ON MAXIMUM STABILITY MARGIN DESIGN OF NONLINEAR UNCERTAIN SYSTEMS: FUZZY CONTROL APPROACH

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ABSTRACT

This paper studies the maximum stability margin design for nonlinear uncertain systems using fuzzy control. First, the Takagi and Sugeno fuzzy model is employed to approximate a nonlinear uncertain system. Next, based on the fuzzy model, the maximum stability margin for a nonlinear uncertain system is studied to achieve as much tolerance of plant uncertainties as possible using a fuzzy control method. In the proposed fuzzy control method, the maximum stability margin design problem is parameterized in terms of a corresponding generalized eigenvalue problem (GEVP). For the case where state variables are unavailable, a fuzzy observer-based control scheme is also proposed to deal with the maximum stability margin for nonlinear uncertain systems. Using a suboptimal approach, we characterize the maximum stability margin via fuzzy observer-based control in terms of a linear matrix inequality problem (LMIP). The GEVP and LMIP can be solved very efficiently via convex optimization techniques. Simulation examples are given to illustrate the design procedure of the proposed method.

KEYWORDS: Maximum stability margin, fuzzy control, GEVP and LMIP.

I. INTRODUCTION

Interest in the stability margin arose from the study of perturbed systems. The stability margin for perturbed systems is an important design parameter since it is an estimate of the distance from instability [1]. The stability margin denotes the maximum perturbation which can be tolerated by the control system. The stability margin for linear systems with linear or nonlinear perturbation has been extensively studied [1-5], and it can be characterized in either the time domain or frequency domain. Nonlinearity and uncertainty in control systems has increasingly become a well-known fact. Studies on stability for perturbed nonlinear systems can be found in [6-8]. However, there are very few works on the stability margin for nonlinear uncertain systems.

In recent years, there has been rapidly growing interest in fuzzy control of nonlinear systems. The stability of fuzzy control systems has been considered extensively in nonlinear stability frameworks [12-15,17-19]. In most of the previous works, a nonlinear plant was approximated by a fuzzy model, and then a model-based fuzzy control was developed to stabilize the fuzzy control system. However, none of them dealt with the maximum stability margin for nonlinear uncertain systems using fuzzy control. In this situation, a fuzzy control design with a stability margin that is large enough to tolerate as much as possible the effects of the approximation error and uncertain dynamics would be more appealing for practical reasons.

In this study, motivated by Takagi and Sugeno’s fuzzy model [9], a nonlinear uncertain system is approximated by a fuzzy model. The approximation error between the original nonlinear system and fuzzy model is considered as the perturbation part of the system. Therefore, a control design with the maximum stability margin to override the perturbed part of the fuzzy system is needed to guarantee stability of the fuzzy system. Based on the fuzzy model, the maximum stability margin problem is parameterized in terms of a generalized eigenvalue problem (GEVP) using fuzzy control. The GEVP is used to minimize the maximum general eigenvalue of a matrix that depends affinely on a variable, subject to some linear matrix inequality (LMI) constraints. For the case where state variables are unavailable, a fuzzy observer-based control scheme is also proposed to deal with the maximum stability margin for nonlinear uncertain systems.
control is also proposed to tackle the maximum stability margin problem for nonlinear uncertain systems. Using a suboptimal approach, the maximum stability margin problem via fuzzy observer-based control is characterized in terms of a corresponding linear matrix inequality problem (LMI). The LMI is used to find a solution such that the corresponding LMIs are feasible. The GEVP and LMI can be solved very efficiently using convex optimization techniques [20-22]. The primary contribution of this paper is to extend the maximum stability margin from linear uncertain systems to nonlinear uncertain systems using fuzzy control.

The paper is organized as follows: The maximum stability margin design of state feedback fuzzy control for nonlinear uncertain systems is introduced in Section 2. The design problem of a fuzzy observer-based control is introduced in Section 3 to investigate the maximum stability margin problem for nonlinear uncertain systems. In Section 4, simulation examples are provided to demonstrate the design procedure. Finally, concluding remarks are made in Section 5.

II. MAXIMUM STABILITY MARGIN DESIGN OF FUZZY CONTROL SYSTEMS

Consider the following nonlinear uncertain system:

$$\dot{x}(t) = f(x(t), \delta) + g(x(t), \varepsilon)u(t),$$

(1)

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^{n \times 1}$ denotes the state vector; $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^{m \times 1}$ denotes the control input; the vector fields $f(x(t), \delta)$ and $g(x(t), \varepsilon)$ are smooth functions; and $\delta = [\delta_1, \delta_2, \ldots, \delta_q] \in \mathbb{R}^q$, $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r] \in \mathbb{R}^r$ denotes uncertain parameter vectors and belongs to the convex polyhedra

$$\Omega_\delta = \{ \delta \mid |\delta_1| \leq \delta_{1, \max}, |\delta_2| \leq \delta_{2, \max}, \ldots, |\delta_q| \leq \delta_{q, \max} \},$$

and

$$\Omega_\varepsilon = \{ \varepsilon \mid |\varepsilon_1| \leq \varepsilon_{1, \max}, |\varepsilon_2| \leq \varepsilon_{2, \max}, \ldots, |\varepsilon_r| \leq \varepsilon_{r, \max} \},$$

respectively, i.e., $\delta \in \Omega_\delta$ and $\varepsilon \in \Omega_\varepsilon$.

A fuzzy linear dynamic model has been proposed by Takagi and Sugeno [9] to represent the local linear input/output relations of nonlinear systems. This fuzzy model is described by fuzzy If-Then rules and will be employed here to deal with the control design problem for the nonlinear uncertain system in (1). The $i$th rule of the fuzzy model for the nonlinear uncertain system (1) takes the following form [14,17-19]:

**Plant Rule i:**

If $z_i(t)$ is $F_{ii}$ and ... and $z_q(t)$ is $F_{iq}$

Then $\dot{x}(t) = A_i x(t) + B_i u(t)$ for $i = 1, 2, \ldots, L,$

(2)

where $F_i$ is the fuzzy set, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$; $L$ is the number of If-Then rules; and $z_i(t), z_2(t), \ldots, z_q(t)$ are the premise variables.

The fuzzy system in (2) is inferred as follows [14,17-19]:

$$\dot{x}(t) = \frac{\sum_{i=1}^L \mu_i(z(t)) [A_i x(t) + B_i u(t)]}{\sum_{i=1}^L \mu_i(z(t))}$$

$$= \sum_{i=1}^L h_i(z(t)) [A_i x(t) + B_i u(t)],$$

(3)

where

$$\mu_i(z(t)) = \prod_{j=1}^q F_{ij}(z_j(t))$$

$$h_i(z(t)) = \frac{\mu_i(z(t))}{\sum_{i=1}^L \mu_i(z(t))}$$

and $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in $F_{ij}$.

In this paper, we assume that

$$\mu_i(z(t)) \geq 0$$

and

$$\sum_{i=1}^L \mu_i(z(t)) > 0 \text{ for } i = 1, 2, \ldots, L$$

for all $t$.

Therefore, we get

$$h_i(z(t)) \geq 0 \text{ for } i = 1, 2, \ldots, L$$

(5)

and

$$\sum_{i=1}^L h_i(z(t)) = 1.$$
\[
\Delta f = f(x(t), \delta) - \sum_{i=1}^{k} h_i(z(t))A_i x(t)
\]
(8)

and

\[
\Delta g = [g(x(t), \varepsilon) - \sum_{i=1}^{k} h_i(z(t))B_i u(t)]
\]
(9)

denote the approximation error between the nonlinear uncertain system (1) and the fuzzy model (3).

**Remark 1.** If we assume that \( g = n \) and \( z_i(t) = x_i(t), \) \( z_n(t) = x_n(t), \) then the plant rule can be represented as follows:

**Plant Rule \( i: \)**

If \( x_i(t) \) is \( F_{i1} \) and ... and \( x_n(t) \) is \( F_{in} \), then

\[
x(t) = A_i x(t) + B_i u(t)
\]
(10)

for \( i = 1, 2, \ldots, L. \)

Suppose the following fuzzy controller is employed to deal with the above control system design.

**Control Rule \( j: \)**

If \( z_j(t) \) is \( F_{ji} \) and ... and \( z_n(t) \) is \( F_{jn} \), then

\[
u(t) = K_j x(t)
\]
(11)

Hence, the fuzzy controller is given by

\[
u(t) = \frac{\sum_{j=1}^{L} \mu_j(z(t)) [K_j x(t)]}{\sum_{j=1}^{L} \mu_j(z(t))} = \sum_{j=1}^{k} h_j(z(t)) [K_j x(t)]
\]
(12)

where \( h_j(z(t)) \) is defined in (4) and \( K_j \) (for \( j = 1, 2, \ldots, L \)) represents the control parameters.

After substituting (12) into (7) and performing some manipulation, (7) can be expressed in the following form:

\[
x(t) = \sum_{i=1}^{k} h_i(z(t)) \sum_{j=1}^{k} h_j(z(t)) [A_i + B_j K_j] x(t) + [\Delta f + \Delta g]
\]
(13)

Suppose that there exist design matrices \( \Delta A \) and \( \Delta B \) such that

\[
\sup_{\varepsilon \in \Omega_\varepsilon} |\Delta f| \leq |\Delta A x(t)|
\]
(14)

and

\[
\sup_{\varepsilon \in \Omega_\varepsilon} |\Delta g| \leq \left| \sum_{j=1}^{k} h_j(z(t)) \Delta B K_j x(t) \right|
\]
(15)

for all trajectories.

According to (14) and (15), we obtain

\[
\sup_{\varepsilon \in \Omega_\varepsilon} (\Delta f) (\Delta g) = \sup_{\varepsilon \in \Omega_\varepsilon} \left[ f(x, \delta) - \sum_{i=1}^{k} h_i(z(t))A_i x(t) \right] \times \left[ f(x, \delta) - \sum_{i=1}^{k} h_i(z(t))A_i x(t) \right] \\
\leq (\Delta A x(t))^T (\Delta A x(t))
\]
(16)

\[
\sup_{\varepsilon \in \Omega_\varepsilon} (\Delta g) (\Delta g) \leq \left| \sum_{j=1}^{k} h_j(z(t)) \Delta B K_j x(t) \right|^2 \\
\leq \left| \sum_{j=1}^{k} h_j(z(t)) \Delta B K_j x(t) \right|
\]
(17)

**Remark 2.** Under the convexity assumption, the superior in (16)-(17) can be reached by one of the vertices of the convex polyhedron. Therefore, only the vertices of the convex polyhedron need to be checked to obtain the superior in (16)-(17).

If the approximation error is scaled by a positive \( \alpha \), the closed-loop nonlinear uncertain system (13) can be rewritten as

\[
x(t) = \sum_{i=1}^{k} h_i(z(t)) \sum_{j=1}^{k} h_j(z(t)) [A_i + B_j K_j] x(t) + \alpha [\Delta f + \Delta g]
\]
(18)

The maximum stability margin for the fuzzy system in (18) is defined as the largest \( \alpha \geq 0 \) for which the fuzzy system (18) is quadratically stable [20], i.e., the largest tolerable approximation error \( \alpha [\Delta f + \Delta g] \) under quadratic stability. Obviously, it is appealing for control engineers to specify control parameters \( K_j \) in the fuzzy controller (12) to achieve the maximum stability margin and, therefore, maximum tolerance of the approximation error \( \alpha [\Delta f + \Delta g] \).

The following well-known lemmas are useful for our design.

**Lemma 1** [20]. If the quadratic Lyapunov function \( V(x) = x^T P x \) is used to establish a quadratic stability margin of the fuzzy system (18), and if there exists a positive definite matrix \( P > 0 \) such that \( dV(x)/dt < 0 \) along every nonzero trajectory, then the dynamic system is quadratically stable.
Lemma 2 \cite{25}. For any matrices (or vectors) \( X \) and \( Y \) with appropriate dimensions, we have
\[
X^{T}Y + Y^{T}X \leq X^{T}FX + Y^{T}F^{-1}Y,
\] (19)
where \( F \) is any positive-definite symmetric matrix. In this paper, we let \( F \) be an identity matrix.

Let us choose a Lyapunov function for the fuzzy system (18) as
\[
V(t) = x^T(t)Px(t),
\] (20)
where the weighting matrix \( P = P^{T} > 0 \).

By differentiating (20), we obtain
\[
\dot{V}(\alpha) = x^T P x + x^T P \alpha \Delta \phi + \alpha (\Delta \phi)^T P \dot{x} = 0.
\]

Step 1.

Proof. From (21) and (22), we obtain
\[
\dot{V}(\alpha) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_{i}(z(t))h_{j}(z(t)) x^{T} \left[ \left( \dot{A}_{i} + B_{i}K_{j} \right)^{T} P + P \left( \dot{A}_{i} + B_{i}K_{j} \right) + \alpha \Delta \phi \right] x.
\] (22)

Then, the fuzzy control nonlinear uncertain system (18) is quadratically stable.

Proof. From (21) and (22), we obtain
\[
\dot{V}(\alpha) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_{i}(z(t))h_{j}(z(t)) x^{T} \left[ \left( \dot{A}_{i} + B_{i}K_{j} \right)^{T} P + P \left( \dot{A}_{i} + B_{i}K_{j} \right) + \alpha \Delta \phi \right] x.
\]

This completes the proof.

If we introduce new variables \( W = P^{-1} \) and \( Y_{j} = K_{j}W \) and the Schur complements [20], (22) will be equivalent to the following LMIs:
\[
\begin{bmatrix}
S & (\alpha \Delta A)^{T} (\alpha \Delta B)_{Y_{j}} \\
\alpha \Delta A W & -I & 0 \\
\alpha \Delta B Y_{j} & 0 & -I
\end{bmatrix} < 0,
\] (23)

where \( S = W^{T}A_{i}W + A_{i}W + B_{i}Y_{j} + Y_{j}^{T}B_{i}^{T} + 2I \).

The maximum quadratic stability margin for the fuzzy nonlinear uncertain system (18) can be computed using the following generalized eigenvalue problem (GEVP) [20] in \( W \) and \( Y_{j} \) (for \( j = 1, 2, \ldots, L \)):
\[
\max \quad \alpha
\]
\[
\text{subject to} \quad W = W^{T} > 0, \quad \alpha \geq 0 \text{ and (23).}
\] (24)
i.e., the maximum value of \( \alpha \) is achieved so that quadratic stability still holds.

The design procedure for the maximum stability margin for fuzzy control systems is summarized as follows:

**Design Procedure 1.**

**Step 1.** Select fuzzy plant rules (2).

**Step 2.** Solve the GEVP in (24) to obtain \( W \) and \( Y_{j} \) (thus, \( K_{j} = W^{-1}Y_{j} \) can also be obtained).

**Step 3.** Check the assumptions of \( \sup_{\Delta \phi \in \Delta \phi} |\Delta \phi| \leq \left| \Delta A x(t) \right| \) and \( \sup_{t \in \Delta \phi} |\Delta \phi| \leq \left| \sum_{j=1}^{L} h_{j}(z(t)) \right| \Delta B K_{j} x(t) \) according to Remark 2. If they are not
satisfied, adjust (expand) the bounds for all elements in $\Delta A$ and $\Delta B$, and then repeat Steps 2-3.

**Step 4.** Obtain the fuzzy control rule in (12).

### III. Maximum Stability Margin Design of Fuzzy Observer-Based Control Systems

In the previous sections, we assumed that all the state variables were available. In practice, this assumption often does not hold. In this situation, we need to estimate the state vector $x$ from the output $y$ for state feedback control.

Consider the following nonlinear uncertain system:

$$\dot{x}(t) = f(x(t), \delta(t)) + g(x(t), \varepsilon)u(t)$$

$$y(t) = h(x(t), \phi),$$

where $y(t) \in \mathbb{R}^{n \times 1}$ is the output of the system and $\phi = [\phi_1, \phi_2, \ldots, \phi_r] \in \mathbb{R}^r$ denotes an uncertain parameter vector and belongs to the convex polyhedron

$$\Omega_\phi = \{ \phi \mid \phi_1 \leq \varnothing_1, \phi_2 \leq \varnothing_2, \ldots, \phi_r \leq \varnothing_r \},$$

i.e., $\phi \in \Omega_\phi$.

The $i$th rule of the fuzzy model for the nonlinear uncertain system (25) takes the following form [14,17-19]:

**Plant Rule $i$:**

If $z_i(t)$ is $F_{ii}$ and ... and $z_i(t)$ is $F_{ig}$

Then $x(t) = A_i x(t) + B_i \mu(t)$

$$y(t) = C_i x$$ for $i = 1, 2, \ldots, L,$

where $C_i \in \mathbb{R}^{p \times n}$.

The state dynamic of the fuzzy system is the same as (3), and the output of the fuzzy system is inferred as follows [14,17-19]:

$$y(t) = \sum_{i=1}^{L} h_i(z(t)) C_i x(t).$$

Therefore, the output of the nonlinear uncertain system in (25) can be rearranged as the following equivalent system:

$$y(t) = h(x(t), \phi) = \sum_{i=1}^{L} h_i(z(t)) C_i x(t) + \Delta_h,$$

$$\Delta_h = h(x(t), \phi) - \sum_{i=1}^{L} h_i(z(t)) C_i x(t)$$

denotes the approximation error between the output of the nonlinear uncertain system (25) and the fuzzy output (27).

Suppose the following fuzzy observer is used to deal with state estimation of the nonlinear uncertain system in (25):

**Observer Rule $i$:**

If $z_i(t)$ is $F_{ii}$ and ... and $z_i(t)$ is $F_{ig}$

Then $\dot{x}(t) = A_i \dot{x}(t) + B_i \mu(t) + L_i(y(t) - \hat{y}(t))$,

where $L_i$ is the observer gain for the $i$th observer rule and

$$\hat{y}(t) = \sum_{i=1}^{L} h_i(z(t)) C_i \dot{x}(t).$$

The overall fuzzy observer is represented as follows:

$$\dot{x}(t) = \sum_{i=1}^{L} h_i(z(t)) [A_i \dot{x}(t) + B_i \mu(t) + L_i(y(t) - \hat{y}(t))]$$

$$= \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) [A_i \dot{x}(t) + B_i \mu(t) + L_i(y(t) - \hat{y}(t))]$$

$$+ L_i C_i [x(t) - \hat{x}(t)] + L_i \Delta h_i.$$

Let us denote the estimation errors as

$$e(t) = x(t) - \hat{x}(t).$$

By differentiating (32), we get

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$

$$= f(x(t), \delta(t)) + g(x(t), \varepsilon)u(t)$$

$$- \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) [A_i \dot{x}(t) + B_i \mu(t) + L_i(y(t) - \hat{y}(t))]$$

$$- \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) [A_i \dot{x}(t) + B_i \mu(t) + L_i(y(t) - \hat{y}(t))]$$

$$+ L_i C_i [x(t) - \hat{x}(t)] + L_i \Delta h_i].$$

Then, the augmented system can be written in the following form:
\[
\begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix} = \begin{bmatrix}
\sum_{j=1}^{k} h_j(z(t)) \sum_{j=1}^{k} h_j(z(t)) [A_j x(t) + B_j u(t) + L_j C_j e(t)] + L_j \Delta h_j \\
\sum_{j=1}^{k} h_j(z(t)) \sum_{j=1}^{k} h_j(z(t)) [(A_j - L_j C_j) x(t) - L_j \Delta h_j] + L_j \Delta g_j
\end{bmatrix}.
\]

(34)

Hence, the fuzzy observer-based controller is modified as follows:

\[
u(t) = \sum_{j=1}^{k} h_j(z(t)) (K_j \hat{x}(t)).
\]

(35)

After manipulation, (34) can be expressed in the following form:

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix} = \sum_{j=1}^{k} h_j(z(t)) \sum_{j=1}^{k} h_j(z(t)) \begin{bmatrix}
A_j + B_j K_j & L_j C_j \\
0 & A_j - L_j C_j
\end{bmatrix} \begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} \Delta f + \begin{bmatrix}
0 \\
0
\end{bmatrix} \Delta g.
\]

(36)

Let us denote

\[
\begin{align*}
x(t) &= \begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix}, \\
\Delta f &= \begin{bmatrix}
0 \\
\Delta f
\end{bmatrix}, \\
\Delta g &= \begin{bmatrix}
0 \\
\Delta g
\end{bmatrix}, \\
\Delta h &= \begin{bmatrix}
\sum_{j=1}^{k} h_j(z(t)) L_j \Delta h_j \\
-\sum_{j=1}^{k} h_j(z(t)) L_j \Delta h_j
\end{bmatrix}, \\
\bar{A}_j &= \begin{bmatrix}
A_j + B_j K_j & L_j C_j \\
0 & A_j - L_j C_j
\end{bmatrix}.
\end{align*}
\]

(37)

Therefore, the augmented system defined in (36) can be expressed in the following form:

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix} = \sum_{j=1}^{k} h_j(z(t)) \sum_{j=1}^{k} h_j(z(t)) \bar{A}_j \hat{x}(t) + \Delta h + \Delta \hat{e} + \Delta g.
\]

(38)

Suppose there exists a design matrix \( \Delta C \) such that

\[
\sup_{\forall \delta \in \Delta \phi} \left\| \sum_{j=1}^{k} h_j(z(t)) L_j \Delta h_j \right\| \leq \sum_{j=1}^{k} h_j(z(t)) L_j \Delta C x(t)
\]

(39)

for all trajectories.

According to (14), (15) and (39), we obtain

\[
\sup_{\forall \delta \in \Delta \phi} (\Delta \hat{e})^T (\Delta \hat{e}) = \sup_{\forall \delta \in \Delta \phi} (\Delta g)^T (\Delta g) \leq (\Delta A x(t))^T (\Delta A x(t)) = (\Delta A, \Delta A) x(t) = (\Phi x(t))^T (\Phi x(t)),
\]

(40)

where \( \Phi = [\Delta A, \Delta A] \).

\[
\sup_{\forall \delta \in \Delta \phi} (\Delta g)^T (\Delta g) \leq \left( \sum_{j=1}^{k} h_j(z(t)) A_j x(t) \right)^T \times \left( \sum_{j=1}^{k} h_j(z(t)) A_j x(t) \right)
\]

\[
(\Delta A x(t))^T (\Delta A x(t)) = (\Delta A, \Delta A) x(t) = (\Phi x(t))^T (\Phi x(t)),
\]

(41)

where \( \Omega_j = [\Delta B K_j, 0] \) for \( j = 1, 2, ..., L \) and

\[
\sup_{\forall \delta \in \Delta \phi} (\Delta \hat{e})^T (\Delta \hat{e}) = \sup_{\forall \delta \in \Delta \phi} 2 \left( \sum_{j=1}^{k} h_j(z(t)) L_j \Delta h_j \right)^T \times \left( \sum_{j=1}^{k} h_j(z(t)) L_j \Delta h_j \right)
\]

\[
\leq 2 \left( \sum_{j=1}^{k} h_j(z(t)) L_j \Delta C x(t) \right)^T \times \left( \sum_{j=1}^{k} h_j(z(t)) L_j \Delta C x(t) \right)
\]

(42)
\[
\times \left\{ \sum_{i=1}^{l} h_i(z(t))\eta_i[L, \Delta C, L, \Delta C|x(t) \right\} \\
= 2 \left\{ \sum_{i=1}^{l} h_i(z(t))\Xi_i x(t) \right\}^T \times \left\{ \sum_{i=1}^{l} h_i(z(t))\Xi_i x(t) \right\} \\
\leq 2 \sum_{i=1}^{l} h_i(z(t))\eta_i^T(\Xi_i)^T \Xi_i\eta, \tag{42}
\]
where \(\Xi_i = [L, \Delta C, L, \Delta C] \) for \(i = 1, 2, \ldots, L\).

If the approximation error is scaled by a positive \(\alpha\), the closed-loop nonlinear uncertain system (38) can be rewritten as

\[
\dot{x}(t) = \sum_{i=1}^{l} h_i(z(t))\sum_{j=1}^{l} h_j(z(t))\overline{A}_{ij} x(t) + \alpha(\Delta h_i + \Delta f + \Delta g). \tag{43}
\]

The maximum stability margin for the augmented fuzzy system in (43) is defined as the largest \(\alpha \geq 0\) for which the augmented fuzzy system (43) is quadratically stable.

Let us choose a Lyapunov function for the system (43) as

\[
V(t) = \dot{x}^T(t)\bar{P} \dot{x}(t), \tag{44}
\]
where the weighting matrix \(\bar{P} = \bar{P}^T > 0\).

Differentiating (44), we obtain

\[
\dot{V}(t) = \dot{x}^T(t)\bar{P} \dot{x}(t) + x^T(t)\bar{P}\dot{x}(t) \\
= \left[ \sum_{i=1}^{l} h_i(z(t))\sum_{j=1}^{l} h_j(z(t))\overline{A}_{ij} x(t) + \alpha(\Delta h_i + \Delta f + \Delta g) \right]^T \bar{P} \dot{x}(t) \\
+ \dot{x}^T(t)\bar{P} \left[ \sum_{i=1}^{l} h_i(z(t))\sum_{j=1}^{l} h_j(z(t))\overline{A}_{ij} \dot{x}(t) \right] \bar{P} \dot{x}(t) \\
+ \dot{x}^T(t)\bar{P}\alpha(\Delta h_i + \Delta f + \Delta g) \bar{P} \dot{x}(t) \\
\leq \left[ \sum_{i=1}^{l} h_i(z(t))\sum_{j=1}^{l} h_j(z(t))\overline{A}_{ij} \dot{x}(t) \right]^T \bar{P} \dot{x}(t) \\
+ \dot{x}^T(t)\bar{P} \left[ \sum_{i=1}^{l} h_i(z(t))\sum_{j=1}^{l} h_j(z(t))\overline{A}_{ij} \dot{x}(t) \right] \\
+ \dot{x}^T(t)\bar{P}\alpha(\Delta h_i + \Delta f + \Delta g) \bar{P} \dot{x}(t) + 2\alpha \left[ \sum_{i=1}^{l} h_i(z(t))\overline{\Xi}_i(x(t)) \right]^T \\
\times \left[ \sum_{i=1}^{l} h_i(z(t))\overline{\Xi}_i(x(t)) \right] + \alpha^2(\Phi \dot{x}(t))^T \Phi \dot{x}(t) \\
+ \alpha^2 \left( \sum_{i=1}^{l} h_i(z(t))\overline{\Omega}_i \right)^T \left( \sum_{i=1}^{l} h_i(z(t))\overline{\Omega}_i \right) + 3\dot{x}^T(t)\bar{P} \dot{x}(t) \leq \sum_{i=1}^{l} \sum_{j=1}^{l} h_i(z(t))h_j(z(t))\eta_i^T(\overline{A}_{ij}^T \bar{P} + \bar{P} \overline{A}_{ij}) \\
+ 2\alpha^2 \overline{\Xi}_i^T \overline{\Xi}_i + \alpha^2 \Phi^T \Phi + \alpha^2 \overline{\Omega}_i^T \overline{\Omega}_i + 3\dot{x}^T(t)\bar{P} \dot{x}(t). \tag{45}
\]

Then, we obtain the following result:

**Theorem 2.** In the augmented fuzzy system (43), suppose there exists a positive definite matrix \(\bar{P} = \bar{P}^T > 0\) such that the following matrix inequalities are satisfied:

\[
\overline{A}_{ij}^T \bar{P} + \bar{P} \overline{A}_{ij} + 2\alpha^2 \overline{\Xi}_i^T \overline{\Xi}_i + \alpha^2 \Phi^T \Phi + \alpha^2 \overline{\Omega}_i^T \overline{\Omega}_i + 3\bar{P} \dot{x}(t) < 0. \tag{46}
\]

for \(i, j = 1, 2, \ldots, L\). Then, the augmented fuzzy system (43) is quadratically stable.

**Proof.** From (45) and (46), we obtain

\[
\dot{V}(t) \leq \sum_{i=1}^{l} \sum_{j=1}^{l} h_i(z(t))h_j(z(t))\eta_i^T(\overline{A}_{ij}^T \bar{P} + \bar{P} \overline{A}_{ij}) \\
+ 2\alpha^2 \overline{\Xi}_i^T \overline{\Xi}_i + \alpha^2 \Phi^T \Phi + \alpha^2 \overline{\Omega}_i^T \overline{\Omega}_i + 3\dot{x}^T(t)\bar{P} \dot{x}(t) < 0.
\]

This completes the proof.

After the quadratic stability of the fuzzy observer-based nonlinear control system is obtained in Theorem 2, the maximum stability margin problem is to maximize \(\alpha\) such that the matrix inequalities in (46) still have a common positive definite solution \(\bar{P}\) (i.e., the quadratic stability still holds).

If \(\bar{P}\) is chosen such that it has the following form

\[
\bar{P} = \begin{bmatrix} \bar{P}_{11} & 0 \\ 0 & \bar{P}_{22} \end{bmatrix}, \tag{47}
\]

where \(\bar{P}_{11} = \bar{P}_{11}^T > 0\) and \(\bar{P}_{22} = \bar{P}_{22}^T > 0\), then by substituting (47) into (46), we can obtain

\[
\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} < 0, \tag{48}
\]

where
\[ R_{11} = A_t \tilde{P}_{11} + \tilde{P}_{11} A_j + (B, K_j)^T \tilde{P}_{11} + \alpha \tilde{C}(L, \Delta C)^T (L, \Delta C) \]
\[ + \alpha \Delta \Delta^T \Delta \Delta + \alpha \Delta K_j^T \Delta B L_i \]
\[ R_{12} = R_{21} = \tilde{P}_{11} L_i C_j + \alpha \tilde{C}(2L, \Delta C)^T (L, \Delta C) + \Delta \Delta^T \Delta \Delta^T \Delta \Delta, \]
\[ R_{22} = (A_j - L_i C_j)^T \tilde{P}_{22} + \tilde{P}_{22} (A_j - L_i C_j) + 2\alpha \tilde{C}(L, \Delta C)^T (L, \Delta C) + \alpha \Delta \Delta^T \Delta \Delta + 3\tilde{P}_{22} \tilde{P}_{22}. \]

By introducing a new matrix
\[
\tilde{W} = \begin{bmatrix} \tilde{W}_{11} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{P}_{11}^{-1} & 0 \\ 0 & I \end{bmatrix},
\]
where \( \tilde{W}_{11} = \tilde{P}_{11}^{-1} \), and by premultiplying and postmultiplying it to (48), we can obtain
\[
\tilde{W} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tilde{W} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0,
\]
where
\[ S_{11} = \tilde{W}_{11} A_t^T + A_j \tilde{W}_{11} + \alpha \tilde{C} \tilde{W}_{11} (2L, \Delta C)^T \]
\[ \times (L, \Delta C) + \Delta \Delta^T \Delta \Delta \tilde{W}_{11} + \tilde{W}_{11} (B, K_j)^T \]
\[ + B, K_j \tilde{W}_{11} + \alpha \tilde{C} \tilde{W}_{11} K_j^T \Delta B L_i \tilde{W}_{11} + 3I, \]
\[ S_{12} = S_{21}^T = L_i C_j + \alpha \tilde{C} \tilde{W}_{11} (2L, \Delta C)^T \]
\[ \times (L, \Delta C) + \Delta \Delta^T \Delta \Delta^T \Delta \Delta, \]
\[ S_{22} = (A_j - L_i C_j)^T \tilde{P}_{22} + \tilde{P}_{22} (A_j - L_i C_j) + 2\alpha \tilde{C}(L, \Delta C)^T (L, \Delta C) + \alpha \Delta \Delta^T \Delta \Delta + 3\tilde{P}_{22} \tilde{P}_{22}. \]

With \( Y_j = K_i \tilde{W}_{11} \) and \( Z_i = \tilde{P}_{22} L_i \), and using the Schur complements, the matrix inequalities (50) can be rearranged in the following forms:
\[
\begin{bmatrix}
M_{11} & \tilde{W}_{11} & Y_j^T & M_{14} & 0 \\
\tilde{W}_{11} & M_{22} & 0 & 0 & 0 \\
Y_j & 0 & M_{33} & 0 & 0 \\
M_{14} & 0 & 0 & M_{44} & (L, \Delta C)^T \\
0 & 0 & 0 & L, \Delta C & -(2\alpha I)^{-1}
\end{bmatrix} < 0,
\]
where
\[ M_{11} = \tilde{W}_{11} A_t^T + A_j \tilde{W}_{11} + (B, Y_j)^T + B, Y_j + 3I, \]
\[ M_{22} = -((\alpha \tilde{C}(2L, \Delta C)^T (L, \Delta C) + \Delta \Delta^T \Delta \Delta)^{-1}, \]
\[ M_{14} = M_{41} = -L_i C_j + \alpha \tilde{W}_{11}(2L, \Delta C)^T (L, \Delta C) + \Delta \Delta^T \Delta \Delta, \]
\[ M_{33} = -(\alpha \Delta \Delta^T \Delta \Delta)^{-1}, \]
\[ M_{44} = A_j^T \tilde{P}_{22} + \tilde{P}_{22} (A_j - (Z_i C_j)^T - Z_i C_j) + \alpha \Delta \Delta^T \Delta \Delta + 3\tilde{P}_{22} \tilde{P}_{22}. \]

Therefore, the maximum stability margin for the augmented fuzzy system (38) can be characterized as the following maximization problem:
\[ \max \alpha \]
\[ \{ \tilde{W}_{11}, Y_j, \tilde{P}_{22}, ..., Y_L, \}
\[ \tilde{P}_{22}, Z_i, Z_j, ..., Z_L \}
\]
subject to \( \tilde{W}_{11} > 0, \tilde{P}_{22} > 0, \alpha \geq 0, \) and (51). (52)

The above analysis shows that when dealing with the maximum stability margin problem of the fuzzy observer-based control system, the most important task is to find the maximum value of \( \alpha \) and solve for common solutions \( \tilde{W}_{11} = \tilde{W}_{11} > 0, \) \( \tilde{P}_{22} = \tilde{P}_{22} > 0 \) from the maximization problem (52). Since four parameters, \( \tilde{P}_{11}, \tilde{P}_{22}, \) and \( L_i \) should be determined from (51), there have been no effective algorithms solving them simultaneously until now. However, a suboptimal (near maximum) stability margin can be obtained by means of the following separation of observer and control designs.

We can easily check that the matrix inequalities (51) imply
\[ A_j^T \tilde{P}_{22} + \tilde{P}_{22} (A_j - (Z_i C_j)^T - Z_i C_j) \]
\[ + \alpha \Delta \Delta^T \Delta \Delta + 3\tilde{P}_{22} \tilde{P}_{22} < 0, \]
which is related to the fuzzy observer design. Using the Schur complements, (53) can be transformed into the following linear matrix inequalities (LMIs) if \( \alpha \) is given in advance:
\[ \begin{bmatrix}
Z & \tilde{P}_{22} \\
\tilde{P}_{22} & -(3I)^{-1}
\end{bmatrix} < 0, \]
where \( Z = A_j^T \tilde{P}_{22} + \tilde{P}_{22} A_j - (Z_i C_j)^T - Z_i C_j + \alpha \Delta \Delta^T \Delta \Delta, \) for \( i, j = 1, 2, ..., L. \)

Note that solving \( \tilde{P}_{22} \) and \( Z \) from (54) is a standard linear matrix inequality problem (LMIP). If we solve the LMIP in (54) to obtain \( \tilde{P}_{22} \) and \( Z \) (thus, \( L_i = \tilde{P}_{11}^{-1} Z \)) and then substitute \( \tilde{P}_{22}, Z, \) and \( L_i \) into (51), (51) becomes a standard
set of linear matrix inequalities (LMIs). Then, we can
solve the LMIP in (51) to obtain $\bar{W}_{11}$ and $Y_j$ (thus, $K_j = Y_j^{-1} \bar{W}_{11}$). Therefore, the maximum stability margin
problem in (52) is reduced to the following suboptimal problem:

$$
\max \alpha \\
\{\bar{W}_{11}, \bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_L, \\
\bar{P}_{22}, Z_1, Z_2, \ldots, Z_L\}
$$

subject to $\bar{W}_{11} > 0$, $\bar{P}_{22} > 0$, (54) and (51). (55)

This problem can be solved by increasing $\alpha$ until
$\bar{P}_{22} > 0$ in (54) and $\bar{W}_{11} > 0$ in (51) cannot be found.

The design procedure for the suboptimal stability
margin for fuzzy observer-based control systems can be
summarized as follows:

**Design Procedure 2.**

**Step 1.** Select fuzzy plant rules (26).

**Step 2.** Start with $\alpha = 0$.

**Step 3.** Solve the LMIP in (54) to obtain $\bar{P}_{22}$ and $Z_i$ (thus, $L_i = F_{22}^{-1}Z_i$ can also be obtained).

**Step 4.** Substitute $\bar{P}_{22}$, $Z_i$, and $L_i$ into (51) and then solve
the LMIP in (51) to obtain $\bar{W}_{11}$ and $Y_j$ (thus $K_j = Y_j^{-1} \bar{W}_{11}$ can also be obtained).

**Step 5.** Increase $\alpha$ and repeat Steps 3-5 until $\bar{P}_{22} > 0$ and
$\bar{W}_{11} > 0$ cannot be found.

**Step 6.** Check the assumptions of

$$
\sup_{\forall \phi \in \Omega} \left| \Delta \phi \right| \leq \left| \Delta \chi(t) \right|
$$

$$
\sup_{\forall \phi \in \Omega} \left| \Delta \phi \right| \leq \sum_{j=1}^{L} h_j(\phi(t)) \Delta B K_j \phi(t)
$$

and

$$
\sup_{\forall \phi \in \Omega} \left| \Delta \phi \right| \leq \sum_{j=1}^{L} h_j(\phi(t)) \Delta C \chi(t)
$$

according to Remark 2. If they are not satisfied, adjust (expand) the bounds for all elements in $\Delta A$, $\Delta B$ and $\Delta C$ and then repeat Steps 3-6.

**Step 7.** Construct the fuzzy observer in (31).

**Step 8.** Obtain fuzzy control rules in (35).

**Remark 3.** In general, it is not easy to solve the GEVP
and LMIP analytically. Fortunately, the GEVP and
LMIP can be solved very efficiently using a convex
optimization technique, such as the interior point algo-

**IV. SIMULATION EXAMPLES**

To demonstrate use of the proposed fuzzy control
approach, a control problem of balancing an inverted
pendulum on a cart was considered in this study. For this
example, the state equations of the inverted pendulum
were given by

$$
\dot{x}_1 = x_2,
$$

$$
\dot{x}_2 = \frac{1.0}{[(M + m)(J + m\ell^2) - (ml \cos x_1)^2]}
$$

$$
\times [-f_1(M + m)x_2 - (mx_2)^2 \sin x_1 \cos x_1 + (M + m)mg \sin x_1 - ml \cos x_1 u],
$$

$$
y = x_1,
$$

where $x_1$ denotes the angle (rad) of the pendulum from the
vertical, $x_2$ is the angular velocity (rad/s), $g = 9.8 m/s^2$ is the
gravity constant, $m$ is the mass (kg) of the pendulum, $M$
the mass (kg) of the cart, $f_1$ is the friction factor (N/rad/s)
of the pendulum, $l$ is the length (m) from the center of mass
of the pendulum to the shaft axis, $J$ is the moment of inertia
(kgm$^2$) of the pendulum, and $u$ is the force (N) applied
to the cart. The parameters in this example were assumed to
be $m = 0.3$, $M = 15$, $l = 0.3$, $J = 0.005 + \eta_1 (kg m^2)$, and
$f_1 = 0.007 + \eta_2 (N/rad/s)$, while the uncertain parameters
were $\eta_1 = 0.0001 \sin(t)$, and $\eta_2 = 0.0005 \sin(t)$, i.e., $\left| \eta_1 \right|
\leq 0.0001$ and $\left| \eta_2 \right| \leq 0.0005$.

**Example 1.** A design case of the maximum stability
margin in fuzzy control systems with state variables
available.

Now, if we employ the Design Procedure 1 de-
scribed in the previous section, then the maximum stability
margin control design will involves the following steps.

**Step 1:** To use the fuzzy control approach, we must have a fuzzy model which represents the dynamics of the nonlinear uncertain plant. Therefore, we first represent
the system (56) using a fuzzy model. The Takagi-Sugeno
fuzzy model is an approximation linear model around a
certain operating point for a nonlinear system. The uncertain part is not considered in the fuzzy model. In other
words, the fuzzy model is represented for the nominal nonlinear system only (without considering the uncertain part). The approximation errors between the fuzzy model
and original nonlinear uncertain system are considered in
the design parameters $\Delta A$, $\Delta B$ and $\Delta C$. Let us consider the state variable $x_2$ in (56) as follows:

$$
x_2 = \frac{1.0}{[(M + m)(J + m\ell^2) - (ml \cos x_1)^2]}
$$
\[
\times [-f_1(M+m)x_2 - (mlx_2)^2 \sin x_1 \cos x_1 \\
+ (M+m)mg \sin x_1 - ml \cos x_1 u] + \left[-\frac{1}{ml \cos x_1} \right] u
\]

If we neglect the high order term \((mlx_2)^2 \sin x_1 \cos x_1\), then the above equation is reduced to

\[
x_2 = \frac{1.0}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \times [-f_1(M+m)x_2 + (M+m)mg \sin x_1 - ml \cos x_1 u]
\]

\[
= \left\{ \frac{[(M+m)mg \sin x_1] \cdot \frac{1}{x_1}}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \right\} \cdot x_1
\]

\[
- \left\{ \frac{f_1(M+m)}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \right\} \cdot x_2
\]

\[
- \left\{ \frac{ml \cos x_1}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \right\} \cdot u
\]

Therefore,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{(M+m)mg \sin x_1}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \\
- \frac{f_1(M+m)}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \\
- \frac{ml \cos x_1}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} 
\end{bmatrix} \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix}
\]

\[
+ \left[-\frac{1}{ml \cos x_1} \right] u
\]

if \(x_1 = 0\),

or

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{(M+m)mg \sin x_1 \cdot \frac{1}{x_1}}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \\
n- \frac{f_1(M+m)}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} \\
- \frac{ml \cos x_1}{[(M+m)(J+ml^2)-(ml \cos x_1)^2]} 
\end{bmatrix} \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix}
\]

To minimize the design effort and complexity, we wish to use as few rules as possible. Hence, we approximate the system using the following four-rule fuzzy model.

**Rule 1**: If \(x_1\) is about 0

Then \(\dot{x} = A_1 x + B_1 u, y = C_1 x\),

where

\[
A_1 = \begin{bmatrix}
0 & 1 \\
28.0262 & -0.2224
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 1 \\
27.4065 & -0.2220
\end{bmatrix}
\]

**Rule 2**: If \(x_1\) is about \(\pm \frac{2\pi}{9}\)

Then \(\dot{x} = A_2 x + B_2 u, y = C_2 x\),

where

\[
A_3 = \begin{bmatrix}
0 & 1 \\
25.6263 & -0.2209
\end{bmatrix}, \quad A_4 = \begin{bmatrix}
0 & 1 \\
22.8887 & -0.2197
\end{bmatrix}
\]

**Rule 3**: If \(x_1\) is about \(\pm \frac{\pi}{3}\)

Then \(\dot{x} = A_3 x + B_3 u, y = C_3 x\),

where

\[
A_5 = \begin{bmatrix}
0 & 1 \\
-0.1869 & -0.1753
\end{bmatrix}, \quad A_6 = \begin{bmatrix}
0 & 1 \\
-0.1422 & -0.0923
\end{bmatrix}
\]

\[
\Delta A = \begin{bmatrix}
0 & 0.01 \\
0.2289 & 0.0022
\end{bmatrix}, \quad \Delta B = \begin{bmatrix}
0 \\
0.0028
\end{bmatrix}
\]

**Rule 4**: If \(x_1\) is about \(\pm \frac{\pi}{3}\)

Then \(\dot{x} = A_4 x + B_4 u, y = C_4 x\),

where

\[
A_7 = \begin{bmatrix}
0 & 1 \\
-0.1422 & -0.0923
\end{bmatrix}, \quad A_8 = \begin{bmatrix}
0 & 1 \\
-0.1869 & -0.1753
\end{bmatrix}
\]

\[
\Delta A = \begin{bmatrix}
0 & 0.01 \\
0.2289 & 0.0022
\end{bmatrix}, \quad \Delta B = \begin{bmatrix}
0 \\
0.0028
\end{bmatrix}
\]

\[
\Delta C = 0, \quad C_i = \begin{bmatrix}I & 0 \end{bmatrix}
\]

for \(i = 1, \ldots, 4\).

For convenience of design, triangle type membership functions are adopted in Rule 1 to Rule 4.

**Step 2**: Solve GEVP using the LMI optimization toolbox in Matlab. In this case, \(\alpha_{\max} = 3.5\) and

\[
W = \begin{bmatrix}
9.1931 & -45.5436 \\
-45.5436 & 229.5717
\end{bmatrix}
\]
Step 3: The assumptions

\[\sup_{\forall \delta \in \Omega} \left| f(x, \delta) - \sum_{i=1}^{4} h_i(z(t))A_i x(t) \right| \leq \Delta x(t)\]

and

\[\sup_{\forall \varepsilon \in \Omega} \left| g(x, \varepsilon) - \sum_{i=1}^{4} h_i(z(t)) \times \sum_{j=1}^{4} h_j(z(t))B_j K_j x(t) \right| \leq \sum_{j=1}^{4} h_j(z(t)) \Delta B K_j x(t)\]

are satisfied (refer to Figs. 2-3). In this case, \(\delta = [\eta_1, \eta_2]\) and \(\varepsilon = [\eta_1]\).

Step 4: The control parameters are found to be

\[K_1 = [2.2986 \times 10^3, 0.4630 \times 10^3]\]
\[K_2 = [2.0631 \times 10^3, 0.4153 \times 10^3]\]
\[K_3 = [1.2143 \times 10^3, 0.2425 \times 10^3]\]
\[K_4 = [1.2220 \times 10^3, 0.2448 \times 10^3]\]

Figures 1 to 3 present the simulation results for the fuzzy control. The initial condition was assumed to be \((x_1(0), x_2(0))^T = (\frac{\pi}{4}, 0)^T\) in the simulations. Figure 1 shows the trajectories of states \(x_1\) and \(x_2\).

**Example 2.** A design case of the suboptimal stability margin in fuzzy observer-based control systems.

If we employ Design Procedure 2 described in the previous section, then the suboptimal stability margin design for fuzzy observer-based control systems involves the following steps.

**Step 1:** The same as in example 1.

**Step 2-5:** Solve LMIP using the LMI optimization toolbox in Matlab. In this case, \(\alpha_{\text{sub max}} = 2\).

\[\tilde{W}_{11} = \begin{bmatrix} 25.3457 & -126.7523 \\ -126.7523 & 634.2065 \end{bmatrix}\]
and
\[
\bar{P}_{21} = \begin{bmatrix}
3.5536 & -0.4084 \\
-0.4084 & 0.3783
\end{bmatrix}.
\]

Step 6: The assumptions
\[
\sup_{\forall \delta \in \Omega} |f(x, \delta) - \sum_{i=1}^{4} h_i(\tau(t))A_i x(t)| \leq |\Delta x(t)|
\]
and
\[
\sup_{\forall \varepsilon \in \Omega} |g(x, \varepsilon) - \sum_{i=1}^{4} h_i(\tau(t)) \sum_{j=1}^{4} h_j(\tau(t))B_i K_j x(t)| \leq \sum_{j=1}^{4} h_j(\tau(t)) |\Delta BK_j x(t)|
\]
are satisfied (refer to Figs. 5-6).

Step 7: The observer parameters are found to be
\[
L_1 = \begin{bmatrix}
20.1793 \\
50.3656
\end{bmatrix},
L_2 = \begin{bmatrix}
20.1792 \\
49.7454
\end{bmatrix},
L_3 = \begin{bmatrix}
20.1791 \\
47.9637
\end{bmatrix},
L_4 = \begin{bmatrix}
20.1789 \\
45.2245
\end{bmatrix}.
\]

Step 8: The control parameters are found to be
\[
K_1 = [8.9943 \times 10^4, 1.7984 \times 10^4].
\]
\[
K_2 = [8.0836 \times 10^4, 1.6163 \times 10^4].
\]
\[
K_3 = [4.6507 \times 10^4, 0.9297 \times 10^4].
\]
\[
K_4 = [4.7119 \times 10^4, 0.9420 \times 10^4].
\]

Figures 4 to 6 present the simulation results. The initial condition was assumed to be \((x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)) = (\frac{\pi}{4}, 0, 0, 0)^T\) in the simulations. Figure 4 shows the trajectories of the states \(x_1\) and \(x_2\) (including estimated states \(\hat{x}_1\) and \(\hat{x}_2\)).

V. CONCLUSIONS

In this paper, a Takagi and Sugeno fuzzy model has been proposed and used to study the maximum stability margin problem for nonlinear uncertain systems to achieve as much tolerance of plant uncertainties as possible using fuzzy control. For the case where the state variables are unavailable, a fuzzy observer-based control scheme has also been proposed to tackle the maximum stability margin problem for nonlinear uncertain systems. The outcome of the maximum stability margin design problem in this study has been characterized in terms of a generalized eigenvalue problem (GEVP) for the state feedback fuzzy control and a linear matrix inequality problem (LMIP) for the fuzzy observer-based control, respectively. The GEVP and LMIP can be solved very efficiently using the convex optimization techniques.

This work has extended the maximum stability margin from linear uncertain systems to nonlinear uncertain systems using fuzzy control. LMI-based design proce-
dures for the state feedback fuzzy control or fuzzy observer-based control have been proposed to tackle the maximum stability margin design problem for nonlinear uncertain systems. The proposed design procedure is very simple and can be performed efficiently using the Matlab toolbox. Simulation examples given to demonstrate use of the design procedure, and the results are satisfactory.

REFERENCES


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