SYSTEMATIC GAIN-SCHEDULING CONTROL DESIGN:
A MISSILE AUTOPILOT EXAMPLE

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ABSTRACT

The missile autopilot was designed using linear parameter-varying (LPV) control techniques. The controller provides exponential stability guarantee and performance bound in terms of induced $\ell_2$ norm for the missile plant. The systematic gain-scheduling approach is motivated by the recent development in LPV control theory and provides a well founded and systematic procedure for high performance missile autopilot design problem.

KeyWords: Gain-scheduling control, linear parameter-varying system, missile autopilot, control design.

I. INTRODUCTION

In this paper, we will apply recently developed LPV control techniques to the pitch-axis autopilot design for a generic tail-controlled missile problem. Missile control problem is an aerospace industrial application which is well suited for advanced control design due to fast and wide parameter variations during its operation. Through the design process, we are trying to demonstrate the advantages of LPV control theory and provide a concrete example for this systematic gain-scheduling control methodology.

The gain-scheduling approach is perhaps one of the most popular nonlinear control design techniques which has been widely used in the fields ranging from aerospace and process control. Although it seems working well in practice, this heuristic design procedure does not take the parameter variations into account [15,17]. In its early practice, the control design comes with virtually no guarantee on performance, robustness, or even nominal stability. Motivated by the gain scheduling control design methodology, a systematic gain-scheduling control technique has been developed recently in the framework of linear parameter-varying (LPV) systems with guaranteed stability and performance properties [2,3,6,12,21].

This class of systems is different from its standard, linear time-varying counterpart due to the causal dependence of its controller gains on the variations of the plant dynamics. LPV control theory simplifies the interpolation and realization problems associated with conventional gain scheduling.

The missile autopilot design problem is a challenging one for its wide range of parameter variations and stringent performance requirements, and hence it has attracted attentions of many researchers. In reference [13], robust control method in the framework of $\mu$-synthesis was applied to the autopilot design. The robust stability of closed-loop system is achieved in the existence of large speed and angle of attack variations, but the performance of such controller is relatively conservative. In references [14] and [5], gain-scheduled controllers in the framework of robust control were sought by fixed point design at several equilibrium points, then different gain-scheduling strategies were used to interpolate and switch those controllers. The concept of “quasi-LPV” system was first introduced in [18] and a state transformation was proposed to convert nonlinear missile dynamics into a linear parameter-varying form. The resulting gain-scheduled controller has an interesting inner-out loop architecture, with inner-loop having linearization effect and outer-loop generating desired output. In reference [11], the nonlinearity in missile models was carefully addressed through gain-scheduled control from extended linearization point of view [15]. By imposing linearization property to gain-scheduling controller, it was hoped that the stability and performance properties of local $\mathcal{H}_\infty$ control design will be inherited by the resulting gain-scheduled controllers. Different from conventional gain-scheduling approaches, LPV control techniques provide explicit
stability and performance properties. In references [4] and [20], single quadratic Lyapunov functions were employed for the missile autopilot design. In this paper, it will be shown that significant performance improvement can be achieved using a parameter-dependent Lyapunov function. Moreover, an efficient control switching scheme based on table lookup and local interpolation is proposed to simplify LPV control implementation and levitate real-time control requirement.

II. LPV SYSTEM CONTROL SYNTHESIS APPROACH

Consider a generalized open-loop LPV system for control synthesis which takes the standard structure

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\
C_1(\rho(t)) & D_{11}(\rho(t)) & D_{12}(\rho(t)) \\
C_2(\rho(t)) & D_{21}(\rho(t)) & D_{22}(\rho(t))
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e(t) \\
u(t)
\end{bmatrix}
\]

where \(x_0, \dot{x} \in \mathbb{R}^n, d \in \mathbb{R}^n, e \in \mathbb{R}^n, u \in \mathbb{R}^n\) and \(y \in \mathbb{R}^n\). All matrix valued state-space functions are continuous and have appropriate dimensions.

It is assumed that the vector-valued parameter \(\rho\) evolves continuously over time and its range is limited to a compact subset \(\mathcal{P} \subseteq \mathbb{R}^n\). In addition, its time derivative is bounded and satisfies the constraint \(\nu_i \leq \frac{d\rho_i}{dt} \leq \nu_i, i = 1, 2, \ldots, s\). For notational purposes, denote \(\mathcal{V} = \{\nu : \nu_i \leq \nu \leq \nu_i, i = 1, 2, \ldots, s\}\), where \(\mathcal{V}\) is a given convex polytope in \(\mathbb{R}^s\) that contains the origin. Given the sets \(\mathcal{P}\) and \(\mathcal{V}\), we can define the parameter \(\nu\)-variation set as

\[\mathcal{F}_\nu = \{\rho \in C^1(\mathbb{R}, \mathbb{R}^n) : \rho(t) \in \mathcal{P}, \ \rho(t) \in \mathcal{V}, \ \forall t \geq 0\}\]

Then \(\mathcal{F}_\nu\) specifies the set of all allowable piecewise continuous parameter trajectories.

For simplicity, we also assume that: (A1). The triple \((A(\rho), B_1(\rho), C_1(\rho))\) is parameter-dependent stabilizable and detectable for all \(\rho \in \mathcal{P}\); (A2). \(D_{12}(\rho)\) and \(D_{21}(\rho)\) have full column and row rank respectively for all \(\rho \in \mathcal{P}\); (A3). \(D_{11}(\rho) = 0\) and \(D_{22}(\rho) = 0\). These technical assumptions can be relaxed at the expense of more complicated formulae, see [19] for details.

The class of LPV controller \(K_\nu\) is in the form of

\[
\begin{bmatrix}
\dot{x}_1(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_1(\rho(t), \rho(t)) & B_1(\rho(t), \rho(t)) \\
C_1(\rho(t), \rho(t)) & D_1(\rho(t), \rho(t))
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
y(t)
\end{bmatrix}
\]

where \(x_1 \in \mathbb{R}^m\). The dimension of controller state \(n_1\) is yet to be determined. Note that the controller gain will be scheduled by parameter \(\rho\) and its derivative \(\dot{\rho}\) in general.

We are now ready to state the main result for the LPV output-feedback control synthesis problem. The detailed control law can be found in [21].

**Theorem 1.** Given compact sets \(\mathcal{P}\) and \(\mathcal{V}\), a performance level \(\gamma > 0\), there exists a LPV controller that stabilizes the open-loop LPV system in (1) and renders the closed-loop performance less than \(\gamma\) if there exist continuously differentiable matrix functions \(R, S : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}\), such that for all \(\rho \in \mathcal{P}\).

\[
\begin{bmatrix}
0 \\
R(\rho)
\end{bmatrix} \begin{bmatrix}
\nu, \mathcal{V} \end{bmatrix} \begin{bmatrix}
dR(d\rho) \\
R(\rho)
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
\nu, \mathcal{V} \\
\mathcal{S}(\rho)
\end{bmatrix} \begin{bmatrix}
dS(d\rho) \\
\mathcal{S}(\rho)
\end{bmatrix} > 0
\]

where \(\mathcal{F}_\mathcal{V} = \{\nu \in \mathcal{V} : \nu_i \leq \nu \leq \nu_i, \forall \nu_i \in \nu\}\).
by the parameter-dependent controller.

III. MISSILE MODELING AND AUTOPilot DESIGN

In this section, we present the design procedure of LPV controllers as missile pitch-axis autopilots. The design is largely simplified due to the systematic approach adopted, and the stability of the closed-loop system is always guaranteed as long as the parameters in the pre-specified range.

3.1 Missile model and performance objective

The missile dynamics are taken from [11], in which the variables $\eta(t)$ and $q(t)$ are measured output and available for feedback use. The input to the plant is commanded tail deflection $\delta(t)$.

$$\alpha(t) = K_a M(t) C_n[\alpha(t), \delta(t), M(t)] \cos(\alpha(t)) + q(t)$$

(6)

$$\dot{q}(t) = K_a M^2(t) C_m[\alpha(t), \delta(t), M(t)]$$

(7)

where the aerodynamic coefficients $C_n$, $C_m$ are given by

$$C_n = \alpha \left[ \alpha_n | \alpha |^2 + b_n | \alpha | + c_n (2 - \frac{M}{3}) \right] + d_n \delta$$

$$C_m = \alpha \left[ \alpha_m | \alpha |^2 + b_m | \alpha | + c_m (\frac{8M}{3} - 7) \right] + d_m \delta$$

Actuator dynamics describing the tail deflection are

$$\frac{d}{dt} \left[ \delta(t) \right] = \left[ \begin{array}{c} 0 \\ -\omega^2 - 2\zeta \omega_g \end{array} \right] \left[ \begin{array}{c} \delta(t) \\ \alpha(t) \end{array} \right] + \left[ \begin{array}{c} 0 \\ \omega_g^2 \end{array} \right] \dot{\alpha}(t),$$

(8)

and the output is normal acceleration

$$\eta(t) = K_a M^2(t) C_n[\alpha(t), \delta(t), M(t)].$$

(9)

The physical meaning of different plant variables are listed below:

$\alpha(t)$ Angle of attack in degrees,
$q(t)$ Pitch rate in degrees per second,
$M(t)$ Mach number,
$\delta(t)$ Commanded tail deflection angle in degrees,
$\dot{\delta}(t)$ Actual tail deflection angle in degrees,
$\eta(t)$ Commanded normal acceleration in g’s,
$\dot{\eta}(t)$ Actual normal acceleration in g’s.

Also, the numerical values of various constants in the plant model are

$$K_a = 1.18587, \ K_q = 70.586, \ K_z = 0.6661697$$

$$a_n = 0.000103 \text{deg}^{-3}, \ b_n = -0.00945 \text{deg}^{-2}$$

$$c_n = -0.1696 \text{deg}^{-1}, \ d_n = -0.034 \text{deg}^{-1}$$

$$a_m = 0.000215 \text{deg}^{-3}, \ b_m = -0.0195 \text{deg}^{-2}$$

$$c_m = 0.051 \text{deg}^{-1}, \ d_m = -0.206 \text{deg}^{-1}$$

These coefficients are valid for the missile traveling between Mach number 2 and 4 at an altitude of 20,000 ft.

The performance goals for the closed-loop system are as follows:

- Maintain robust stability over the operating range specified by $(\alpha(t), M(t))$ such that $-25^\circ \leq \alpha(t) \leq 25^\circ$ and $2 \leq M(t) \leq 4$. Robust stability is shown by varying $\pm 25\%$ and $\pm 10\%$ of angle of attack part and tail-deflection part in coefficient $C_n$ and $C_m$ independently.
- Track step commands in $\eta(t)$ with time constant no more than 0.35 sec, maximum overshoot no greater than 10%, and steady-state error less than 1%.
- Maximum tail deflection rate for $1^g$ step command in $\eta(t)$ should not exceed 25 deg/sec.

3.2 LPV control design

The angle of attack $\alpha$ and Mach number $M$ are scheduling parameters. Note that angle of attack is not directly measurable, therefore we have to estimate it from measured output.

Let $\rho_1 := \alpha, \rho_2 := M$, then the missile pitch-axis dynamic model can be written in the quasi-LPV form like

$$\frac{d}{dt} \left[ \begin{array}{c} \alpha \\ \dot{\alpha} \\ q \\ \dot{q} \end{array} \right] = \left[ \begin{array}{c} a_{11} \rho_2 \alpha^2 + b_1 \rho_2 + c_1 (2 - \frac{M}{3}) \cos(\rho_1) \\ a_{21} \rho_2 + b_2 \rho_2 + c_2 (8 \rho_2 - 7) \\ d_1 \rho_2^2 + b_3 \rho_2 + c_3 (2 - \frac{\rho_1}{3}) \end{array} \right] \delta$$

(10)

and

$$\eta(t) = \left[ \begin{array}{c} c_{11} \rho_2 + b_4 \rho_2 + c_4 (2 - \frac{\rho_1}{3}) \\ K_i \rho_2^2 \end{array} \right] \delta$$

(11)

where

$$a_{11} = K_a \rho_1 \left[ a_n \rho_1^2 + b_n \rho_1 + c_n \left( 2 - \frac{\rho_2}{3} \right) \right] \cos(\rho_1)$$

$$a_{21} = K_a \rho_2 \left[ a_m \rho_2^2 + b_m \rho_2 + c_m \left( 8 \rho_2 - 7 \right) \right]$$

$$c_{11} = K_i \rho_1 \left[ a_n \rho_1^2 + b_n \rho_1 + c_n \left( 2 - \frac{\rho_1}{3} \right) \right]$$
and the parameter set $\mathcal{P} = [-25.0, 25.0] \times [2.0, 4.0]$. Such a formulation has advantages over conventional linearization-based gain-scheduling approach, which requires the gain-scheduled controller to have a family of linearizations that agrees with the linear controller family from fixed parameter design. The linearization property is imposed to ensure successful application of gain-scheduling control techniques to nonlinear systems. Thus the performance prediction from fixed point control design will be inherited by the resulting gain-scheduled controller by eliminating the hidden coupling terms [16]. In our approach, we convert the nonlinear missile dynamics directly into a LPV system description, therefore automatically satisfying linear equivalence requirement for nonlinear gain-scheduling control. In addition, the quasi-LPV model represents the global missile dynamics. So the stability and performance guarantee derived from LPV control synthesis also provides global stability property for the missile.

The design objective is quantified from frozen parameter design viewpoint by weighting functions, and the weighted open-loop interconnection is given in Fig. 1

where all the weighting functions are given as

$$
W_c = \frac{144(-0.05s + 1.0)}{s^2 + 2 \times 0.8 \times 12s + 144}, \quad \text{Act} = 1.0
$$

$$
W_\delta = \frac{s}{25(0.005s + 1)}, \quad W_e = \frac{0.5(s + 34.642)}{s + 0.057735},
$$

$$
W_{n1} = W_{n2} = 0.001.
$$

The weighting function $W_c$ reflects the step response we are aiming to, which have less than 0.35 sec time constant and incorporate the non-minimum phase characteristics of missile plant. The non-minimum phase phenomena can be verified by local Jacobian linearization, in which the right-half zero ranges from 19 to 46. For now, with no real theory to guide us, we simply put the slowest zero ($s = 20$) in the desired command response filter. $W_c$ has a low frequency gain 300, which corresponds to 1/3% tracking error, and high frequency gain 0.5 to limit overshoot less than 5%. At this stage, actuator dynamics is not included to keep the order of open-loop system as small as possible. The actuator model will be used for simulation.

Because of nonlinear dependence of equations (10)-(11) on parameters, generally we need to grid the two-dimensional space $(\alpha, M)$ for synthesis purpose. It is observed that the state-space entries are symmetric in terms of parameter $\rho_1$, therefore only positive $\rho_1$ values need to be considered. Specifically, gridding $(\alpha, M)$ space by $5 \times 5$ points uniformly as

$$
\mathcal{P} = \{(\rho_1, \rho_2) : \rho_1 \in \{0, 6.25, 12.5, 18.75, 25\}, \rho_2 \in \{2.0, 2.5, 3.0, 3.5, 4.0\}\}.
$$

Then the LPV synthesis problem can be solved using either a single or parameter-dependent quadratic Lyapunov function over all gridding points in two-dimensional parameter space [6, 21]. The performance thus obtained is relatively conservative and $\gamma = 3.309$ for SQLF [20]. In the parameter-dependent Lyapunov function case, three basis functions were used to parameterize the functional space, which are

$$
f_1 = 1, \quad f_2 = |\rho_1|, \quad f_3 = \rho_2
$$

The maximum variation rates for parameters $\alpha, M$ are determined as 200 deg/s and 0.5s respectively. Then the optimal achievable performance using PDLF with different combination of basis functions is provided in Table 1. It can be seen that incorporating scheduling parameter variation information has significant effect on control performance; while restricting matrix function $S$ as constant does not cause observable performance degradation. However, the LPV controller derived from such a solution will eliminate its functional dependency on parameter derivative, which is advantageous from controller implementation point of view [2].

A discretization scheme was proposed for LPV controller implementation in [1] and the controller gains will be updated on-line to conduct real-time actuation. This imposes significant demand on CPU power and requires dedicated software code. Considering the controller gains can be calculated at any desired gridding density off-line, we propose a simple LPV control implementation technique through table lookup. In this way, all of the major computation will be done off-line, and gain-scheduling control becomes a simple matter of local interpolation. It greatly reduces the requirement on computer hardware to achieve real-time performance.

### Table 1. Induced $\ell_2$ performance using PDLF method.

<table>
<thead>
<tr>
<th>Functional form</th>
<th>Performance level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R, S$ parameter-dependent</td>
<td>1.319</td>
</tr>
<tr>
<td>$S$ fixed</td>
<td>1.319</td>
</tr>
<tr>
<td>$R$ fixed</td>
<td>3.307</td>
</tr>
</tbody>
</table>
3.3 Simulation results

For simulation purposes, Mach number is generated by

\[ M(t) = \frac{1}{\eta} \left[ \left\lfloor \eta(t) \right\rfloor \sin(\alpha(t)) + A_M \int \cos(\alpha(t)) \right] \]

\[ M(0) = 3.0 \]

to provide a reasonably realistic speed profile. Recall for LPV systems, the parameter is assumed to be measurable in real-time. But for this missile plant, angle of attack \( \alpha \) is not available. So we must estimate \( \alpha \) in terms of \( \eta, \delta, M \) from output equation

\[ \eta = f(\alpha, \delta, M) = K_M C_n [\alpha, \delta, M] \]

Normalizing variables as \( \eta_N := \eta/60.0, \delta_N := (\delta - 10.0)/25.0 \) and \( M_N := M - 3.0 \). The polynomial approximation of inverse function \( \alpha = f^{-1}(\eta, \delta, M) \) is given by

\[ \alpha = -1.396 - 0.33421 M_N - 3.7653 \delta_N - 0.91681 \delta_N M_N + \eta_N ( -46.03 + 21.26 M_N - 8.8362 M_N^2 - 0.33564 \delta_N + 0.385 \delta_N M_N + 0.32892 \delta_N M_N^2 ) \]

\[ + \eta_N^3 (61.367 - 69.756 M_N + 30.44 M_N^2 + 3.9589 \delta_N - 15.668 \delta_N M_N + 11.498 \delta_N M_N^2 + 18.807 \delta_N M_N^3 ) \]

\[ (12) \]

The relative approximation error ranges from 0.002% to 38%, and the mean of errors is about 8%. So the curve fitting is acceptable but not perfect. The \( \alpha \) estimator equation (12) is kept the same throughout simulations even in the cases where we perturb the missile aerodynamic coefficients \( C_n \) and \( C_m \).

Figure 2(a) shows the tracking performance of \( \eta(t) \) to a series of step commanded acceleration. In Figs. 2(b) and 2(c), the corresponding angle of attack \( \alpha(t) \) and tail-deflection rate \( \delta(t) \) are plotted. From the simulation results, we see clearly that the performance goals are satisfied.

Fig. 2. Missile dynamic response to a sequence of step acceleration commands.
The LPV controlled performance is comparable to a conventional gain-scheduled control design [11] as shown in Fig. 3. However, the LPV control theory provides a systematic gain-scheduling control design framework that overcomes deficiency of the conventional gain-scheduling approaches, such as stability, controller implementation issues.

To demonstrate comparative performance, the robustness property of our LPV controller is verified by perturbing aerodynamic coefficients $C_m$ and $C_n$ independently. For $C_m$, we simultaneously change constants $a_m$, $b_m$, $c_m$ from their nominal values by factors of 1.25 and 0.75, $d_m$ by a factor of 1.25 and 0.75 from its nominal value. Similar perturbations to $C_n$ are carried out, except the variations are limited to $\pm 10\%$. Totally, 16 plots result from the combination of all of these variations. From Figs. 4(a)-(c), we observe reasonable performance degradation of missile under perturbation. Moreover, all cases examined presents stable dynamic behavior.

VI. CONCLUSIONS

In this paper, we presented a detailed missile autopilot design example using systematic gain-scheduling approach. Through the design process of missile autopilot, we demonstrate the usefulness of LPV control theory for real engineering problems, especially, the high performance flight control problem. As an alternative to Jacobian linearization, it was shown that the nonlinear missile dynamics can be transformed into a quasi-LPV system.
Such a formulation satisfies the linearization property imposed by nonlinear gain-scheduling control and provides global stability and performance for missile problem. By incorporating scheduling parameter variation information, we achieved large improvement in controlled system performance. The LPV control approach simplifies the design procedure greatly compared with conventional gain-scheduling control designs [11] and still processes adequate control performance.

REFERENCES