PERFORMANCE ANALYSIS OF PERIODIC CONTROL FOR $l_1$ AND $l_\infty$ DISTURBANCE REJECTION

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ABSTRACT

This paper presents a performance analysis of discrete time periodically time varying controllers for the rejection of $l_p$ specific and uniform disturbances. Earlier results on $l_2$ performance are extended to $l_1$ and $l_\infty$ performance to present a unified treatment of $l_p$ performance for all $p \in [1, \infty]$. For a given linear time varying periodic controller, a linear time invariant controller is constructed and necessary and sufficient conditions are presented under which the linear time invariant controller gives strictly better $l_p$ disturbance rejection performance than the time varying periodic controller.

KeyWords: Disturbance rejection, discrete linear systems, $l_p$ space, periodically time-varying systems.

I. INTRODUCTION.

The use of linear periodically time-varying (LPTV) controllers for the control of linear time-invariant (LTI) plants has been studied extensively in recent years. The results on this problem have shown a number of distinctive properties of LPTV control in comparison with the conventional LTI control, for example [1,4,5,8,9]. For the problem of disturbance rejection, it has been shown that time-varying controllers offer no advantages over LTI controllers [2,5,6,7,10,11]. In particular, a frequency domain approach is used in [11] to show that discrete LPTV controllers offer strictly worse control for $l_1$ disturbance rejection. This result is extended to $l_p$ performance for $p \in [1, \infty]$ in [10] using a time domain approach.

In this paper, we extend the $l_1$ performance analysis given in [10] to the analysis of the $l_1$ and $\infty$ performance of discrete LPTV control for disturbance rejection to provide further understanding of the LPTV control properties. In order to unify the results, the analysis in this paper is formulated for $l_p$ performance for $p \in [1, \infty]$.

In the performance analysis, the $l_p$ disturbance is classified into specific disturbance and uniform disturbance. Then, the performance of LPTV control subject to these disturbances is evaluated in comparison with the performance of LTI control. The underlying concept of the performance analysis in this paper is to distinguish LTI and strictly LPTV discrete controllers. For a given strictly LPTV discrete controller, this paper presents necessary and sufficient conditions under which an LTI discrete controller can be found to provide strictly better $l_p$ disturbance rejection performance than the strictly LPTV discrete controller.

II. PROBLEM FORMULATION

2.1 Preliminaries

Consider a discrete LPTV system with time-varying state dimension written as

$$
\begin{align*}
x(t + 1) &= A(t)x(t) + B(t)u(t), \\
y(t) &= C(t)x(t) + D(t)u(t),
\end{align*}
$$

where $t \in \mathbb{Z}$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ are the discrete time system input and output signals, and $x(t) \in \mathbb{R}^{n_0}$ is the discrete time system state vector with time-varying state dimension $n(t)$. The matrices $A(t) \in \mathbb{R}^{n(t) \times n(t)}$, $B(t) \in \mathbb{R}^{n(t) \times m}$, $C(t) \in \mathbb{R}^{r \times n(t)}$ and $D(t) \in \mathbb{R}^{r \times m}$, where $\mathbb{R}^{r \times m}$ is the vector space of all $p \times q$ real matrices. The system (1) is said to be a discrete LPTV system with period $N$ if these matrices are $N$-periodically time-varying matrices with time-varying dimensions, i.e. they satisfy

$$
\begin{align*}
A(t + iN) &= A(t), & B(t + iN) &= B(t), & C(t + iN) &= C(t), \\
D(t + iN) &= D(t),
\end{align*}
$$

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for some least $N \in \mathbb{Z}$ and for any $i \in \mathbb{Z}$.

An LPTV system in the form (1) is LTI if and only if it has period $N = 1$. An LPTV system is strictly LPTV if and only if $N > 1$.

The system (1) is a finite order system if the time-varying dimension of the system state is bounded by a finite integer $\tilde{n}$, i.e., $n(t) \leq \tilde{n}$ for all $t$. Note that every discrete LPTV system in the form (2) has finite order, because $n(t)$ is also $N$-periodic and hence bounded. The discrete LPTV system (1) is a stable system if its state matrix $A(t)$ satisfies

$$\lim_{t \to \infty} \left( \prod_{i=0}^{N-1} A(i) \right)' = \lim_{t \to \infty} (A(N-1)A(N-2) \cdots A(0))' = 0.$$  

(3)

For any $p \in [1, \infty)$, let $l_p$ be the space of all discrete signals $u : \mathbb{Z} \to \mathbb{R}^n$ with each $u \in l_p$ having bounded $l_p$ norm

$$\|u\|_p = \left( \sum_{i=-\infty}^{\infty} |u(i)|^p \right)^{1/p} < \infty,$$

where $\cdot \cdot^p$ is the $p$-norm on $\mathbb{R}^n$, i.e.

$$\|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

Also let $l_\infty$ be the space of all discrete signals with bounded $l_\infty$ norm

$$\|u\|_\infty = \sup \{|u(t)| : t \in \mathbb{Z}\} < \infty,$$

where $\cdot \cdot^\infty$ is the supremum norm on $\mathbb{R}^n$:

$$\|x\|_\infty = \max \{|x_i| : 1 \leq i \leq n\}.$$

For any $\tau \in \mathbb{Z}$ and $p \in [1, \infty)$, let $q^{-\tau} : l_p \to l_p$ be the back shift operator defined by

$$q^{-\tau} u(t) = u(t - \tau).$$

For any $u \in l_p$, and for any integer $\tau$,

$$\|q^{-\tau} u\|_p = \|u\|_p.$$  

(4)

Let $G : l_p \to l_p$ represent the input-output mapping of the discrete LPTV state equation (1). With respect to $q^{-\tau}$, the $N$-periodicity of $G$ may be stated as

$$G = q^{N}\mathcal{G} q^{-N}.$$  

(5)

We further define the $l_p$-induced norm of a discrete LPTV system $G$ as

$$\|G\|_p = \sup \{|Gu| : u \in B_p\},$$

where $B_p$ is the closed unit ball in $l_p$. If $G$ is stable, then $\|G\|_p$ is bounded.

A Banach space $V$ is said to satisfy the strict convexity property if, for any finite set $S = \{u_1, u_2, \ldots, u_n\} \subseteq V$ containing at least two distinct elements,

$$\frac{1}{N} \sum_{i=1}^{N} u_i \leq \max \{|u_i| : 1 \leq i \leq N\}.$$  

(6)

It is well known, e.g., [3], that $l_p$ satisfies the strict convexity property for $p \in (1, \infty)$, but not for $p = 1$ and $p = \infty$.

### 2.2 LPTV control for $l_p$ disturbance rejection

We now consider the feedback control system in Fig. 1, where $P : [w] \to [z]$ represents the input-output mapping of a finite order discrete LTI plant, $w(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^r$, $z(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^r$ are the plant exogenous input, control input, controlled output and measurement output, respectively, and $K : y \to u$ is the input-output mapping of a finite order discrete strictly LPTV feedback controller with period $N > 1$.

Let $S : w \to z$ represent the input-output mapping of the closed loop system. The problem of disturbance rejection can be stated as finding a controller $K$ which stabilizes the closed loop system and minimizes the effect of the input $w$ on the output $z$ with respect to a certain performance specification.

For the problem of specific disturbance rejection, the system exogenous input is assumed to be a specific single disturbance signal $w \in l_p$, which can be applied to the system at any integer time $t = \tau$, for $0 \leq \tau \leq N - 1$. Subject to this specific disturbance, the analysis is to evaluate the greatest $l_p$ norm of the system output $z_t = S \mathcal{G}^{-\tau} w$ for $0 \leq \tau \leq N - 1$. For the problem of uniform disturbance rejection, the system input $w$ is assumed to be all signals in $l_p$. Subject

![Fig. 1. The closed loop LPTV control system.](image-url)
to these disturbances, the analysis is to evaluate the induced $l_p$ norm of the closed loop system, i.e. the worst possible output in terms of the system $l_p$ norm.

For the above disturbance rejection problems, it will be shown in this paper that, for a given discrete strictly LPTV controller $K$ and the closed loop system $S$, a finite order discrete LTI controller $K_{TI}$ can be constructed which yields and stabilizes the LTI closed loop system $S_{TI}$ when applied to the LTI plant $P$. In general, the LTI controller gives superior performance than the strictly LPTV controller in the sense that, for $l_p$ specific disturbance rejection

\[ |S_{TI}w|_p < \max_{0 \leq r \leq N-1} |Sw|_p, \]

and for $l_p$ uniform disturbance rejection

\[ |S_{TI}|_p < |S|_p. \]

III. PROPERTIES OF DISCRETE LPTV SYSTEMS

In this section, we discuss some properties of discrete LPTV systems and introduce an LTI system that will be used for comparison with LPTV systems in the performance analysis in Section 4. In the following lemmas and definitions, $G$ denotes a finite order strictly LPTV system with period $N > 1$, and with state space realization given by (1).

Lemma 3.1. Let $p \in [1, \infty]$. Then, for all $0 \leq \tau \leq N - 1$,

\[ \|q^\tau Gq^{-\tau}\|_p = \|G\|_p. \]  \hspace{1cm} (7)

Proof. For any $\tau$, let $\{u^{[1]}\} \subseteq B_p$, be a sequence of signals on which $q^\tau Gq^{-\tau}$ attains its norm. Then, by (4),

\[ \|q^\tau Gq^{-\tau}\|_p = \lim_{k \to \infty} \|q^\tau Gq^{-\tau}u^{[1]}\|_p = \lim_{k \to \infty} \|Gq^{-\tau}u^{[1]}\|_p < \|G\|_p. \]

The reverse inequality is obtained similarly.

Definition 3.1. Define

\[ G_{TI} = \frac{G + qGq^{-1} + \cdots + q^{N-1}Gq^{-N+1}}{N}. \]  \hspace{1cm} (8)

and also define

\[ G_{TV} = \frac{(G - qGq^{-1}) + \cdots + (G - q^{N-1}Gq^{-N+1})}{N}. \]  \hspace{1cm} (9)

It follows immediately that $G$ can be written in terms of $G_{TI}$ and $G_{TV}$ as

\[ G = G_{TI} + G_{TV}. \]  \hspace{1cm} (10)

Lemma 3.2. $G_{TI}$ in the form (8) is a stable finite order LTI system.

Proof. See [10].

The following definitions relate to situations where the time-varying dynamics of the closed loop system do not appear to act on the specific or uniform input signal(s). To characterize these situations precisely, we introduce the notion of $l_p$ time invariance. Equipped with this definition, we will be able to show that the LTI controller gives strictly better disturbance rejection performance, provided the closed loop system is not $l_p$ time invariant to the input signal(s).

For $p \in (1, \infty)$, the definition of $l_p$ time invariance is straightforward, mostly because the strict convexity property applies to these $l_p$ spaces. As $l_1$ and $L_p$ do not satisfy the strict convexity property, they require a somewhat more technical definition of time invariance.

Definition 3.2. For any $p \in [1, \infty]$, $G$ is $l_p$ norm time invariant to an input signal $u \in l_p$ if, for all $0 \leq \tau \leq N - 1$,

\[ \|Gu\|_p = \|q^\tau Gq^{-\tau}u\|_p. \]  \hspace{1cm} (11)

and for any $p \in [1, \infty]$, $G$ is $l_p$ norm-time invariant to a sequence of input signals $\{u^{[1]}\} \subseteq B_p$, if for all $0 \leq \tau \leq N - 1$,

\[ \lim_{k \to \infty} \|Gu^{[1]}\|_p = \lim_{k \to \infty} \|q^\tau Gq^{-\tau}u^{[1]}\|_p. \]  \hspace{1cm} (12)

Definition 3.3. A set $U = \{u_1, u_2, \ldots, u_m\} \subseteq \mathbb{R}^n$ is sign invariant if for any two points $u_i, u_j \in U$,

\[ u_i u_j \geq 0 \]

for all $1 \leq k \leq n$,

where $u_k$ denotes the $k$-th entry in the vector $u$.

Definition 3.4. A set $U = \{u_1, u_2, \ldots, u_m\} \subseteq \mathbb{R}^n$ has a common peak if there exists a $1 \leq k \leq n$ such that for all $1 \leq i, j \leq m$,

\[ \|u_i\|_\infty = \|u_j\|_\infty \quad \text{and} \quad u_i u_j \geq 0. \]

We now have sufficient preliminary technical definitions to introduce the definition of $l_p$ time invariance. In the following definitions, note that $(q^\tau Gq^{-\tau}u)(t) \in \mathbb{R}^n$. Also $Z = \mathbb{Z} \cup \{\pm \infty\}$, and for convenience we denote $\lim_{t \to \pm \infty} y(t)$ by $y(\pm \infty)$. 
Definition 3.5.1. 
1. \( p = 1 \). \( G \) is \( l_1 \) time invariant to an input signal \( u \in l_1 \), if (a) \( G \) is norm time invariant to \( u \), and (b) the sets \( U_r \) with
\[
U_r = \{(q^*Gq^{-1}u)(t) : 0 \leq \tau \leq N - 1\},
\]
are sign invariant for all \( r \in \mathbb{Z} \).

2. \( p \in (1, \infty) \). For any \( p \in (1, \infty) \), \( G \) is \( l_p \) time invariant to an input signal \( u \in l_p \), if for all \( 0 \leq \tau \leq N - 1 \),
\[
G u = q^* Gq^{-1} u.
\]

3. \( p = \infty \). \( G \) is \( l_\infty \) time invariant to an input signal \( u \in l_\infty \), if (a) \( G \) is norm time invariant to \( u \), and (b) there exists a \( t_0 \in \mathbb{Z} \) such that
\[
(i) \left| \left( G u \right)(t_0) \right| = \left| G u \right|, \quad \text{and}
(ii) \left( q^* Gq^{-1} u \right)(t_0) = \left( q^* Gq^{-1} u \right) \ 	ext{has a common peak.}
\]

In the following definition, we consider sequences \( \{u^{(i)}\} \subseteq B_p \) with convergent subsequences \( \{u^{(i)}\} \). If the subsequence has limit point \( u \in B_p \), then for all \( 0 \leq \tau \leq N - 1 \), the limit point \( \lim_{k \to \infty} q^* Gq^{-1} u^{(k)} = q^* G q^{-1} u \), because the linear operators \( q^* G q^{-1} \) are all bounded and hence continuous. Thus, \( q^* G q^{-1} u(t) \in \mathbb{R}^p \).

Definition 3.5.2. For any \( p \in [1, \infty] \), \( G \) is \( l_p \) time invariant to a sequence of input signals \( \{u^{(i)}\} \subseteq B_p \) if (a) \( G \) is norm time invariant to \( \{u^{(i)}\} \), and if \( \{u^{(i)}\} \) has a convergent subsequence \( \{u^{(k)}\} \) with limit point \( u \in B_p \), then (b) \( G \) is \( l_p \) time invariant to \( u \).

If \( G \) is norm time invariant to \( \{u^{(i)}\} \), there are then three possibilities: (i) \( \{u^{(i)}\} \) has no convergent subsequences, (ii) \( \{u^{(i)}\} \) has one or more convergent subsequences, and \( G \) is \( l_p \) time invariant to all their limit points, and lastly (iii) \( \{u^{(i)}\} \) has one or more convergent subsequences, and \( G \) is not \( l_p \) time invariant to all their limit points. Definition 3.5.2 includes case (i) implicitly and (ii) explicitly. Thus if \( G \) is not \( l_p \) time invariant to \( \{u^{(i)}\} \), then either it is not norm time invariant to \( \{u^{(i)}\} \), or case (iii) applies.

Lemma 3.3 gives elementary properties of the 1-norm and \( \infty \)-norm.

Lemma 3.3. For any set \( U = \{u_1, u_2, \ldots, u_m\} \subseteq \mathbb{R}^p \),
\[
\left| \sum_{i=1}^{m} u_i \right| = \sum_{i=1}^{m} \left| u_i \right|,
\]
if and only if \( U \) is sign invariant, and
\[
\left| \sum_{i=1}^{m} u_i \right| \leq \sum_{i=1}^{m} \left| u_i \right|,
\]
if and only if \( U \) has a common peak.

The next two lemmas are crucial for the performance analysis in Section 4. Conveniently, the form of these lemmas is the same for all values of \( p \in [1, \infty] \). As the definition of \( l_p \) time invariance differs for \( p = 1, p \in (1, \infty) \) and \( p = \infty \), the proof treats these cases separately.

Lemma 3.4.1. For any \( p \in [1, \infty] \), if \( G \) is not \( l_p \) time invariant to an input signal \( u \in l_p \), then the LT1 system in (8) satisfies
\[
\left| G \tau u \right| \leq \max_{0 \leq \tau \leq N-1} \left| q^* Gq^{-1} u \right|,
\]
and if \( G \) is not norm time invariant to \( u \), then the inequality in (18) is strict. Next suppose that \( G \) is \( l_1 \) norm time invariant to \( u \). By assumption, the sets \( U_t \), are not all positive invariant, i.e. there exists a \( t_0 \in \mathbb{Z} \) such that \( U_{t_0} = \{(q^* Gq^{-1})u(t_0) : 0 \leq \tau \leq N - 1\} \) is not sign invariant. By Lemma 3.3,
\[
\left| (G \tau u)(t_0) \right| = \frac{1}{N} \sum_{\tau=0}^{N-1} (q^* Gq^{-1} u)(t_0)
\]
\[
< \frac{1}{N} \sum_{\tau=0}^{N-1} (q^* Gq^{-1} u)(t_0).
\]
By absolute convergence, (19) and norm time invariance,
\[
\left| G \tau u \right| = \sum_{\tau=-\infty}^{\infty} \left| (G \tau u)(t) \right|
\]
\[
= \frac{1}{N} \sum_{\tau=-\infty}^{\infty} \sum_{\tau=0}^{N-1} (q^* Gq^{-1} u)(t)
\]
\[
< \frac{1}{N} \sum_{\tau=-\infty}^{\infty} \sum_{\tau=0}^{N-1} (q^* Gq^{-1} u)(t)
\]
\[
= \frac{1}{N} \sum_{\tau=0}^{N-1} \sum_{\tau=0}^{N-1} (q^* Gq^{-1} u)(t)
\]
\[
= \max_{0 \leq \tau \leq N-1} \left| q^* Gq^{-1} u \right|.
\]

2. \( p \in (1, \infty) \). If \( G \) is not \( l_p \) time invariant to \( u \), then \( \{q^* Gq^{-1} u : 0 \leq \tau \leq N - 1\} \) contains at least two distinct elements, and hence by strict convexity (6),
\[ \left\| G_{\tau}u \right\|_p = \frac{1}{N} \left\| \sum_{t=0}^{N-1} q^t G q^{-t} u \right\|_p \leq \max_{0 \leq s \leq N-1} 1 \left\| q^s G q^{-s} u \right\|_p. \]

3. \( p = \infty \). Assume \( G \) is not \( l_\infty \) time invariant to \( u \). As in the case \( p = 1 \), if \( G \) is not norm time invariant to \( u \), then the inequality in (18) is strict. Assume that \( G \) is norm time invariant to \( u \). Let \( t_0 \in \mathbb{Z} \) be such that \( \left\| G_{\tau}u \right\|_\infty = \left\| (G_{\tau}u)(t_0) \right\|_\infty \). Then, \( t_0 \) does not satisfy both Definition 3.5.1.3 (i) and (ii). If Definition 3.5.1.3 (i) fails, then \( \left\| (G_u(t_0)) \right\|_\infty < \left\| Gu \right\|_\infty \), which yields (17) by the triangle inequality, and if Definition 3.5.1.3 (ii) fails, then \((q^i G q^{-i} u)(t_0) : 0 \leq i \leq N - 1\) has no common peak. By Lemma 3.3, \( \left\| G_{\tau}u \right\|_\infty = \left\| (G_{\tau}u)(t_0) \right\|_\infty = \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_\infty \leq \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_\infty = \max_{0 \leq i \leq N-1} \left\| q^i G q^{-i} u \right\|_\infty. \]

**Lemma 3.4.2.** For any \( p \in [1, \infty) \), if \( G \) is not \( l_p \) time invariant to a sequence of input signals \( \{u^{(i)}\} \subseteq B_p \), then
\[
\lim_{k \to \infty} \left\| G_{\tau}u^{(i)} \right\|_p < \max_{0 \leq \tau \leq N-1} \lim_{k \to \infty} \left\| q^\tau G q^{-\tau} u^{(i)} \right\|_p. \quad (20)
\]

**Proof.** Firstly suppose that \( G \) is not norm time invariant to the sequence \( \{u^{(i)}\} \). Then the set of real numbers \( \left\{ \left\| q^\tau G q^{-\tau} u^{(i)} \right\|_p : 0 \leq \tau \leq N - 1 \right\} \) contains at least two distinct elements, and so by the triangle inequality and strict convexity (6) in \( \mathbb{R} \),
\[
\lim_{k \to \infty} \left\| G_{\tau}u^{(i)} \right\|_p \leq \lim_{k \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u^{(i)} \right\|_p
\]
\[
< \max_{0 \leq \tau \leq N-1} \lim_{k \to \infty} \left\| q^\tau G q^{-\tau} u^{(i)} \right\|_p.
\]

Alternatively suppose that \( G \) is norm time invariant to the sequence \( \{u^{(i)}\} \), but \( \{u^{(i)}\} \) has a subsequence \( \{u^{(i_k)}\} \) that converges to some limit \( u \in B_p \). Then, by assumption, \( G \) is not \( l_p \) time invariant to \( u \). The result follows from Lemma 3.4.1.

**Lemma 3.5.1.** For any \( p \in [1, \infty) \), let \( u \in l_p \) be an input signal to \( G \) such that \( G \) is \( l_p \) time invariant to \( u \). Then, the LTI system in (8) satisfies
\[
\left\| G_{\tau}u \right\|_p = \left\| Gu \right\|_p. \quad (21)
\]

**Proof.** 1. \( p = 1 \). As \( G \) is \( l_1 \) time invariant to \( u \), (11) and (13) hold, and it follows by Lemma 3.3 that for all \( t \in \mathbb{Z} \),
\[
\left\| (G_{\tau}u)(t) \right\|_1 = \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_1 \leq \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_1 = \| Gu \|_1.
\]

As \( G \) is norm time invariant to \( u \), \( \left\| q^\tau G q^{-\tau} u \right\|_1 \) holds, and (22)
\[
\left\| G_{\tau}u \right\|_1 = \frac{1}{N} \sum_{t=0}^{N-1} \left\| (G_{\tau}u)(t) \right\|_1 = \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_1 = \| Gu \|_1.
\]

2. \( p \in (1, \infty) \). Here (21) follows immediately from the definition of \( l_p \) time invariance.

3. \( p = \infty \). As \( G \) is \( l_\infty \) time invariant to \( u \), \( G \) is norm time invariant to \( u \), and there is a \( t_0 \in \mathbb{Z} \) such that \( \left\| (G_u(t_0)) \right\|_\infty = \| Gu \|_\infty \), and \( \| (q^i G q^{-i} u)(t_0) : 0 \leq i \leq N - 1 \| \) has a common peak. Hence, by Lemma 3.3,
\[
\left\| (G_{\tau}u)(t_0) \right\|_\infty = \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_\infty \leq \frac{1}{N} \sum_{t=0}^{N-1} \left\| q^t G q^{-t} u \right\|_\infty = \| Gu \|_\infty.
\]

So, by (11) we achieve \( \| G_{\tau}u \|_\infty \geq \| q^\tau G q^{-\tau} u \|_\infty \) for all \( 0 \leq \tau \leq N - 1 \), and the reverse inequality is obtained from the triangle inequality. Hence (21) is established.

**Lemma 3.5.2.** For any \( p \in [1, \infty) \), let \( \{u^{(i)}\} \subseteq B_p \) be a sequence of input signals to \( G \) such that \( G \) is \( l_p \) time invariant to \( \{u^{(i)}\} \), and further assume that \( \{u^{(i)}\} \) has a convergent subsequence. Then, the LTI system in (8) satisfies
\[ \lim_{k \to \infty} \left| G_{r;\mu}^{[\text{max}]} \right|_\rho = \lim_{k \to \infty} \left| G^{[\text{max}]} \right|_\rho. \]  
(23)

**Proof.** Let \( u \in B_r \) be the limit point of the convergent subsequence \( \{ u^{[k]} \} \). Then by assumption, \( G \) is \( l_p \) time invariant to \( u \). The result then follows from Lemma 3.5.1.

### IV. PERFORMANCE ANALYSIS OF DISCRETE LPTV SYSTEMS

#### 4.1 Construction of an LTI controller

The controller parametrization technique for time-varying control systems in [2,5,6,7] may be applied to the closed loop LPTV system \( S \) to construct an LTI stabilizing controller \( K_{\text{TI}} \) for the plant \( P \). Details of the procedure are given in [10] and may be summarized as follows:

**Theorem 4.1.** Suppose that \( K \) is a finite order discrete time strictly LPTV controller with period \( N > 1 \) which stabilizes the closed loop system \( S \). Then, there exists a finite order discrete time LTI controller \( K_{\text{TI}} \) which, when applied to the discrete time plant \( P \), yields and stabilizes the LTI closed loop system \( S_{\text{TI}} \) defined in terms of \( S \) according to (8).

#### 4.2 Performance of closed loop discrete LPTV systems

We now present the main results of the paper on LPTV closed loop system performance for \( l_p \) disturbance rejection. In the analysis, it is supposed that \( K \) is a discrete finite order strictly LPTV controller with period \( N > 1 \). \( S \) is the closed loop system resulting from the controller \( K \) and \( S_{\text{TI}} \) is the LTI closed loop system subject to the LTI controller \( K_{\text{TI}} \) given in Theorem 4.1.

For the problem of specific disturbance rejection, let \( w \in l_p \) be a specific disturbance signal. Suppose that \( w \) can be applied to the system at any \( \tau \), for \( 0 \leq \tau \leq N - 1 \), yielding the system output \( z = SQ^{-1}w = SW \). Correspondingly, let \( z_{\text{TI}} = S_{\text{TI}}w \) be the output of the LTI closed loop system \( S_{\text{TI}} \) subject to \( w \). The result on \( l_p \) specific disturbance rejection is as follows.

**Theorem 4.2.** For any \( p \in [1, \infty) \), the LTI controller \( K_{\text{TI}} \) gives strictly better control than the strictly LPTV controller \( K \) for \( l_p \) specific disturbance rejection in the sense that

\[ \left| z_{\text{TI}} \right|_p < \max_{0 \leq \tau < N - 1} \left| z_{\tau} \right|_p. \]  
(24)

if and only if the closed loop system \( S \) is not \( l_p \) time invariant to the specific signal \( w \in l_p \).

**Proof.** Firstly, suppose the closed loop LPTV system \( S \) is not \( l_p \) time invariant to the specific disturbance \( w \). Then, by Lemma 3.4.1 and (4),

\[ \left| z_{\text{TI}} \right|_p = \max_{0 \leq \tau < N - 1} \left| z_{\tau} \right|_p, \]

Next, suppose \( S \) is \( l_p \) time invariant to \( w \). Then, by Lemma 3.5.1 and (4) we have

\[ \left| z_{\text{TI}} \right|_p = \max_{0 \leq \tau < N - 1} \left| z_{\tau} \right|_p, \]

if the closed loop system \( S \) is not \( l_p \) time invariant to any sequence \( \{ w^{[k]} \} \subseteq B_p \) of inputs on which \( S \) attains its \( l_p \) norm. If any such sequence \( \{ w^{[k]} \} \) has a convergent subsequence, then this condition is also necessary.

**Proof.** (Sufficiency) Let \( \{ w^{[k]} \} \subseteq B_p \) be a sequence of input signals to the LTI system \( S_{\text{TI}} \) on which \( S_{\text{TI}} \) attains its \( l_p \) norm, i.e.

\[ \lim_{k \to \infty} \left| S_{\text{TI}}w^{[k]} \right|_p = \left| S_{\text{TI}} \right|_p. \]  
(27)

Suppose firstly that the LPTV system \( S \) is \( l_p \) time invariant to this input sequence \( \{ w^{[k]} \} \). By assumption, \( S \) does not attain its \( l_p \) norm on \( \{ w^{[k]} \} \). The triangle inequality yields

\[ \left| S_{\text{TI}} \right|_p \leq \lim_{k \to \infty} \left| S_{\text{TI}}w^{[k]} \right|_p \leq \lim_{k \to \infty} \left| Sw^{[k]} \right|_p < \left| S \right|_p. \]  
(28)

If \( S \) is not \( l_p \) time invariant to \( \{ w^{[k]} \} \), then (26) follows from Lemmas 3.1 and 3.4.2:

\[ \left| S_{\text{TI}} \right|_p = \lim_{k \to \infty} \left| S_{\text{TI}}w^{[k]} \right|_p < \max_{0 \leq \tau < N - 1} \lim_{k \to \infty} \left| q^\tau S_\tau^{-1} w^{[k]} \right|_p \leq \max_{0 \leq \tau < N - 1} \left| q^\tau S_\tau^{-1} \right|_p \left| S \right|_p. \]

(Necessity) Let \( \{ w^{[k]} \} \) be a sequence of input signals
VI. EXAMPLE

This section presents an example to illustrate the results of Section 4. Referring to Fig. 1, we consider the LTI plant

\[ P = \begin{bmatrix} 1 & (1-q^{-1})^{-1}q^{-1} \\ 1 & (1-q^{-1})^{-1}q^{-1} \end{bmatrix} \]

and the 2-periodic controller

\[ K = -\alpha(t)(1 + q^{-1})(1 + \alpha(t)q^{-1})^{-1}, \]

where \( \alpha(t) \) is a 2-periodic coefficient with \( \alpha(0) = 0 \) and \( \alpha(1) = 1 \). The closed loop LPTV system \( S \) can be found to be

\[ S = 1 - q^{-2} - q^{-1}Q(1-q^{-1}), \]

where the parameter \( Q \) is 2-periodic and satisfies \( Q = \alpha(t) \). Following the procedure for constructing the LTI system \( S_{TI} \) and controller \( K_{TI} \), we can obtain

\[ S_{TI} = 1 - \frac{1}{2}q^{-1} - \frac{1}{2}q^{-2}, \]

\[ K_{TI} = -\frac{1}{2}(1 + q^{-1})(1 + \frac{1}{2}q^{-1})^{-1}. \]

\( S \) and \( S_{TI} \) may be represented in matrix operator form by

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & 1 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & -1 & 1 & 0 & \cdots \\
0 & -1 & 0 & 1 & \cdots \\
0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
S_{TI} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & \frac{1}{2} & 1 & 0 & \cdots \\
0 & -1 & \frac{1}{2} & 1 & \cdots \\
0 & -1 & 0 & \frac{1}{2} & 1 & \cdots \\
0 & 0 & 0 & -\frac{1}{2} & 1 & \cdots \\
0 & 0 & 0 & -\frac{1}{2} & 1 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\] (30)

It is straightforward to verify that

\[ \| S \|_1 = 3, \| S \|_\infty = 2, \| S_{TI} \|_1 = 2, \| S_{TI} \|_\infty = 2. \]

Consider an input signal

\[ w = [1, -1, 1, 0, 0, \ldots]^T \in L_\infty, \]

with \( \| w \|_1 = 1 \). Applying \( w \) to the LPTV closed loop system \( S \) yields

\[ Sw = [1, -1, 0, -1, 0, 0, \ldots]^T, \]

\[ q^1Sq^{-1}w = [1, -2, -2, 1, 0, 0, \ldots]^T. \]

Apparently, \( \| Sw \|_1 = \| q^1Sq^{-1}w \|_1 = 2 \). It follows from Definition 3.5.1 that \( S \) is \( L_\infty \) time invariant to \( w \). Hence, also \( S \) is \( L_\infty \) time invariant to the constant sequence \( \{w[k]\} = w \), and \( S \) also attains its \( L_\infty \) norm on this sequence.

As \( \| S \|_\infty = \| S_{TI} \|_\infty \), this verifies Theorem 4.3 with respect to the \( L_\infty \) performance of \( S \).

It follows from the matrix expression of \( S \) in (30) that any input sequence \( \{w[k]\} \), with \( \| w[k] \|_1 = 1 \), on which \( S \) attains its \( l_1 \) norm, converges to a signal of the type

\[ w = [0, \ldots, 0, \pm 1, 0, 0, 0, \ldots]^T, \]

where the only non-zero element of \( w \) is \( \pm 1 \) in the \((2i + 1)\)th entry. It is easily verified that for such a signal,

\[ \| Sw \|_1 = 3 \text{ and } \| q^1Sq^{-1}w \|_1 = 1. \]

Hence, \( S \) is not \( l_1 \) time invariant to any sequence \( \{w[k]\} \) converging to \( w \), and so \( \| S \|_1 < \| S_{TI} \|_1 \) by Theorem 4.3. In fact, we have already seen that \( \| S_{TI} \|_1 = 2 < \| S \|_1 \).

VI. CONCLUSION

In this paper, the problem of discrete LPTV control of LTI plants for \( l_p \) disturbance rejection is studied for all \( p \in [1, \infty) \). By distinguishing LTI and strictly LPTV systems in the set of LPTV systems, it is shown that strictly LPTV controllers strictly degrade the closed loop system performance. For a given strictly LPTV controller with active time-varying dynamics on the system performance, an LTI controller can be constructed to provide strictly better performance for the rejection of \( l_p \) specific and uniform disturbances. The authors regard time invariance as a rarity, i.e. in general a closed loop system with a strictly LPTV controller will be time invariant to very few inputs. This analysis provides further understanding of the relative performance of LPTV and LTI control systems.
REFERENCES


