HIGHER RELATIVE DEGREE NONLINEAR SYSTEMS WITH ILC USING LOWER-ORDER DIFFERENTIATIONS

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ABSTRACT

This paper addresses the problem of iterative learning control (ILC) for a class of nonlinear continuous-time systems with higher relative degree. The proposed ILC solution is a family of updating laws using differentiations of tracking error with the order less than the system relative degree. A unified convergence condition for this family of ILC updating laws is provided and proved to be independent of the highest order of differentiation. The application to path tracking of a robotic manipulator is presented to illustrate the effectiveness of the proposed method.

KeyWords: Iterative learning control, convergence, robustness, relative degree, nonlinear systems.

I. INTRODUCTION

Iterative learning control is applicable to dynamic systems performing a task repeatedly over a finite operation cycle. Through learning, the imperfect knowledge of the dynamics structure and/or parameters is overcome and zero-error tracking is achieved over the whole operation interval. The control input is updated by using the information obtained from current cycle and/or previous cycle(s). ILC for nonlinear continuous-time systems has been an area of increasing research activity for years and many types of continuous-time learning schemes have been extensively studied; see, e.g., [3,7,10,18,23] and references therein. Roughly speaking, there are two major categories of continuous-time learning schemes. The one is ILC using differentiations where the highest order of error derivatives is required to be the same as the system relative degree, see, e.g., [1,6,12,15,19,22]. The other one refers to continuous-time learning schemes where the highest order of error derivatives is one less than the system relative degree or the system state variables are required, for example, [2,5,11,16,20,26]

Unified analyses for ILC using differentiations were developed for systems with direct transmission term [22] and well-defined relative degree [1]. Error derivative signals are required for control input update, the highest order being equal to the system relative degree. As a special type, the so-called D-type ILC requires the highest-order error derivative only. On the other hand, perturbed initial conditions as well as state disturbances and measurement noises are unavoidable in the implementations. Robustness of ILC in the presence of these uncertainties is an issue of critical importance due to its iteration feature. In [4], the robustness problem was first studied with the aid of linearization. In [8], uniform boundedness of the trajectories throughout the operation cycles was established through combining linear output feedback. In [13], a forgetting factor was introduced in the D-type ILC [12] and a global analysis was developed to deal with the robustness problem for a class of nonlinear systems. Uniform boundedness of the resulting input, state and output trajectories was established and these trajectories were proved to converge to the neighborhoods of their desired ones as operations repeat. The error bounds are proportional to the bounds linearly and independently on initial condition errors, state disturbances and measurement noises. The same problem has been investigated for different systems and applications in [9, 21,25]

Without requiring acceleration measurements, several learning schemes were proposed for robotic motion in early works [2]. The learning schemes where the highest order of error derivatives is required to be one less than the system relative degree are now referred to as P-type ILC. For systems with relative degree one, such learning schemes use only the output error. In [13], however, a P-type learning scheme was illustrated not robust to perturbation from the initial condition errors. To gain robustness, a
forgetting factor was introduced in the P-type learning scheme [5,11,20]. The robustness problem has been investigated based on passivity analysis of robot dynamics [5]. And technical analyses were presented by assuming the boundedness of time derivative of input-output coupling matrix [11,20]. More efforts are still needed for defining and analyzing P-type ILC. Recently in [24], anticipatory ILC was proposed in parallel to the P-type and D-type ILCs. The convergence and robustness results were developed still for a class of nonlinear systems with relative degree one.

In this paper, we are interested in ILC using lower-order differentiations for a class of nonlinear systems with higher relative degree. Higher-order differentiations are avoided for easy implementation. A causal relationship between the input taken and its produced results is developed and a family of ILC updating laws is proposed based on the developed causality. This scheme uses the error derivatives with the order being less than the system relative degree. A unified convergence proof is given to analyze the tracking performance and a sufficient condition for choosing the learning gain is derived, which is shown to be independent of the highest differentiation order used in the scheme. In the presence of initial condition errors, the robustness is guaranteed under the same sufficient condition.

II. PROBLEM FORMULATION

The class of nonlinear continuous-time systems under consideration is described by the following state-space equations

\[
\dot{x}(t) = f(x(t)) + B(x(t))u(t),
\]

\[
y(t) = g(x(t)),
\]

where \(x(t) \in R^r\) is the state, \(u(t) \in R^r\) is the control input, and \(y(t) \in R^m\) is the output of the system. The nonlinear functions \(f(\cdot) \in R^r, B(\cdot) = [b_1(\cdot), \ldots, b_i(\cdot)] \in R^{r \times r}\) and \(g(\cdot) = [g_1(\cdot), \ldots, g_m(\cdot)]^T \in R^m\) are smooth in their domain of definition. The systems perform the same operation repeatedly which ends in a finite time \(T\), i.e., \(t \in [0, T]\). For each fixed \(x(0), S\) denotes a mapping from \((x(0), u(t), t \in [0, T])\) to \((x(t), t \in [0, T]\)) and \(O\) a mapping from \((x(0), u(t), t \in [0, T])\) to \((y(t), t \in [0, T])\). In these notations, \(x(0) = S(x(0), u(\cdot))\) and \(y(\cdot) = O(x(0), u(\cdot))\).

Given a realizable trajectory \(y_d(t) = [y_{1d}(t), \ldots, y_{md}(t)]^T\), \(t \in [0, T]\) and a tolerance error bound \(\varepsilon > 0\) for the system (1)-(2), the objective of this paper is to find an input profile \(u(t), t \in [0, T]\) by applying iterative learning control method, so that the error between the system output \(y(t) = [y_1(t), \ldots, y_m(t)]^T\) and the desired trajectory \(y_d(t)\) is within the tolerance error bound, i.e. \(|y_d(t) - y(t)| < \varepsilon, t \in [0, T]\), where \(|\cdot|\) is the vector norm defined as \(|a| = \max_{1 \leq i \leq n} |a_i|\) for an \(n\)-dimensional vector \(a = [a_1, \ldots, a_n]^T\). Throughout the paper, for a matrix \(A = [a_{ij}] \in R^{m \times n}\), the induced norm \(\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|\) is used.

In order to solve this problem, we need the derivative notations and the definition for system relative degree [14]. The derivative of a scalar function \(g(x)\) along a vector \(f(x)\) is defined as

\[
L_g(x) = \frac{dg(x)}{dx}. 
\]

The repeated derivatives along the same vector are

\[
L_g^q(x) = L_{g}^{q-1}(g(x)), 
\]

The derivative of \(g(x)\) taken first along \(f(x)\) and then along a vector \(b(x)\) is

\[
L_bL_g(x) = \frac{d(L_{g}(g(x)))}{dx} b(x). 
\]

Definition 2.1. The system (1)-(2) is said to have (vector) relative degree \(\{\mu_1, \ldots, \mu_m\} \in X \subset R^r\), if, for all \(x \in X\) and \(1 \leq q \leq m\),

(i) \(L_{b_q} L_{g_{q}}^q(x) = 0\), \(0 \leq j \leq \mu_q - 2\), \(1 \leq p \leq r\), and

(ii) \(L_{b_p} L_{g_{q}}^{q-p} \neq 0\) for some \(p, 1 \leq p \leq r\).

Remark 2.1. Note that \(\mu_q (1 \leq p \leq m)\) is the times that the \(q\)th output component needs to be differentiated so that the terms involving the input appear, as follows

\[
y_q(t) = g_q(x(t)), \quad \dot{y}_q(t) = L_{g_q}(x(t)), \\
\vdots \\
y^{q-p-1}_q(t) = L_{g_q}^{q-p-1}(g_q(x(t))) \\
y^{q-p}_q(t) = L_{g_q}^{q-p}(g_q(x(t))) \\
\quad + [L_{b_q} L_{g_q}^{q-p-1} g_q(x(t)), \ldots, L_{b_p} L_{g_q}^{q-p-1} g_q(x(t))] u(t)
\]

Furthermore, the output component at the instant \(t + \sigma, \sigma > 0\), is directly affected by at least one component of the input, in the integration form

\[
y_q(t + \sigma) = y_q(t) + \frac{\sigma}{1} \dot{y}_q(t) + \frac{\sigma^2}{2} \ddot{y}_q(t) + \ldots + \frac{\sigma^{q-1}}{(q-1)!} L_{g_q}^{q-1}(t) + \ldots + \frac{\sigma^p}{(p-1)!} L_{g_q}^{p-1}(t) + \ldots + \frac{\sigma^{q-p-1}}{(q-p-1)!} L_{g_q}^{q-p-1}(t) \\
\quad + \int_{t}^{t+\sigma} \int_{t}^{t+\sigma} \ldots \int_{t}^{t+\sigma} L_{g_q}^{q-p-1}(x(t_{p+1})) \\
\quad + [L_{b_q} L_{g_q}^{q-p-1} g_q(x(t_{p+1})), \ldots, 
\]

M. Sun and D. Wang: Higher Relative Degree Nonlinear Systems with ILC Using Lower-order Differentiations
\[ + L_s L_p^{-1} g_q (x(t_{\mu_q})) u(t_{\mu_q}) \mathrm{d}t_{\mu_q} \ldots \mathrm{d}1. \]  

(3)

It is logical to form the updating law that the next input action is updated on the basis of the action and its produced results in the current operation cycle. In view of (3), \( u(t), y_q(t) + \sigma - \Sigma_{q=0}^{\mu_q+1} \sigma_q(t), 0 \leq \sigma_q \leq \mu_q - 1, 1 \leq q \leq m \) is one causal pair of dynamically related cause and effect. This observation is lent to the following updating law, with input saturation being taken into account,

\[
\begin{align*}
 u_{k+1}(t) &= \begin{cases}
 u(t) + \Gamma(t)
 & e_{i, k}(t + \sigma) - \Sigma_{i=0}^{\mu_q} \sigma_i(t), t \in [0, T - \sigma], \\
 v(T - \sigma),
 & t \in (T - \sigma, T]
\end{cases} \quad (4)
\end{align*}
\]

\( u(t) = \text{sat}(u(t)) \),

(5)

where \( 0 \leq \theta_q \leq \mu_q - 1, 1 \leq q \leq m, k \) indicates the number of operation cycles, \( e_q(t) = [e_1, e_2, \ldots, e_m] = \Theta_k(t) = y_q(t) - y_q(t), \) is the output error, or the tracking error, \( y_q(t) \) the \( q \)-th component of the desired trajectory at the instant \( t \) and \( y_{q, j}(t) \) the \( q \)-th output component at \( k \)-th cycle at the instant \( t \). \( \Gamma_k(t) \in R^{m \times m} \) is the learning gain matrix piecewise continuous and bounded. The input saturation function sat: \( R^r \rightarrow R^r \) is defined as sat \( (v) = [\text{sat}(v_1), \ldots, \text{sat}(v_i)]^T \), where

\[
\text{sat} (v_p) = \begin{cases}
 v_p, & |v_p| \leq \delta_p, \\
 \text{sgn}(v_p) \delta_p, & |v_p| > \delta_p
\end{cases}
\]

(6)

for the input saturation bound \( \delta_p > 0, p = 1, \ldots, r \). Define \( \delta = \max_{1 \leq p \leq r} \{ \delta_p \} \). The iterative learning control system using lower-order differentiations is illustrated in Fig. 1.

**Remark 2.2.** In the updating law (4) and (5), \( \theta_q, 1 \leq q \leq m \), are design parameters can be chosen in the range, \( 0 \leq \theta_q \leq \mu_q - 1 \). Thus, equations (4) and (5) define a family of updating laws requiring lower-order derivatives of the tracking error and these signals are just part state variables for mechanical systems.

**Remark 2.3.** In contrast to existing learning schemes, the scheme (4) and (5) is easier to implement because the required highest-order of error derivatives can be less than the system relative degree. For instance, in case of second-order mechanical systems, P-type ILC uses velocity variables which needs velocity sensors to measure. Under the same measurement requirements, we can apply (4) in the form

\[
u_{k+1}(t) = u(t) + \Gamma_k(t) [e_1(t + \sigma) - e_1(t) - \sigma e_1(t)], \quad (7)
\]

If the systems are equipped with joint position sensors but not velocity sensors, P-type ILC can be just implemented approximately, not exactly. Our scheme, however, provides alternative as follows

\[
u_{k+1}(t) = u(t) + \Gamma_k(t) [e_1(t + \sigma) - e_1(t)], \quad (8)
\]

which uses position measurements only.

**Assumptions:**

(A1) The mappings \( S \) and \( O \) are one to one.
(A2) The desired trajectory \( y_d(t), t \in [0, T] \) is achievable by the desired input within the input saturation bounds, i.e., \( u_d(t) = \text{sat}(u_d(t)), t \in [0, T] \).
(A3) There exists a compact set \( X \subset R^r \) such that the system state \( x(t), t \in [0, T] \) produced by any input \( u(t) \in U, t \in [0, T] \) belongs to \( X \), where \( U \subset R^r \) is a bounded set, i.e., \( x(t) \in X, t \in [0, T] \).
(A4) The system has relative degree \( \{ \mu_1, \ldots, \mu_m \} \) for all \( x \in X \).
(A5) The operations start from the initial condition \( x_1(0) = x_d(0) \) for \( k = 0, 1, 2, \ldots, \), where \( x_d(0) \) is the initial condition corresponding to the desired trajectory.

**Remark 2.4.** (A1) implies that there exists a unique control input \( u_d(t) \), driving the system output to follow the desired trajectory so that

\[
y_d(t) = g(x_d(t)), \quad (9)
\]

\[
\dot{x}_d(t) = f(x_d(t)) + B(x_d(t))u_d(t), \quad (10)
\]

where \( x_d(t) \) is the resultant state. Iterative learning control with an input saturator can be still effective by assuming (A2) as argued in [15].
III. CONVERGENCE OF ILC USING LOWER-ORDER DIFFERENTIATIONS

In this section, we shall established the convergence results when the proposed updating law (4)-(5) is applied to the system (1)-(2) with relative degree \( \{ \mu_1, \ldots, \mu_m \} \).

One of the main results is summarized in the following theorem.

**Theorem 3.1.** Given a desired trajectory \( y_d(t), t \in [0, T] \), for the system (1)-(2), let the system satisfy assumptions (A1)-(A5) and use the updating law (4)-(5). The system output converges to the desired trajectory on the interval \( [0, T - \sigma] \) in the sense that

\[
\lim \sup_{t \to \infty} \sup_{t \in [0, T - \sigma]} \| e(t) \| \leq \beta_1 \delta \max_{1 \leq q \leq m} \left\{ \frac{\sigma^{\mu_q - 1}}{\mu_q} \right\},
\]

with positive constants \( \lambda \) and \( \beta_1 \) to be defined, if, for all \( k \) and \( t \in [0, T - \sigma] \),

\[
| I - \Gamma_k(t)D_k(t) | \leq \rho < 1,
\]

where

\[
D_k(t) = \left[ \int_i^t \int_i^{t_{i-1}} \cdots \int_i^{t_{k-2}} \int_i^{t_{k-1}} \{ L_f^{\mu_q} g_q(x_d(t_{\mu_q})) \} \right] \mu_q \cdots \mu_k \right] dr_{\mu_k} \cdots dr_1. \]

**Proof.** It follows from (3) that, for \( 1 \leq q \leq m \),

\[
e_q(t + \sigma) - \sum_{i=0}^{q} \left( \frac{\sigma^{\mu_q - 1}}{\mu_q} \right) \left( \frac{\mu_q - 1}{\mu_q - 1} \right)^i (t) + \left( \frac{\mu_q - 1}{\mu_q - 1} \right)^i (t) + \int_i^{t+\sigma} \int_i^{t_{i-1}} \cdots \int_i^{t_{k-2}} \int_i^{t_{k-1}} \{ L_f^{\mu_q} g_q(x_d(t_{\mu_q})) \} \right] \mu_q \cdots \mu_k \right] dr_{\mu_k} \cdots dr_1.
\]

For \( t \in [0, T - \sigma] \), substituting (13) into (14) gives

\[
\Delta u_{k+1}(t) = (I - \Gamma_k(t)D_k(t))\Delta u_k(t) - \Gamma_k(t)\xi(t) + \zeta(t) + \sigma(\xi(t)),
\]

where \( \Delta u_{k+1}(t) = u_k(t) - u_{k+1}(t), \Delta u_k(t) = u_d(t) - u_k(t), \xi(t) = [\xi_1(t), \ldots, \xi_m(t)]^T, \zeta(t) = [\zeta_1(t), \ldots, \zeta_m(t)]^T, \sigma(\xi(t)) = [\sigma_1(\xi(t)), \ldots, \sigma_m(\xi(t))]^T. \]

Denote by \( C_f \) the norm bound for \( \Gamma_k(t) \). Taking norms and applying the bounds yield

\[
\| \Delta u_{k+1}(t) \| \leq \rho \| \Delta u_k(t) \| + C_f \| \xi(t) \| + \| \zeta(t) \| + \| \sigma(\xi(t)) \|.
\]

Note that the functions \( f(\cdot), b(\cdot), g(\cdot), \mu \) and \( L_f \) are local Lipschitz in \( x \in X \) since they are smooth functions. Both \( [L_0, L_0^{\mu_q - 1} - 1] \) are bounded on \( X \) due to the same reason.

In the rest of the proof, \( L_0, L_1, L_2, \ldots, L_m \) and \( c_0 \) denote the Lipschitz constants and the norm bounds, respectively. Therefore,

\[
\| \xi(t) \| \leq (l_{\mu_0} + mL_c) \int_i^{t_{i-1}} \cdots \int_i^{t_{k-2}} \int_i^{t_{k-1}} \| \Delta u_k(t_{\mu_q}) \| \mu_q \cdots \mu_k \right] dr_{\mu_k} \cdots dr_1.
\]

For \( t \in [0, T - \sigma] \), substituting (13) into (14) gives
where $\Delta x_i(t) = x_j(t) - x_i(t)$. Defining $c_1 = c_\tau \max_{0 \leq s \leq \tau} \{ \frac{\sigma^{q+1}}{(\theta_q + 1)!} + \ldots + \frac{\sigma^{q+1}}{(\theta_q - 1)!} \} f_j$, $c_2 = c_\tau (l_f + r l_{\delta g} c_{ud})$ and $c_3 = c_\tau c_{bg}$ leads to

$$\left| \Delta u_{k+1}(t) \right| \leq \rho \left| \Delta u_k(t) \right| + c_1 \left| \Delta x_i(t) \right|$$

where

$$c_1 = c_\tau \max_{0 \leq s \leq \tau} \{ \frac{\sigma^{q+1}}{(\theta_q + 1)!} + \ldots + \frac{\sigma^{q+1}}{(\theta_q - 1)!} \} f_j$$

$$c_2 = c_\tau (l_f + r l_{\delta g} c_{ud})$$

$$c_3 = c_\tau c_{bg}$$

Assuming $c_\tau > 0$, by applying Bellman-Gronwall lemma results in

$$\left| \Delta u_{k+1}(t) \right| \leq c_1 \left| \Delta u_k(t) \right| + c_2 \int_0^t \int_{t'}^t |\Delta u_k(s) - \Delta u_k(t)| \, ds \, dr$$

Multiplying both sides by $e^{-\lambda t}$ ($\lambda > 0$) gives

$$e^{-\lambda t} \left| \Delta u_{k+1}(t) \right| \leq c_1 e^{-\lambda t} \left| \Delta u_k(t) \right|$$

To evaluate the state errors in the right-hand side of (14), we integrate the state equations to obtain

$$\Delta x_i(t) \leq \Delta x_i(0) + \int_0^t \left[ f(x_j(s)) - f(x_i(s)) + (B(x_i(s)) - B(x_j(s))) u_i(s) + B(x_i(s)) \Delta u_i(s) \right] \, ds$$

Using the assumption (A5) and taking norms give rise to

$$\left| \Delta x_i(t) \right| \leq c_\delta \int_0^t \left| f(x_j(s)) - f(x_i(s)) \right| + c_\delta \left| u_i(s) \right| + c_\delta \left| \Delta u_i(s) \right| \, ds$$

Applying Bellman-Gronwall lemma results in

$$\left| \Delta x_i(t) \right| \leq c_\delta \int_0^t e^{\epsilon d s} \left| \Delta u_i(s) \right| \, ds$$

where $c_\delta = l_f + l_{\delta g} c_{ud}$. Then, substituting (16) into (14) produces

$$\left| \Delta u_{k+1}(t) \right| \leq \rho \left| \Delta u_k(t) \right| + c_1 c_\delta \int_0^t e^{\epsilon d s} \left| \Delta u_i(s) \right| \, ds$$

and by the assumption (A2),

$$\left| \Delta u_i(t) \right| \leq \left| \Delta v_i(t) \right|.$$

Thus,
\[
\sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_{k+1}(t) \right\| \} \leq \overline{\sigma} \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \} + c_5 \delta \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \},
\]

where \( \overline{\sigma} = \sigma + c_5 e_n \frac{1 - e^{c_4 T}}{\lambda - c_4} + c_5 e_n e^{c_4 T} - 1 \frac{c_4}{\lambda - c_4} + 4 c_5 \).

Since \( 0 \leq \sigma < 1 \), we can find a \( \lambda > c_3 \) sufficiently large such that \( \overline{\sigma} < 1 \). Then, (17) is a contraction in \( \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \} \), which implies

\[
\limsup_{k \to \infty} \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \} \leq \frac{c_3 \delta}{1 - \overline{\sigma}} \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \}.
\]

Furthermore, from (16), using the similar manipulations produces

\[
\sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \} \leq c_4 e_n \frac{1 - e^{c_4 T}}{\lambda - c_4} \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \}.
\]

Thus,

\[
\limsup_{k \to \infty} \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| \Delta \nu_k(t) \right\| \} \leq c_4 e_n \frac{1 - e^{c_4 T}}{\lambda - c_4} c_5 \delta \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \}.
\]

Now, the result for the tracking error \( e_i(t) \), \( t \in [0, T - \sigma] \), is established as

\[
\limsup_{k \to \infty} \sup_{t \in [0, T - \sigma]} \{ e^{-d t} \left\| e_i(t) \right\| \} \leq c_4 e_n \frac{1 - e^{c_4 T}}{\lambda - c_4} c_5 \delta \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \}.
\]

By defining

\[
\beta_1 = 1 \frac{1 - e^{c_4 T}}{\lambda - c_4} \frac{1}{1 - \overline{\sigma}},
\]

the theorem follows.

\[\text{Remark 3.1.} \] In Theorem 3.1, the convergence is established for the time interval \([0, T - \sigma]\). The error bounds in (18), (19) and (20) are proportional to the term \( \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \} \). The proposed scheme requires \( \sigma > 0 \), but it can be small enough to specify the bounds as required.

For example, we consider SISO case where \( \tau = m = 1 \) and \( \mu_1 = \ldots = \mu_m = \mu \) so that the condition (12) reduces to

\[
1 - \gamma(t) \int_{t}^{t+\sigma} \int_{t}^{t+1} \ldots \int_{t}^{t+\mu-1} d \{ \mu_\nu \} d \nu \ldots d \nu_1 \leq \rho < 1,
\]

where \( d \{ \mu_\nu \} = L_d L_d^{-1} g(x(t_\nu)) \) and \( \gamma(t) = \Gamma_\nu(t) \). If \( \sigma \) is set to be small enough so that (22) can be approximately written as

\[
1 - \gamma(t) \int_{t}^{t+\sigma} \int_{t}^{t+1} \ldots \int_{t}^{t+\mu-1} d \{ \mu_\nu \} d \nu \ldots d \nu_1 \leq 1 - \frac{\sigma}{\mu_\nu} \gamma \int_{t}^{t+\mu} d \{ \mu_\nu \} d \nu \ldots d \nu_1
\]

Assume that \( d \{ \mu_\nu \} > 0 \) and is modeled to be \( \tilde{d} \{ \mu_\nu \} \tilde{d} \{ \mu_\nu \} > 0 \).

And we choose \( \gamma(t) = \frac{\mu_\nu}{\sigma_\nu} \gamma \int_{t}^{t+\mu} d \{ \mu_\nu \} d \nu \ldots d \nu_1 \), where \( \gamma \) is an adjustable parameter to provide compensation for model discrepancy. The condition

\[
1 - \frac{\sigma}{\mu_\nu} \gamma \int_{t}^{t+\mu} d \{ \mu_\nu \} d \nu \ldots d \nu_1 \leq \rho < 1
\]

holds if

\[
0 < \gamma < 2 \tilde{d} \{ \mu_\nu \} \int_{t}^{t+\mu} d \{ \mu_\nu \} d \nu \ldots d \nu_1.
\]

The term \( \frac{\mu_\nu}{\sigma_\nu} \gamma \) in \( \gamma(t) \) should take an appropriate value for the update in the learning algorithm. When \( \sigma \) is small we need to choose \( \gamma \) small enough simultaneously to keep the input from wandering excessively, which is further illustrated by the numerical example in Section 5. However, from (11) and (23), if \( \gamma \) is chosen to be positive but small enough, (23) will be held and the error bound can be tight. In practice, convergence to a specified neighborhood of the desired trajectory will be sufficient for most applications.

Now, the convergence result for the time interval \([T - \sigma, T]\) can be easily established as stated in the following theorem.

\[\text{Theorem 3.2.} \] Let the system (1)-(2) satisfy assumptions (A1)-(A5) and use the updating law (4)-(5). The learning gain \( \Gamma_\nu(t) \) is chosen satisfying (12) for all \( k \) and \( t \in [0, T] \).

The system output converges to the desired trajectory on the interval \([T - \sigma, T]\) in the sense that

\[
\limsup_{k \to \infty} \sup_{t \in [T - \sigma, T]} \{ e^{-d t} \left\| e_i(t) \right\| \} \leq \beta_2 \delta \max_{1 \leq q \leq m} \{ \frac{\sigma q}{\mu_q} \} + \beta_2 \delta \sigma
\]

with positive constants \( \lambda, \beta_1, \beta_2 \) and \( \beta_2 \) to be defined.

\[\text{Proof.} \] For \( t \in (T - \sigma, T) \), (16) still holds so that

\[
\left\| \Delta \nu_{k+1}(t) \right\| \leq \frac{c_3 \delta}{1 - \overline{\sigma}} \frac{1}{1 - \overline{\sigma}}
\]
\[ + c_B \int_{-\tau}^{t} e^{-\lambda (t-s)} \| \Delta u_k(s) \| ds, \]

which results in, for \( \lambda > c_4 \),

\[
\sup_{t \in [T-\sigma, T]} \{ e^{-\lambda t} \| \Delta x(t) \| \} \\
\leq c_B \frac{1 - e^{-(\lambda - 4)T}}{\lambda - c_4} \sup_{t \in [0, T]} \{ e^{-\lambda t} \| \Delta u(t) \| \} + 2c_B \delta \sigma. \tag{25}
\]

Substituting (18) into (25) gives

\[
\limsup_{k \to \infty} \sup_{t \in [T-\sigma, T]} \{ e^{-\lambda t} \| x_k(t) \| \} \\
\leq c_B \frac{1 - e^{-(\lambda - 4)T}}{\lambda - c_4} \frac{c_B \delta}{1 - \beta} \max_{1 \leq q \leq m} \{ \sigma^2_{q_j} \} + 2c_B \delta \sigma. \tag{26}
\]

Therefore,

\[
\limsup_{k \to \infty} \sup_{t \in [T-\sigma, T]} \{ e^{-\lambda t} \| e_\sigma(t) \| \} \\
\leq l_e c_B \frac{1 - e^{-(\lambda - 4)T}}{\lambda - c_4} \frac{c_B \delta}{1 - \beta} \max_{1 \leq q \leq m} \{ \sigma^2_{q_j} \} \tag{27}
\]

By defining

\[
\beta_1 = l_e c_B \frac{1 - e^{-(\lambda - 4)T}}{\lambda - c_4} \frac{c_B \delta}{1 - \beta}, \tag{28}
\]

\[
\beta_2 = 2 l_e c_B, \tag{29}
\]

the theorem follows. \( \blacksquare \)

**Remark 3.2.** In Theorem 3.1 and Theorem 3.2, unified convergence condition (12) is derived for the family of updating law (4)-(5). It is proved to be independent of the highest order of differentiation. The relative degree of the systems under consideration is required to be known explicitly. There may be points where relative degree cannot be defined for some class of systems and the proposed scheme fails to work. This would be the major limitation on the learning control approach which is applicable to the systems with well defined relative degree. In the practical implementation, \( \{ \theta_1, \ldots, \theta_m \} \) in the updating law (4)-(5) can be determined according to the measurable variables of the system.

**IV. ROBUSTNESS TO INITIAL CONDITION ERRORS**

In the following development, robustness of the proposed learning scheme with respect to initial condition errors is established.

(A5') All operations start within a neighborhood of \( x_0(0) \) in the sense of \( \| x_0(0) - x(0) \| \leq c_{x_0} \) for all \( k \) and a positive constant \( c_{x_0} \).

**Theorem 4.1.** Let the system satisfy (1)-(2) assumptions (A1)-(A4) and (A5') and use the updating law (4)-(5). The learning gain \( \Gamma_k(t) \) is chosen satisfying (12) for all \( k \) and \( t \in [0, T] \). As operations repeat, \( k \to \infty \), the tracking error converges into the following error bounds

\[
\limsup_{k \to \infty} \sup_{t \in [T-\sigma, T]} \{ e^{-\lambda t} \| e_\sigma(t) \| \} \leq \beta_1 \delta \max_{1 \leq q \leq m} \{ \sigma^2_{q_j} \} + \alpha c_{x_0}, \tag{30}
\]

\[
\limsup_{k \to \infty} \sup_{t \in [T-\sigma, T]} \{ e^{-\lambda t} \| e_\sigma(t) \| \} \leq \beta_2 \delta + \alpha c_{x_0}. \tag{31}
\]

with positive constants \( \lambda, \beta_1, \beta_2 \) and \( \alpha \) to be defined.

**Remark 4.1.** Theorem 4.1 ensures boundedness of the tracking error in the presence of bounded initial condition errors. The first term in error bound (30) and the first and second terms in error bound (31) can be specified by the time shift \( \sigma \) whereas the second terms cannot. Thus, the first terms in these bounds are not dominant when \( \sigma \) is chosen small enough.

**Proof of Theorem 4.1.** For \( t \in [0, T - \sigma] \), parallel to the same manipulations for arriving at (14), this inequality still holds in the presence of initial condition errors with the same constants \( c_1, c_2 \) and \( c_3 \). Integrating the state equations and applying Bellman-Gronwall lemma give rise to

\[
\| \Delta x_k(t) \| \leq e^{\lambda t} c_{x_0} + c_B \int_{0}^{t} e^{-(\lambda - 4)s} \| \Delta u_k(s) \| ds, \tag{32}
\]

where \( c_4 = l_e + l_e c_{ad} \). Then substituting (32) into (14) produces

\[
\| \Delta u_k(t) \| \leq \rho \| \Delta u_k(t) \| + c_3 (e^{\lambda t} c_{x_0}) + \int_{0}^{t} e^{-(\lambda - 4)s} c_B \| \Delta u_k(s) \| ds, \tag{33}
\]

\[
+ c_3 \left( \int_{0}^{t} e^{-(\lambda - 4)s} c_B \| \Delta u_k(s) \| ds \right) + \int_{0}^{t} e^{-(\lambda - 4)s} c_B \| \Delta u_k(s) \| ds.
\]
\[
\begin{align*}
+ c_3 & \left( \int_{t_0}^{t_1} \cdots \int_{t_m}^{t_{m+1}} \left| \Delta u_4(t_{m}) - \Delta u_4(t) \right| \, dt_{m+1} \cdots \, dt_1 \right) \\
& \left( \int_{t_0}^{t_1} \cdots \int_{t_m}^{t_{m+1}} \left| \Delta u_4(t_{m}) - \Delta u_4(t) \right| \, dt_{m+1} \cdots \, dt_1 \right).
\end{align*}
\]

Multiplying both sides by \( e^{-\lambda t} \) and using the saturation feature lead to

\[
\sup_{t \in [T - \sigma, T]} \left( e^{-\lambda t} \left| \Delta u_{k+1}(t) \right| \right) \leq \rho \sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_k(t) \right| \right) + c_3 \delta \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + c_3 e^{c_3 t}.
\]

(33)

where \( \rho = \rho + c_6 c_2 e^{-\lambda T} \frac{e^{c_3 T} - 1}{c_4} + 4 c_3 \) and \( c_6 = c_1 + c_2 + \frac{e^{c_3 T} - 1}{\lambda} \max_{1 \leq q \leq m} \left( \frac{1 - e^{-\lambda T}}{\lambda} \sigma_{pq} \right) \). Since \( 0 \leq \rho < 1 \), we can find a \( \lambda > c_3 \) sufficiently large such that \( \rho < 1 \). Then (17) is a contraction in \( \sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_k(t) \right| \right) \), which implies

\[
\lim_{k \to \infty} \sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_k(t) \right| \right) \leq \frac{c_6 \delta}{1 - \rho} \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + \frac{c_3}{1 - \rho} e^{c_3 t}.
\]

(34)

Furthermore, it follows from (32) that

\[
\sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_1(t) \right| \right) \leq \frac{c_6}{1 - \rho} \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + \frac{c_3}{1 - \rho} e^{c_3 t}.
\]

Thus,

\[
\lim_{k \to \infty} \sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_k(t) \right| \right) \leq \frac{c_6}{1 - \rho} \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + \frac{c_3}{1 - \rho} e^{c_3 t}.
\]

(35)

For \( t \in [T - \sigma, T] \), (32) still holds so that

\[
\left| \Delta u_1(t) \right| \leq e^{c_3 T} \Delta u_1(t) + \int_{0}^{T - \sigma} e^{c_3 s} \left| \Delta u_4(s) \right| \, ds + c_B \int_{T - \sigma}^{T} e^{c_3 s} \left| \Delta u_4(s) \right| \, ds.
\]

which results in, for \( \lambda > c_4 \),

\[
\sup_{t \in [T - \sigma, T]} \left( e^{-\lambda t} \left| \Delta u_1(t) \right| \right) \leq \frac{c_6}{1 - \rho} \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + \frac{c_3}{1 - \rho} e^{c_3 t} + 2 c_B \delta \sigma + c_{s,0} e^{c_3 T}.
\]

Substituting (34) into (36) gives

\[
\lim_{k \to \infty} \sup_{t \in [0, T - \sigma]} \left( e^{-\lambda t} \left| \Delta u_k(t) \right| \right) \leq \frac{c_6}{1 - \rho} \max_{1 \leq q \leq m} \left( \frac{\sigma_{pq}}{\mu_q} \right) + \frac{c_3}{1 - \rho} e^{c_3 T} – 1 + 1 c_{s,0} e^{c_3 T}.
\]

(37)

Therefore, combining (35) and (37) and defining

\[
\beta_1 = l c B c_6 \frac{1 - e^{-c_4 T}}{\lambda - c_4} \frac{1}{1 - \rho},
\]

(38)

\[
\beta_2 = 2 l c B c_6, \quad \alpha = l c B c_6 \frac{1 - e^{-c_4 T}}{\lambda - c_4} \frac{c_6}{1 - \rho} + 1.
\]

(39)

(40)

the theorem follows.

\section*{V. APPLICATION EXAMPLE}

Consider the dynamic motion equation of a rigid-link manipulator with \( n \) degree of freedom described in joint space by the following equation

\[
D(q) \ddot{q} + c(q, \dot{q}) = u,
\]

(41)

where \( q \in \mathbb{R}^n \) represents the joint angle; \( D(q) \in \mathbb{R}^{n \times n} \) the inertia matrix; \( c(q, \dot{q}) \in \mathbb{R}^n \) the vector of centrifugal, Coriolis and gravitational forces; and \( u \) the input torque. Note that the system has relative degree \( \{ \mu_1, \ldots, \mu_n \} = \{ 2, \ldots, 2 \} \) if only the joint angle is assumed to be the output variable. Let the operation time \( t \in [0, T] \) and \( \sigma \) be the time shift. Given the desired trajectory \( q_d(t) \) for the interval, the control objective is to find the input \( u(t) \), so that the tracking error \( q_d(t) - q(t) \) is within given small error bound \( \varepsilon \), i.e.

\[
\left| q_d(t) - q(t) \right| < \varepsilon, \quad t \in [0, T].
\]

Based on (4)-(5), we propose an updating law for the
iterative learning control in the form of
\[
v_{k+1}(t) = u(t) + \gamma \frac{2}{\sigma^2} \hat{D}(q_k(t))(q_k(t + \sigma) - q_k(t + \sigma)) - (q_k(t) - q_k(t)),
\]
(42)
\[
u_k(t) = \text{sat}(v_k(t)),
\]
(43)
where the scalar \(\gamma > 0\) is an adjustable factor to provide compensation for model discrepancy, \(\hat{D}\) is an estimate of the inertia matrix and \(\text{sat}(\cdot)\) is defined in (6). For this case, the convergence condition given by (12) reduces to
\[
\left| I - \frac{2}{\sigma^2} \hat{D}(q_k(t)) \right|_{t_0}^{t_1} \leq \rho < 1.
\]
(44)

Usually, there exist errors between \(D(q_k(t))\) and \(\hat{D}(q_k(t))\). It is easy to see from the condition that like conventional learning control methods, our scheme also allows larger model discrepancies. The dynamics for a planar robot with two revolute joints are described by [17]
\[
\begin{bmatrix}
m_1 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 c_2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\
m_2 l_2^2 + m_2 l_2 l_1 c_2 & m_2 l_2^2 + m_2 l_1 l_2 c_2
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+
\begin{bmatrix}
-m_2 l_1 s_2 (\dot{q}_2^2 + 2 \dot{q}_1 \dot{q}_2) \\
m_2 l_1 s_2 \dot{q}_1^2
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]
(45)
where \(c_2 = \cos(q_2)\) and \(s_2 = \sin(q_2)\). The link masses and lengths are given in Table 1, in which the estimated values of the parameters have 10% errors from the true values. The desired trajectories of Joint 1 and 2 are given as
\[
\begin{bmatrix}
q_1, d(t) \\
q_2, d(t)
\end{bmatrix} = \begin{bmatrix}
0.7854(6r^5 - 15r^4 + 10r^3) \\
0.5326(6r^5 - 15r^4 + 10r^3)
\end{bmatrix}.
\]
(46)

Let the operation interval be \([0, 1]\). The updating law (42)-(43) is applied with \(\sigma = 0.01\) and \(\gamma = \frac{1}{100}\). The saturation bounds are \(\delta_1 = 8\) and \(\delta_2 = 2.5\). Zero initial conditions of the system are used, i.e., \(q_1(0) = 0\) and \(\dot{q}_1(0) = 0\), which match the desired initial conditions. No feedback control is used and the initial control \(u_0(t), t \in [0, 1]\), is obtained by applying the inverse of the model to the desired trajectories. Define the performance index
\[
J_k = \max_{s \in [0, 1]} \max \left\{ \left| q_{1, d}(t) - q_1(t) \right|, \left| q_{2, d}(t) - q_2(t) \right| \right\}.
\]
Via simulation, the control objective \(J_k < 0.003(\text{rad})\) is achieved at the cycle \(k = 62\) and the convergence is shown in Fig. 2. To examine robustness of the proposed scheme with respect to initial condition errors, let the initial conditions be \(q_1(0) = 0.01\text{rand}(\text{rad})\) and \(\dot{q}_1(0) = 0\). The randn is a generator of random scalar with normal distribution (mean = 0 and variance = 1) but bounded on the interval \([-5, 5]\). Repetition is done until \(k = 1000\). Figure 3 shows the resultant tracking errors.

**VI. CONCLUSION**

In this paper, the problem of ILC using lower-order differentiations is studied for a class of nonlinear continuous-time systems with higher relative degree. The proposed family of ILC updating laws uses a causal pair of the input taken and its produced variable measurements to compute the required input action. The convergence result of the learning scheme is established with a unified analysis and a sufficient condition for choosing the learning gain is derived, which is shown independent of the highest-order differentiation. Under the same sufficient condition, the robustness with respect to initial condition errors is guaranteed. The numerical results show the effectiveness of the proposed approach for path tracking of a robotic manipulator.

**REFERENCES**

Fig. 2. Path tracking results of Joint 1 and Joint 2: (a) at the cycle $k = 0$; (b) at the cycle $k = 20$; (c) at the cycle $k = 40$; (d) at the cycle $k = 62$.

Fig. 3. Robustness with respect to initial condition errors.


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