AN OUTPUT TRACKING VSS CONTROL IN THE PRESENCE OF A CLASS OF MISMATCHED UNCERTAINTIES

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ABSTRACT

A variable structure control system for output tracking in the presence of a class of mismatched uncertainty is studied in this paper. For the mismatched uncertainty, we use an unknown input observer to estimate the states and then apply the VSS control methodology to attain asymptotic output tracking for an arbitrary desired trajectory. The robustness of the VSS control system in the presence of mismatched uncertainty is analyzed using the quadratic stability approach.

KeyWords: Variable structure system, output tracking, mismatched uncertainty, unknown input observer.

I. INTRODUCTION AND PROBLEM DEFINITION

The variable structure system (VSS) theory has been widely studied and used in control system design over the past forty years [17,33,38]. It is a useful tool for dealing with structured system uncertainties and external disturbances, especially under the matching condition. Earlier results concerning the stability of VSS control systems based on the state feedback can be found in [8,32,33]. VSS control systems using only input and output information to attain stabilization can be classified into two categories: static feedback and dynamic feedback systems. Many VSS control systems based on dynamic feedback control use an observer to estimate the states and then design the VSS control based on the estimated states. The advantage of using static output feedback [12,15,35] is that it makes the VSS controller easy to implement. The main stability problem with such VSS control systems is finding a gain matrix by solving a specific pole placement problem in which the Kimura-Davison condition [7,21,22] plays a key role. This condition necessitates use of the minimum-phase assumption with respect to the plant and imposes constraints on the numbers of inputs, outputs, and states. It should be noted that in the above mentioned literature [12,15,35], it was assumed that the plants were all of relative degree one. Actually, the relative degree of a plant is an important factor in VSS control using output information to achieve system stabilization or output tracking. The case in which the system is of higher relative degree was dealt with in [14,13,27,28]. However, high-order differentials of the sliding manifold are needed to synthesize the observer and the VSS control law. This makes it difficult to implement these proposed VSS control frameworks. Conventionally, this problem is indirectly overcome by using low pass filters to approximate the differentials. A solution for the differential problem was proposed in [25], where a robust exact differentiation was proposed to approximate the differential operator. It was shown that such robust exact differentiation provides exact differentiation in finite time. Another approach [26] is to dilate the system dynamics to obtain an augmented system which is of minimum phase and relative degree one. The dilation of system dynamics is achieved through the design of interactors.

In the presence of uncertainty and disturbance, the so-called matching condition or invariance condition is conventionally assumed [4,8,9,13,14,16,27,29,33,34]. A key advantage of the matching assumption is that the uncertain term does not appear in the dynamics in sliding mode. Compared with the state feedback scheme, it becomes more difficult to design a VSS control law when not all of the state information is available. In [13], the standard Luenberger observer was used to estimate the states to achieve stabilization of the VSS control system. The resultant state estimation error acts as an uncertain term in the dynamics in sliding mode. The stability of the overall system can be ensured if two related gain matrices have sufficiently high gains. In [34], a sliding observer
was introduced to asymptotically estimate the states. The key problem in the development of a sliding observer is that one can only use input and output information while the states appear in the estimation error equation. This problem can be solved by assuming an SPR (strictly positive real) condition for the nominal plant. On the other hand, several methods using static output to construct stable VSS control systems in the presence of matched uncertainties were proposed in [1,2,9]. In [9], based on the minimum-phase assumption, a related SPR condition was deduced, and then a VSS control law depending on the plant output was developed to obtain the sliding condition. In addition, a design procedure for selecting output feedback gain matrices subject to the Kimura-Davison condition was proposed so that the VSS control system would be quadratically stable. It was further shown in [2] that a dynamic feedback VSS control system using only output information could be reduced to an equivalent system by using only static output feedback, and that the constraint due to the Kimura-Davison condition could not be avoided. In [1], an LMI approach was formulated to find the static output feedback gain in order to achieve stability in sliding mode. It should be noted that the minimum-phase assumption is essential for either output tracking or output stabilization using only output information. It was pointed out in [18] that if an unstable system is implemented by means of an anticausal realization, then it is possible to achieve output tracking with respect to arbitrary command signals in systems with unstable zero dynamics.

Now we will discuss VSS control systems in the presence of mismatched uncertainty. Unlike matched uncertainty, mismatched uncertainty and the equivalent control influence the dynamics in sliding mode, thus leading to the issue of robust stability of VSS control systems. In [24], uncertainty was assumed to be a linear function of an unknown parameter vector, and an identifier was used to identify the vector. However, an analogous matching condition had be be imposed. In [5,6], which considered both matched and mismatched uncertainties, the linear matrix inequality method was used to design a stable VSS control system based on state feedback. It was pointed out in [30] that if measurements of both the state vector and the mismatched uncertainty are available, then the mismatched uncertainty can be transformed into matched uncertainty through nonlinear coordinate transformation. This nonlinear coordinate transformation can be constructed by solving a set of differential-algebraic equations. In [36], the robust stability of a VSS control system in the presence of mismatched uncertainties in both system and input matrices was investigated. The quadratic stability method was adopted to analyze robust stability, and a sufficient condition could be obtained by solving a linear matrix inequality.

For the case in which the states are not all available, two kinds of observers, including an unknown input observer and a sliding mode observer, have been proposed to estimate the states of the plants with mismatched uncertainty. The idea of using a reduced-order unknown input observer [23] to achieve robust stability of a VSS control system with mismatched uncertainties was originally presented in [36]. The unknown input observer is a generalization of the standard Luenberger observer. The main feature of an unknown input observer is that if the mismatched uncertainty satisfies some matrix conditions, then the state estimation error exponentially converges to zero. Such matrix conditions must also be maintained for a sliding mode observer [10]. In [10], the states were transformed and partitioned into two subsets using special coordinate transformation. State estimation of the first subset of states exponentially converges to zeros, and that of the second subset is forced to zero in finite time.

Due to its linear structure, the unknown input observer will be adopted in this paper. We shall establish a VSS control framework for output tracking in the presence of a class of mismatched uncertainties by using a full-order unknown input observer. It is assumed that the plant is of relative degree one, and thus that the proposed VSS controller is of 1-order sliding mode. The plant to be controlled is given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \\
z(t) &= Wx(t), \\
y(t) &= Cx(t),
\end{align*}
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, E \in \mathbb{R}^{r \times n}, W \in \mathbb{R}^{r \times n}\) and \(C \in \mathbb{R}^{r \times n}\). The terms \(z(t)\) and \(y(t)\) are the controlled output and the measured output, respectively. The mismatched uncertain term \(Ed(t)\) is a function of the states and disturbance of the plant. Without loss of generality, we assume that matrices \(B\) and \(E\) are of full column rank, and that matrixes \(C\) and \(W\) are of full row rank. Note that both \(WB\) and \(CE\) are square matrices. For the given plant, we make the following significant assumptions:

\( (O1) \) The pair \((C, A)\) is observable.
\( (O2) \) \(\text{rank}(CE) = \text{rank}(E)\).
\( (O3) \) All the invariant zeros, if some exist, of the system \((sl - A)^{-1}E\) are located in the left half plane.
\( (T1) \) The pair \((W, A)\) is observable.
\( (T2) \) The product \(WB\) is nonsingular; i.e., the controlled plant is of relative degree one.
\( (T3) \) All the invariant zeros, if some exist, of the system \(W(sl - A)^{-1}B\) are located in the left half plane, and the maximum of the real parts of the invariant zero is equal to \(-\lambda_0\) for some positive number \(\lambda_0\).
\( (T4) \) For the mismatched uncertainty \(Ed(t)\), there exist two constants \(C_1 \geq 0\) and \(C_2 \geq 0\) such that

\[ |Ed(t)| \leq C_1|x(t)| + C_2. \]

Assumptions (O1)-(O3) are required for synthesis of the
unknown input observer, while assumptions (T1)-(T4) are used in VSS control law design.

With the special condition imposed on the mismatched term $E$, we shall use an unknown input observer [3,19,37] to estimate the states of the plant and construct a VSS control system to attain asymptotic output tracking for an arbitrary desired trajectory. The robustness of the VSS control system in the presence of mismatched uncertainties will be analyzed using the quadratic stability method. Our objective is to use the signals $u(t)$, $z(t)$, and $y(t)$ to design a VSS control system such that

(i) $\lim z(t) - z_d(t) = 0$ where $z_d(t)$ is a desired controlled output trajectory.

(ii) The VSS control system possesses some degree of robustness; i.e., if the constant $C_1$ is within some range, then the VSS control system is robustly stable.

The remainder of this paper is organized as follows. The design of the unknown input observer used to estimate the states of the plant in the presence of uncertainty is developed in Section II. A VSS control scheme based on use of the unknown input observer to achieve output tracking in the presence of a specific mismatched uncertainty is presented in Section III. A simulation study is presented in Section IV. Conclusions and a discussion are presented in Section V. Some technical results for a special case of the VSS control system, are given in the Appendix.

II. THE UNKNOWN INPUT OBSERVER

Usually, a Luenberger observer is used to estimate the states of the plant when the state vector is not available. However, if there are uncertainties, the Luenberger observer fails to correctly estimate the states. To overcome this problem, in the section, we shall present an unknown input observer (UIO) for a plant in the presence of a special mismatched uncertainty. The topic of unknown input observer has been vigorously studied in the field of fault detection and diagnosis. Development of such an observer can be traced back to [19]. Interested readers are referred to [3,37] and the references therein. With reference to the results given in [3,37], we shall present a systematic and tractable approach to designing an unknown input observer. Moreover, the structure of the solutions of the gain matrices in the unknown input observer will be clearly characterized.

For the plant given in (1), we shall use the control input $u(t)$ together with the measured output $y(t)$ to estimate the state vector $x(t)$. The unknown input observer of the system (1) is implemented by

$$\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) + L_2 \xi(t) + L_1 \epsilon_2(t), \\
\epsilon_1(t) &= y(t) - C \hat{x}(t), \\
\epsilon_2(t) &= y(t) - C (A \hat{x}(t) + B u(t)).
\end{align*}$$

Now we will analyze the stability of the unknown input observer and determine matrices $L_1$ and $L_2$. The state estimation error $\hat{x}(t)$ is defined as $\hat{x}(t) = x(t) - \tilde{x}(t)$. Based on Eqs. (1) and (4), the derivative of the state estimation error can be expressed as

$$\dot{\hat{x}}(t) = (A - L_2 CA - L_1 C) \hat{x}(t) + (E - L_2 CE) d(t).$$

The state estimation error may be affected by the uncertainty $Ed(t)$. In order to remove the effect of the uncertainty $d(t)$ upon state estimation, the term $E - L_2 CE$ in Eq. (5) must be set to zero. The influence of the uncertainty $d(t)$ upon the state estimation error can be eliminated if we choose the matrix $L_2 \in \mathbb{R}^{n \times p}$ such that

$$L_2 CE = E.$$  

Furthermore, if we choose the gain matrix $L_1$ such that the eigenvalues of $(I - L_2 C) A - L_1 C$ are in the left half plane, then the state estimation error will exponentially converge to zero. Before solving Eq. (6), we must consider the conditions under which Eq. (6) has a solution.

Lemma 1. The following linear equation

$$\Xi v = \psi,$$

where $\Xi \in \mathbb{R}^{p \times k}$, $\psi \in \mathbb{R}^{n \times 1}$, and $v \in \mathbb{R}^{n \times 1}$, has a solution if and only if $col(\psi) \subset col(\Xi)$.

Proof. This is a fundamental result in linear algebra; thus, the proof is omitted. ■
Lemma 2. Equation (6) has a solution if and only if rank \((CE) = \text{rank}(E)\).

Proof. This result directly follows from Lemma 1. The reader may also refer to [23] for a detailed discussion.

We will now summarize the above results in the following theorem.

Theorem 1. Consider the plant in (1) and the unknown input observer given in (4). Under assumptions (O1)-(O3), we can find gain matrices \(L_2 = E(CE)^{-1}\) and \(L_1\) such that the observer can asymptotically estimate the state vector in the presence of the special uncertainty \(Ed(t)\), i.e.,

\[
\lim_{t \to \infty} \hat{x}(t) - x(t) = 0.
\]

Proof. With assumption (O2) and Lemma 2, the unique solution to equation (6) is

\[
L_2 = E(CE)^{-1}.
\]

Using the gain matrix \(L_2\), the state estimation error of the unknown input observer can be described as

\[
\dot{x}(t) = (P_o A - L_1 C)\hat{x}(t),
\]

where \(P_o = I - E(CE)^{-1}C\) is a projector. Note that for the system matrix \(N_0\) in (2), we have \(N_0 = P_o A - L_1 C\). The evolution of the state estimation error is given as

\[
\ddot{x}(t) = e^{P_o A - L_1 C} \epsilon(0),
\]

where \(\dot{x}(0)\) is the initial value of the error \(\ddot{x}(t)\).

Now the remaining problem is to find a suitable gain matrix \(L_1\) that can be used to place the eigenvalues of \(P_o A - L_1 C\) in the left half plane such that \(\ddot{x}(t)\) will exponentially converge to zero. Let \(\lambda\) be a positive number such that the real part of any invariant zero of the triple \((C, A, E)\) is less than \(-\lambda\). Note that assumptions (O1)-(O3) coincide with assumptions (S1)-(S3) given in the Appendix. Therefore, by applying Lemma 7 given in the Appendix, we can find a gain matrix \(L_1\) such that

\[
\max_{i = 1, \ldots, n} R_i(\lambda, [P_o A - L_1 C]) < -\lambda_f
\]

and

\[
\|\hat{x}(t)\| \leq M e^{-\lambda_f t}\|\hat{x}(0)\|
\]

for some constant \(M > 0\). This completes the proof.

Remark 1. The term \(\epsilon(t)\) appears in the standard Luenberger observer. The other term \(\epsilon(t)\) in Eq. (4) can be rewritten as

\[
\epsilon(t) = CA\dot{x}(t) + CEd(t).
\]

Applying identity (6), it is easy to see that

\[
L_2 \ddot{x}(t) = L_2 CA\dot{x}(t) + Ed(t).
\]

Now with the exponential decay property of \(\ddot{x}(t)\), we see that the term \(L_2 \ddot{x}(t)\) can be used to approximate the unknown input term \(Ed(t)\).

Remark 2. A difficult problem in synthesis of the unknown input observer encountered in previous works, such as in [3,37], was to determine gain matrix \(L_1\) so that \(P_o A - L_1 C\) is Hurwitz. For square systems, a procedure for finding gain matrix \(L_1\) can be constructed based on the results shown in the Appendix. Moreover, the structure of the solutions of gain matrix \(L_1\) is characterized in Lemma 7.

Remark 3. A reduced-order unknown input observer was proposed in [23]. Our unknown input observer is of full order. As indicated in Remark 1, the term \(L_2 \ddot{x}(t)\) is asymptotically equal to \(Ed(t)\); thus, a filtered version of this term can be used to measure the unknown input \(d(t)\). Since the entire unknown input vector \(d(t)\) is to be measured, a full-order structure is adopted here. The goals of the observer are not only to estimate the state vector, but also to determine the unknown input. This is especially important in fault diagnosis and fault isolation. In [23], the problem of determining the unknown input was not explored.

III. THE VSS CONTROL SCHEME

Based on the state estimates provided by the unknown input observer, we shall propose a VSS control scheme for achieving output tracking in the presence of mismatched uncertainty. The robustness of the VSS control system will be analyzed using the quadratic stability method.

Using the estimated state vector \(\ddot{x}(t)\), we define a variable \(\sigma(t)\) as

\[
\sigma(t) = \ddot{x}(t) - z_d(t).
\]

where \(\ddot{x}(t) = W\ddot{x}(t)\) and \(z_d(t)\) is the desired control output trajectory. We assume that \(z_d(t)\) is differentiable, and that both \(z_d(t)\) and \(\ddot{z}_d(t)\) are bounded. The equation \(\sigma(t) = 0\) defines the sliding surface of the VSS control system. With the unknown input observer in (4), the derivative of Eq. (14) can be expressed as

\[
\sigma(t) = W(A\ddot{x}(t) + Bu(t) + L_1 \epsilon(t) + L_2 \ddot{x}(t)) - \ddot{z}_d(t).
\]

Lemma 3. Let the VSS control law be
\[ u(t) = -(WB)^{-1}[WA\dot{x}(t) + WL_1(y(t) - C\tilde{x}(t)) \]
\[ - \dot{z}_d(t) + (k + D(t)) \frac{\sigma(t)}{\sigma(t)} \]  
(16)

where \( k > 0 \), \( \tilde{x}(t) \) is obtained from the unknown input observer in (2), and
\[ D(t) = C_1\left\| W\right\| \tilde{\varepsilon}(t) + C_2\right\| W \]  

Then, the sliding condition
\[ \sigma^T(t)\sigma(t) < 0, \text{ for } \left\| \sigma(t) \right\| \neq 0 \]  
(17)
holds after a finite period of time, and the VSS control system enters the sliding mode in a finite period of time. Therefore, for \( t \geq t_0 \), \( \sigma(t) \) is uniquely obtained from the Lyapunov equation (17) and (16), the derivative of \( V \) is
\[ \dot{V}(t) = \sigma^T(t)\left\{ WA\ddot{x}(t) + Bu(t) + L_1\varepsilon(t) + L_2\varepsilon(t) \right\} \]
\[ - \dot{z}_d(t) + (k + D(t)) \frac{\sigma(t)}{\sigma(t)} + WL_2\varepsilon(t) \]
\[ = \sigma^T(t)\left\{ (k + D(t)) \frac{\sigma(t)}{\sigma(t)} + WL_2\varepsilon(t) \right\} \]
\[ \leq - (k + D(t)) \left\| \sigma(t) \right\| + \left\| WL_2CA \right\| \left\| \tilde{x}(t) \right\| \]
\[ + C_1\left\| W \right\| \left\| \tilde{x}(t) \right\| + C_2\left\| W \right\| \left\| \sigma(t) \right\| \]
\[ \leq - k \left\| WL_2CA \right\| + C_1\left\| W \right\| \left\| \tilde{x}(t) \right\| \left\| \sigma(t) \right\| . \]  
(18)

Since \( \left\| \tilde{x}(t) \right\| \) exponentially decays to zero, there exists a finite period of time \( t_0 \) such that for \( t \geq t_0 \),
\[ \left\| WL_2CA \right\| + C_1\left\| W \right\| \left\| \tilde{x}(t) \right\| \leq \frac{k}{2} \]
Therefore, for \( t \geq t_0 \), we have
\[ \frac{d}{dt} \left\| \sigma(t) \right\| \leq - \frac{k}{2} \]  
(19)

Therefore, the sliding condition (17) holds after \( t = t_0 \), and VSS control system enters the sliding mode before some finite period of time has passed:
\[ t_{\text{reach}} = \frac{2\left\| \tilde{x}(t_0) - z_d(t_0) \right\|}{k} + t_0 \]

Now, for \( t \geq t_{\text{reach}} \), we get \( \tilde{x}(t) = z_d(t) \). On the other hand, it is guaranteed by the proposed observer that the state estimation error \( x(t) - \hat{x}(t) \) exponentially converges to zero. Thus, the output tracking error \( z(t) - z_d(t) \) also exponentially converges to zero. \[ \blacksquare \]

Now, we will discuss the stability of the dynamics in sliding mode. In sliding mode, the following conditions hold:
\[ \sigma(t) = 0, \quad \dot{\sigma}(t) = 0. \]

In the following, we shall first present a technical lemma and then present our main result concerning the robust stability of the variable structure control system.

**Lemma 4.** Consider the system
\[ \gamma(t) = A_\gamma \gamma(t) + \eta_1(t), \]
where \( A_\gamma \in \mathbb{R}^{2 \times n} \) and the dimension of \( \gamma(t) \) is \( n \). The vector \( \eta_1(t) \) is subject to the following condition:
\[ \eta_1(t) \leq K_1 + K_2 \gamma(t) + K_3 e^{-\lambda t}, \quad \lambda > 0, \]
where \( K_1, K_2, \) and \( K_3 \) are positive constants. Suppose that \( A_\gamma \) is a Hurwitz matrix such that \( \max_{i=1,\ldots,n} Re(\lambda_i(A_\gamma)) < -\lambda_0 > 0 \), for some \( \lambda_0 > 0 \). If \( K_3 \) satisfies the condition
\[ K_3 < \frac{\lambda_{\text{min}}(X)}{\lambda_{\text{max}}(X)} + \frac{1}{2} \frac{\lambda_{\text{max}}(X)}{\lambda_{\text{min}}(X)} \]
where the symmetric positive-definite matrix \( X \) is uniquely obtained from the Lyapunov equation
\[ (A_\gamma + \bar{X}J)^T X + X(A_\gamma + \bar{X}J) = -I, \]
then \( \gamma(t) \) is bounded.

**Proof.** Since the condition \( \max_{i=1,\ldots,n} Re(\lambda_i(A_\gamma)) < -\lambda_0 \) is satisfied, it is seen in [31] that the symmetric positive-definite matrix \( X \) can be uniquely determined from Eq. (23). From Eq. (23), we can obtain
\[ A_\gamma^T X + X A_\gamma = -2\lambda_0 X - I. \]
(24)

Now, consider the following Lyapunov function:
\[ V(t) = \gamma^T(t) \bar{X} \gamma(t). \]
(25)

Differentiating \( V(t) \) with respect to \( t \) yields
\[ \dot{V}(t) = \gamma^T(t)(A_\gamma^T \bar{X} + X A_\gamma \gamma(t) + \eta_1^T(t) \bar{X} \gamma(t) + \gamma^T(t) \bar{X} \eta_1(t)) \]
\[ \leq - (2\lambda_0 \lambda_{\text{min}}(X) + 1) \gamma(t) + 2\lambda_{\text{max}}(X) \gamma(t). \]
(26)
From Eq. (21), Eq. (26) can be rewritten as
\[
V \leq (2 \lambda_{\text{max}}(X) + 1) \left[ \gamma(t) \right]^2 + 2 \lambda_{\text{max}}(X) K \| \gamma(t) \|^2 \\
+ 2 \lambda_{\text{max}}(X) [K_1 + K_\beta e^{-\beta t}] \| \gamma(t) \| \\
\leq -\alpha V(t) + \beta \sqrt{V(t)} + C_\beta e^{-\beta t} \sqrt{V(t)},
\]
where \( \alpha = 2 \lambda_{\text{max}}(X) \), \( \beta = 2K_1 \lambda_{\text{max}}(X) \), and \( C_\beta = 2K_1 \lambda_{\text{max}}(X) \). Then, the evolution of \( V(t) \) can be determined from Eq. (27) as follows:
\[
\sqrt{V(t)} \leq \left( \sqrt{V(0)} - \frac{\beta \sqrt{V(0)}}{\alpha} - \frac{C_\beta}{2(-\lambda + \alpha/2)} \right) e^{-\frac{\beta t}{2}} \\
+ \frac{\beta}{\alpha} \left[ -\frac{C_\beta}{2(-\lambda + \alpha/2)} \right] e^{-\frac{\beta t}{2}},
\]
where \( V(0) \) is the initial value of \( V(t) \) and \( \gamma(t) \) obeys the following inequality:
\[
\| \gamma(t) \| \leq \frac{1}{\sqrt{\lambda_{\text{min}}(X)}} \left[ \sqrt{\lambda_{\text{max}}(X)} \right] \| \gamma(0) \| - \frac{\beta}{\alpha} \\
- \frac{C_\beta}{2(-\lambda + \alpha/2)} e^{-\frac{\beta t}{2}} + \frac{\beta}{\alpha} \left[ -\frac{C_\beta}{2(-\lambda + \alpha/2)} \right] e^{-\frac{\beta t}{2}},
\]
where \( \gamma(0) \) is the initial state of \( \gamma(t) \). Therefore, if \( \alpha \) is positive, i.e., condition (22) holds, then \( \gamma(t) \) is bounded.

Now, we define a projector as \( P_T = I - B(WB)^{-1}W \). With assumptions (T1)-(T3), Lemma 7 ensures that we can find a matrix \( L \) such that
\[
\max_{i=1, \ldots, m} R_\sigma(\tilde{A}) = -\lambda_{\text{in}}
\]
where \( \tilde{A} = P_TA - LW \). The procedure for determining the matrix \( L \) is given in the Appendix.

**Theorem 2.** Based on assumptions (O1)-(O3) and (T1)-(T4), the VSS control system in sliding mode is internally stable if, for some \( \tilde{A}_0 \) with \( 0 \leq \tilde{A}_0 < \lambda_{\text{in}}, \)
\[
C_1 < \frac{\lambda_{\text{max}}(X)}{P_T \lambda_{\text{max}}(X)} \left( \frac{\lambda_{\text{max}}(X)}{\lambda_{\text{min}}(X)} + \frac{1}{2 \lambda_{\text{max}}(X)} \right)
\]
where \( X \) is a symmetric positive definite matrix satisfying the following Lyapunov equation:
\[
(\tilde{A}^T + \tilde{A})X + X(\tilde{A} + \tilde{A}) = -I.
\]

**Proof.** Based on the condition \( \sigma = 0 \) in (15), the equivalent control can be determined from Eq. (15) as
\[
u_{eq}(t) = -(WB)^{-1} [WA\tilde{x}(t) + WL_1\varepsilon(t) + WL_2\varepsilon(t) - \varepsilon_d(t)].
\]
Substituting Eqs. (32) and (13) into Eq. (4), the dynamics of the unknown input observer in sliding surface can be rewritten as
\[
\dot{\tilde{x}}(t) = P_TA\tilde{x}(t) + P_T(L_2CA + L_1C)\tilde{x}(t) \\
+ B(WB)^{-1}\tilde{z}_d(t) + P_T\varepsilon(t).
\]
With the condition \( \sigma = 0 \), we add a term \(-L_2\sigma(t)\) to the right hand side of Eq. (33) to obtain
\[
\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \eta_1(t),
\]
where \( \eta_1(t) \) is defined as
\[
\eta_1(t) = P_T(L_2CA + L_1C)\tilde{x}(t) + P_T\varepsilon(t) + L_2\sigma(t)B(WB)^{-1}\tilde{z}_d(t).
\]
Since \( z_d(t) \) and \( \dot{z}_d(t) \) are assumed to be bounded, there exists a constant \( K_{\sigma} \) such that
\[
\| L_2\sigma(t)B(WB)^{-1}\tilde{z}_d(t) \| \leq K_{\sigma}.
\]
Then, with inequality (12) and assumption (T4), an upper bound for \( \| \eta_1(t) \| \) can be given as
\[
\| \eta_1(t) \| \leq \tilde{K}_1 + \tilde{K}_2 \| \tilde{x}(t) \| + \tilde{K}_3 e^{-\lambda t},
\]
where
\[
\tilde{K}_1 = K_{\sigma} + C_4 \| P_T \|, \\
\tilde{K}_2 = C_4 \| P_T \|, \\
\tilde{K}_3 = M( \| P_T L_2CA + P_T L_1C \| + C_4 \| P_T \| ) \| \tilde{x}(0) \|.
\]
Under condition (30), direct application of Lemma 4 leads to the following inequality:
\[
\| \tilde{x}(t) \| \leq \frac{1}{\sqrt{\lambda_{\text{min}}(X)}} \left( \sqrt{\lambda_{\text{max}}(X)} \right) \| \tilde{x}(0) \| - \frac{\beta}{\alpha} \\
- \frac{C_\beta}{2(-\lambda + \alpha/2)} \| \tilde{x}(0) \| + \frac{\beta}{\alpha} \left[ -\frac{C_\beta}{2(-\lambda + \alpha/2)} \right] e^{-\frac{\beta t}{2}}.
\]
where

\[ \alpha = 2 \lambda_{\max}(X) + \frac{1}{\lambda_{\min}(X)} - 2C \| P \| \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}. \]

\[ \beta = 2(K_d + C_M P_T) \frac{\lambda_{\max}(X)}{\sqrt{\lambda_{\min}(X)}}, \]

\[ C_{\beta} = 2 \left[ C_1 + M \right] P_T L CA + P_T L \frac{\lambda_{\max}(X)}{\sqrt{\lambda_{\min}(X)}}. \]

Therefore, if condition (31) holds, then \( \tilde{x}(t) \) is bounded. Moreover, inequality (12) implies that \( x(t) \) is bounded. Finally, it follows from the definition of the VSS control law (16) that \( u(t) \) is also bounded.

**IV. SIMULATION STUDY**

Consider the following plant:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t)
\end{bmatrix} =
\begin{bmatrix}
-9 & 4 & 1 \\
1 & 7 & 3 \\
-7 & -5 & -3
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} u(t) \\
+ \begin{bmatrix}
0 \\
-1 \\
0
\end{bmatrix} (\sin(t) + x_2(t)),
\]

\[ y(t) = \begin{bmatrix}
0 & -5 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}, \]

\[ z(t) = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}, \quad (36)
\]

The mismatched uncertainty \( E_d(t) = \begin{bmatrix}
0 \\
-1 \\
0
\end{bmatrix} \sin(t) + x_2(t) \) satisfies the following inequality:

\[ \| E_d(t) \| \leq \| x(t) \| + 1, \]

where \( C_1 = 1 \) and \( C_2 = 1 \). The initial state of the system is given as \( x(0) = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T \). The gain matrices \( L_1 \) and \( L_2 \) in the unknown input observer are chosen as

\[ L_1 = \begin{bmatrix}
2 \\
-10 \\
3
\end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix}
0 \\
-0.2 \\
0
\end{bmatrix}, \]

and the initial state of the unknown input observer is \( \tilde{x}(0) = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}^T \). The transmission zeros of the triple \((C, A, E)\) are \(-7.4142 \text{ and } -4.5858\). The state estimation error \( \tilde{x}(t) \) is shown in Fig. 1. The VSS control law is defined as follows:

\[ u(t) = - \left( \begin{bmatrix}
1 & 7 & 3 \\
0 & -5 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1(t) \\
\tilde{x}_2(t) \\
\tilde{x}_3(t)
\end{bmatrix} - 10 \right) \begin{bmatrix}
y(t) \\
\tilde{y}(t) \\
\tilde{y}_{d}(t)
\end{bmatrix} - \tilde{\sigma}(t) + (60 + D(t)) \frac{\sigma(t)}{\| \sigma(t) \|}, \]

\[ D(t) = \| \tilde{x}(t) \| + 1. \]

The transmission zeros of the triple \((W, A, B)\) are \(-7.5 \pm 2.3979i\); thus, \( \lambda_0 = 7.5 \). We choose

\[ L = \begin{bmatrix}
4 & 7.5 & -12
\end{bmatrix}^T, \]

so that condition (30) holds. In (31), we choose \( \bar{\lambda}_0 = 0.5 \) and determine the matrix \( X \) as

\[ X = \begin{bmatrix}
0.0972 & 0 & -0.0408 \\
0 & 0.0714 & 0 \\
-0.0408 & 0 & 0.0835
\end{bmatrix}. \]

![State Estimation Error](image-url)

Fig. 1. The state estimation error \( \tilde{x}(t) \) of the UIO: \( \tilde{x}_d(t) \) is the solid line, \( \tilde{x}_2(t) \) is the dotted line, and \( \tilde{x}_3(t) \) is the dashdot line.
The upper bound for $C_1$, shown on the right hand side of the inequality (31), is equal to 1.0481. Therefore, the VSS control system is robustly stable since $C_1$ is less than 1.0481. The control output $z(t)$ and the desired output trajectory $z_d(t)$ are compared in Fig. 2. The desired output trajectory is selected as the output of a low-pass filter with the unit step as its input. The VSS control input $u(t)$ and the variable $\sigma(t)$ are shown in Fig. 3. The evolution of the state vector $x(t)$ is illustrated in Fig. 4.

**V. CONCLUSIONS AND DISCUSSION**

In this paper, we have used the unknown input observer to estimate the states of the system and proposed a VSS control scheme for achieving asymptotic output tracking and robust stability in the presence of mismatched uncertainty. The following issues should be explored in the future.

1. When the known input observer is used to estimate the states of the system, it is limited by the condition that the invariant zeros of the triple $(C, A, E)$ must be located in the left half plane. It is seen that the same constraint also applies to a sliding mode observer for systems with mismatched uncertainty [10]. How to relax this limitation by using a more general observer is a good topic for future study.

2. In our VSS control design, the relative degree of the plant is assumed to be one. We can extend our results to plants with higher relative degrees in the future. This is a topic concerning high order sliding modes for output tracking control. Using the technique which employs high order sliding modes, we can not only eliminate the chattering phenomenon of the VSS control law, but also can avoid use of the derivative of the control output.

**APPENDIX**

In this appendix, we shall discuss a special pole placement problem. The results will be used both in unknown input observer design and stability analysis of the proposed VSS control system. Consider the matrices $A \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times s}$, and $H \in \mathbb{R}^{n \times r}$, where matrix $H$ is of full column rank and $G$ is of full row rank. Let matrix $P$ be a projector defined as

$$P = I - H(GH)^{-1}G.$$ 

Here, we shall discuss the pole placement problem for the matrix $PA - FG$ by choosing a suitable gain matrix $F \in \mathbb{R}^{s \times s}$ based on the following assumptions:
(S1) The pair \((G, A)\) is observable.
(S2) \(\text{rank}(GH) = \text{rank}(H)\).
(S3) All the invariant zeros, if any exist, of the triple \((G, A, H)\) are in the left half plane.

Assumption (S2) implies that \(GH\) is nonsingular, and that \(\text{null}(G) \cap \text{col}(H) = 0\).

Note that under assumption (S2), it is easy to verify that
\[
\text{null}(P) = \text{col}(H),
\]
\[
\text{col}(P) = \text{null}(G).
\]

Now, assume that the matrix \(F\) is of the form \(F = PF_0 + F_1\). Substituting \(F = PF_0 + F_1\) into \(PA = FG = P\overline{A} - F_1G\),
\[
PA = FG = P\overline{A} - F_1G,
\]
where \(\overline{A} = A - F_0G\). Since the pair \((G, A)\) is observable, we can find a matrix \(F_0\) such that \(A\) is nonsingular. In addition, since the pair \((G, A)\) is observable, so is the pair \((G, \overline{A})\) by the PBH rank test. It is also easy to check that the invariant zeros of the two triples \((G, \overline{A}, H)\) and \((G, A, H)\) are identical.

The relationship between the unobservable modes of the pair \((G, P\overline{A})\) and the invariant zeros of the triple \((G, \overline{A}, H)\) is given in the following lemma.

**Lemma 5.** A number \(\lambda\) is an invariant zero of the triple \((G, \overline{A}, H)\) if and only if it is an unobservable mode of the pair \((G, P\overline{A})\).

**Proof.** For this fact, readers can be referred to [11] or Lemma 1 in [23].

Now, we will consider the pole placement problem of the matrix \(P\overline{A} - F_1G\) by choosing a suitable matrix \(F_1\). Consider a matrix \([U_G, H]\) with which the columns of \(U_G\) form a basis of the null space of \(G\). Since \(\text{col}(U_G) \cap \text{col}(H) = 0\), the columns of the matrix \([U_G, H]\) form a basis of the space \(\mathbb{R}^n\). Therefore, there exists a matrix \(\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}\)

so that
\[
\overline{A} \begin{bmatrix} U_G & H \end{bmatrix} = \begin{bmatrix} U_G & H \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},
\]
where \(\Gamma_{11} \in \mathbb{R}^{(n-s) \times (n-s)}, \Gamma_{12} \in \mathbb{R}^{(n-s) \times s}, \Gamma_{21} \in \mathbb{R}^{s \times (n-s)},\) and \(\Gamma_{22} \in \mathbb{R}^{s \times s}\). With the properties described in (37), left multiplying matrix \(P\) to both sides of equation (39) gives
\[
P\overline{A} \begin{bmatrix} U_G & H \end{bmatrix} = \begin{bmatrix} U_G \Gamma_{11} & H \Gamma_{12} \end{bmatrix}.
\]

In particular, from equation (40), we have
\[
P\overline{A} U_G = U_G \Gamma_{11},
\]
which indicates that \(\text{col}(U_G)\) is an invariant subspace of \(P\overline{A}\). Since \(\overline{A}\) is nonsingular, \(U_G\) is of full column rank, and \(\text{rank}(P) = n - s\), it follows that \(\text{rank}(\Gamma_{11}) = n - s\) and that \(\Gamma_{11}\) is nonsingular. Let matrix \(\Gamma_{11}\) be decomposed as
\[
\Gamma_{11} = N \Lambda N^{-1},
\]
where \(\Lambda \in \mathbb{R}^{(n-s) \times (n-s)}\) is a diagonal or a block-diagonal matrix, the eigenvalues of \(\Gamma_{11}\) are the entries on the diagonal and the columns of \(N\) are the eigenvectors of \(\Gamma_{11}\). Then, equation (40) can be rewritten as
\[
P\overline{A} U_G N = U_G N \Lambda. \tag{43}
\]

Now, define a transformation matrix
\[
T = \begin{bmatrix} U_G N & \overline{A}^{-1} H \end{bmatrix},
\]
we can obtain
\[
T^{-1} P\overline{A} T = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad GT = \begin{bmatrix} 0 & G \overline{A}^{-1} H \end{bmatrix}. \tag{45}
\]

**Lemma 6.** For the pair \((G, P\overline{A})\), we have
(i) the \((n - s)\)-tuple non-zero eigenvalues of \(P\overline{A}\) correspond to the unobservable modes of the pair \((G, P\overline{A})\);
(ii) the other \(s\)-tuple zero eigenvalues of \(P\overline{A}\) correspond to observable modes of the pair \((G, P\overline{A})\).

**Proof.** One can refer to [11] for these facts.

Now, from the identity
\[
\begin{bmatrix} \overline{A} & H \\ 0 & -G \overline{A}^{-1} H \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -(G \overline{A}^{-1}) & I_s \end{bmatrix} \begin{bmatrix} \overline{A} & H \\ G & 0 \end{bmatrix},
\]
we have
\[
\det(\overline{A}) \det(-G \overline{A}^{-1} H) = \det \begin{bmatrix} \overline{A} & H \\ G & 0 \end{bmatrix}.
\]

Note that the invariant zeros of the two triples \((G, A, H)\) and \((G, \overline{A}, H)\) are identical. Since all the invariant zeros of the triple \((G, A, H)\) are located in the left half plane, we have
\[
det \begin{bmatrix} \overline{A} & H \\ G & 0 \end{bmatrix} \neq 0. \quad \text{As } \overline{A} \text{ is nonsingular, it follows}
\]
that matrix $G \widetilde{A}^{-1}H$ is nonsingular. The previous results are summarized in the following.

**Lemma 7.** Consider the matrix $PA - FG$ based on assumptions (S1)-(S3). Suppose that matrix $F$ is given by

$$F = PF_0 + T \bar{F}_1, \quad \bar{F}_1 = \begin{bmatrix} \bar{T}_{11} \\ \bar{T}_{12} \end{bmatrix}.$$  \hspace{1cm} (46)

where $T$ is defined in (44), $\bar{T}_{11}$ is an arbitrary $(n - s) \times s$ matrix, and $\bar{T}_{12} \in \mathbb{R}^{s \times s}$ is determined such that the $s$ eigenvalues of $-\bar{T}_{12}G \widetilde{A}^{-1}H$ are located at specified locations in the left half plane. Then, $PA - FG$ is a Hurwitz matrix whose eigenvalues consist of two parts:

(i) the $n - s$ invariant zeros of the triple $(G, A, H)$ and
(ii) the $s$ eigenvalues of $-\bar{T}_{12}G \widetilde{A}^{-1}H$.

**Proof.** With the definition of the transformation matrix $T$ and Eq. (45), we have

$$T^{-1}(PA - FG)T = T^{-1}(P\bar{A} - T\bar{F}_1G)T$$

$$= \begin{bmatrix} \Lambda & -\bar{T}_{11}G \widetilde{A}^{-1}H \\ 0 & -\bar{T}_{12}G \widetilde{A}^{-1}H \end{bmatrix}.$$  \hspace{1cm} (47)

From the results of Lemma 6, it follows that the eigenvalues of $\Lambda$ correspond to the unobservable modes of $(G, P\bar{A})$, which are the invariant zeros of the triple $(G, A, H)$ located in the left half plane by the assumption. Therefore, all the eigenvalues of $\Lambda$ are stable. Since matrix $G \widetilde{A}^{-1}H$ is nonsingular, the pair $(G \widetilde{A}^{-1}H, 0_{s \times s})$ is observable by the PBH rank test. Therefore, we can choose a matrix $\bar{T}_{12}$ such that the eigenvalues of $-\bar{T}_{12}G \widetilde{A}^{-1}H$ are placed at specific locations in the left half plane. \hfill \Box

**NOTATIONS AND DEFINITIONS**

- $col(B)$: the vector space spanned by the columns of matrix $B$
- $null(A)$: the null space of matrix $A$
- $\|A\|$: the spectral norm of matrix $A$
- $(C, A, B)$: a triple used to represent a system $C(sl - A)^{-1}B$
- $\lambda_i(A)$: the $i$-th eigenvalue of matrix $A$
- $\Re(c)$: the real part of complex number $c$

**REFERENCES**