| > c_{1} \), for every control \( u \in H_{K,N} \), \( \nu_{0} \) will be attained for some time \( t(u) \), depending on \( u \). From this observation and equation (69) we can conclude that for every \( u \in H_{K} \),
\[ C > h(\nu_{0}) > 0. \]

In both cases, we have shown that \( C > 0 \), concluding the theorem.

The condition in the theorem above is not very restrictive. The only situation that we cannot find the required set \( K \) is when the first components of \( x_{0} \) and \( x_{x} \) are equal. In that case, we simply choose an intermediate point \( x_{i} \) whose first component is different from \( x_{i} \) and move from \( x_{0} \) to \( x_{i} \) first and then from \( x_{i} \) to \( x_{x} \).

5.2 Utilization of Null Space of \( D \)

It is also possible to use the null space of \( D \) to try to reduce the control effort \( \bar{u} \). If in equation (4) we use \( \beta = u \), we obtain the iteration update
\[
\frac{du}{dT} = -\alpha D_{N}^{*} - \bar{D} u,
\]
which guarantees that the end-point map error \( e = x(T) - x_{d} \rightarrow 0 \). At the same time, as \( e \) converges to zero, the control effort norm decreases (strictly) until \( u \) reaches the null-space of the \( mN \times mN \) matrix \( \bar{D}_{N} \). This is a result of the fact that \( D_{N} \) is symmetric and semi-positive definite, with \( n \) zero eigenvalues, and the rest all equal to \( 1 \). Hence, its null space has size \( n \) and if \( mN > n \) the norm of \( u \) will decrease as long as \( \bar{u} \) is not in the range of \( D_{N}^{T} \). As an illustration, consider the path-planning iteration for a front-wheel driven car with \( x_{0} = [0, 0, 0, 0] \) to \( x_{d} = [0, 1, 0, 0] \). The control generated using the original update (2) is larger than the one found using (70), as shown in Fig. 7.
Comment 4. For “one-chain” systems, we can also shown that the singular controls do not act like an attractor. The explicit form of the path-iteration matrix $D_j(u)$ has on the upper half of the first row entries equal to $\frac{1}{N}$, and the rest zeroes, while for the second row the lower half are equal to $\frac{1}{N}$. Because the error vector $e$ decays exponentially to zero, we can obtain the following equation:

$$\sum_{i=1}^{N} u_j(\tau) = \sum_{i=1}^{N} u_j(0) - \frac{N}{\alpha} \alpha - e^{-\alpha t},$$

(71)

$j = 1, 2$. If the initial control guess does not have all non-zero entries such that the sum $\sum_{i=1}^{N} u_j(0)$ is non-zero, and if $\alpha$ is sufficiently large, the iterated sum $\sum_{i=1}^{N} u_j(\tau)$ stays approximately constant and away from zero, therefore the singular control is not encountered during the iteration of the algorithm.

VI. CONCLUSION

In this paper we presented several approaches to characterize singular controls for nonlinear affine systems. Various gradient based iterative open loop control methods may not converge if the singularity-free and wellposedness conditions are violated. The singularity condition reduces to $P(\ddot{u}) = D(\ddot{u})D'(\ddot{u})$ being full rank, while wellposedness reduces to $P(\ddot{u})$ being uniformly positive definite over all trajectories $\ddot{u}$. This paper has completed the first row of Table 1. The wellposedness condition is also proven for the general one-chain system. The knowledge of singular controls would be helpful in choosing the initial guess for the iteration (away from the singular controls). Certain continuation algorithm, such as the one presented in [33], also requires the knowledge of the singular control. However, general methods for avoiding the singular controls are still lacking. In the specific example of a one-chain system, the singular control is given by $u_0(t) = 0$. During the path-planning iteration, we must ensure $u(t) > 0$ at least for some interval of $t$. In [19], we have numerically enforced the constraint $0 < u_{\min} < |u_i|$, but the constraint may itself introduce additional singularities.

In addition to wellposedness and singularity avoidance, additional issues that are currently under investigation include the effect of discretization (approximating $\ddot{u}$ by a finite dimensional signal) and incorporation of constraints (which would result in a different singularity structure).

REFERENCES


<table>
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Table 1. The singularity and wellposedness problems for different types of affine systems.

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