EXPONENTIAL STABILIZATION OF BILINEAR SYSTEMS WITH OPEN-LOOP UNSTABLE DYNAMICS

Yean-Ren Hwang

ABSTRACT

The quadratic control (or its modification) can stabilize a homogeneous bilinear system asymptotically (or exponentially) if its open-loop dynamics are neutrally stable. When the open-loop dynamics are unstable, the quadratic control can only bring the system state to a bounded region around the origin. In this paper, we propose a new nonlinear control which can drive the state of an open-loop unstable system to the origin exponentially. This control algorithm can be easily extended to time varying systems with slight modification.

KeyWord: Bilinear systems, exponentially stable, nonlinear control.

I. INTRODUCTION

This paper considers the control of a bilinear system:

\[
x(t) = Ax(t) + Nx(t)u(t),
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}\) is the control input, and \(A \in \mathbb{R}^{n \times n}\) and \(N \in \mathbb{R}^{n \times n}\) are constant matrices.

Slemrod [1], Jurdjevic [2] and Quinn [3] showed that when the open-loop system is neutrally stable (i.e., all the eigenvalues of the \(A\)-matrix are on the imaginary axis) the closed-loop system can be made asymptotically stable by the quadratic control:

\[
u = -\alpha x^T(t)N^TQx(t),
\]

where \(Q\) is a symmetric positive definite matrix and \(\alpha\) is a positive number. Outbib and Sallet [4] showed that the algorithm developed by Jurdjevic and Quinn can also be applied to a broad class of nonlinear systems, such as the angular velocity of rigid bodies. Furthermore, Quinn [3] showed that the state convergent rate of the closed-loop system is approximately

\[
\|x(t)\| \leq \frac{1}{\sqrt{t}}.
\]

Chen [5] developed an exponentially stable control algorithm for the system (1) to improve the closed-loop dynamics. The achieved exponential stability results in faster time response of the system state as well as better robustness of the controlled system (Chen [5], Callier and Desoer [6]). However, to apply this algorithm, the open-loop dynamics still must be neutrally stable.

Real systems in general are not neutrally stable. Hence, the control algorithms developed by previous researchers have limited applications. If the open-loop system is stable (i.e., all the eigenvalues of the \(A\)-matrix are in the left half plane), no control at all (i.e. \(u(t) \equiv 0\)) is needed to stabilize the system exponentially. Because the quadratic control (2) is proportional to \(|x(t)|^2\), the second term on the right hand side of (1) drops much faster than the first term when the state approaches the origin. Therefore, the closed-loop system is not asymptotically stable if the \(A\)-matrix has eigenvalues in the right half plane. For this type of system, Gutman [7] provided sufficient conditions for the closed-loop system (1) with the quadratic control (2) to approach a small region around the origin. However, the closed-loop system with Gutman’s controller is not asymptotically stable.

The objective of this research was to design a new exponentially stable control for the bilinear system (1) with unstable open-loop dynamics. This exponentially stable control can be easily extended to time-varying systems with little modification. The remainder of this paper is organized as follows: in Section 2, we propose a nonlinear control for the system (1); in Section 3, we analyze the stability of this controller and prove the sufficient conditions for the closed-loop system to be exponentially stable; typical examples are also given in Section 3 and we conclude in Section 4.
II. NONLINEAR STABILIZING CONTROL

The proposed nonlinear control is defined as follows:

\[ u(t) = \begin{cases} 
- \beta \frac{x^T(t)(N^TP + PN)x(t)}{x^T(t)x(t)} & \text{if } x(t) \neq 0 \\
0 & \text{if } x(t) = 0, 
\end{cases} \]  

(3)

where \( \beta \) is a positive control gain. Notice that the control (3) is uniformly bounded for all \( x(t) \):

\[ |u(t)| \leq u_{\text{max}} = \beta \sigma_{NP} \quad \forall t > 0, \]  

(4)

where \( \sigma_{NP} = \|N^TP + PN\| \) is the norm of the symmetric matrix \( N^TP + PN \). Also, the control is the same for those states in the same direction, and the system (1) is a homogeneous system of degree one:

\[ x'(t) = Ax(t) - \beta \frac{x^T(t)(N^TP + PN)x(t)}{x^T(t)x(t)}Nx(t). \]  

(5)

III. STABILITY ANALYSIS

For a positive definite matrix \( P \) and a selected positive number \( \epsilon_N \), we define two sets for the states in \( \mathbb{R}^n \) space as follows:

\[ S_N(P, \epsilon_N) = \{ x \mid x \neq 0 \text{ and } \|x(N^TP + PN)x\| \leq \epsilon_N \| x \| \}, \]

\[ S_A(P) = \{ x \mid x \neq 0 \text{ and } x^T(A^TP + PA)x < 0 \}. \]

These two sets represent two sectors in the state space. Furthermore, we denote \( S_N^c(P, \epsilon_N) \) as the complement set of \( S_N(P, \epsilon_N) \).

**Theorem 1.** If there exists a positive definite matrix \( P \) and a positive number \( \epsilon_N \) such that \( S_N(P, \epsilon_N) \) is a subset of \( S_A(P) \), then the closed-loop system (1) with the control (3) is made exponentially stable by selecting

\[ \beta > \frac{\sigma_{AP}}{\epsilon_N^2}, \]  

(6)

where \( \sigma_{AP} = \|A^TP + PA\| \).

**Proof.** Define a positive definite function:

\[ V(t) = x^T(t)Px(t). \]

Applying the derivative to (5), we have

\[ \dot{V}(t) = x^T(t)(A^TP + PA)x(t) - \beta \frac{x^T(t)(N^TP + PN)x(t)^2}{x^T(t)x(t)}. \]  

(8)

This implies that

\[ \|x(t)\|^2 \geq \frac{V(t)}{\|P\|}. \]  

(7)

The derivative of \( V(t) \) is

\[ \dot{V}(t) = x^T(t)(A^TP + PA)x(t) - \beta \frac{(x^T(t)(N^TP + PN)x(t))^2}{x^T(t)x(t)}. \]

Case 1. \( x(t) \in S_N^c(P, \epsilon_N) \).

Because \( x^T(t)(A^TP + PA)x(t) \leq \sigma_{AP} \| x(t) \|^2 \), we have that

\[ \dot{V}(t) \leq \sigma_{AP} \| x(t) \|^2 - \beta \frac{(x^T(t)(N^TP + PN)x(t))^2}{x^T(t)x(t)}. \]

Choose \( \beta = (1 + \epsilon_\beta) \frac{\sigma_{AP}}{\epsilon_N^2} \) and \( \epsilon_\beta > 0 \); then,

\[ \dot{V}(t) \leq \left( \sigma_{AP} - (1 + \epsilon_\beta) \frac{\sigma_{AP}}{\epsilon_N^2} \right) \| x(t) \|^2 = - \epsilon_\beta \sigma_{AP} \| x(t) \|^2. \]

From the inequality (7), we have

\[ \dot{V}(t) \leq - \frac{\epsilon_\beta \sigma_{AP}}{\|P\|} V(t). \]

Case 2. \( x(t) \in S_N(P, \epsilon_N) \).

The second term of Equation (8) is always less or equal to zero. Therefore,

\[ \dot{V}(t) \leq x^T(t)(A^TP + PA)x(t). \]

Consider a function:

\[ f(y) = y^T(A^TP + PA)y, \]

where \( y \in \Omega \), the unit sphere of \( \mathbb{R}^n \) space. Define the set \( \Omega^c \):

\[ \Omega^c = \{ y \mid y = \frac{x}{\|x\|} \text{ and } x \in S_N(P, \epsilon_N) \}. \]

For the state \( x \in S_N(P, \epsilon_N) \subset S_A(P) \), the corresponding \( y = \frac{x}{\|x\|} \in \Omega^c \). Since \( f \) is a continuous function and \( \Omega^c \) is a compact set (see Lemma 1 in the Appendix),

\[ \max_{y \in \Omega^c} \{ f(y) \} = -\sigma_{in} \| 8 \|, \]

where \( \sigma_{in} \) is a positive number. Hence,
Y.-R. Hwang: Exponential Stabilization of Bilinear Systems with Open-Loop Unstable Dynamics

\[ f(\frac{X}{\|X\|}) = \frac{X^T}{\|X\|} (A^T P + PA) \frac{X}{\|X\|} \leq -\sigma_\alpha. \]

This implies that when \( x \in S_\alpha(P, \epsilon_\alpha) \subset S_\alpha(P), \)
\[ x^T(A^T P + PA)x \leq -\sigma_\alpha \|x\|^2. \]

Therefore,
\[ \dot{V}(t) \leq -\sigma_\alpha \|x(t)\|^2 \leq -\frac{\sigma_\alpha}{\|P\|} V(t). \]

Based on the discussion of Case 1 and 2, we can conclude that
\[ V(t) \leq e^{-\gamma t} V(0) \] and \( \gamma = \min \left( \frac{\epsilon_\beta \sigma_\alpha}{\|P\|}, \frac{\sigma_\alpha}{\|P\|} \right). \)

This proves that the closed-loop system is exponentially stable. ■

Since the control input depends only on the state’s direction, analysis of the states on the unit sphere will be enough to verify the statement in Theorem 1. Notice that there may be more than one \( P \) and \( \epsilon_\alpha \) to make the statement in Theorem 1 true. Also, notice that the bound of the control input \( u_{\text{max}} \) is proportional to \( 1/\epsilon_\alpha^2 \). Therefore, a smaller \( \epsilon_\alpha \) will induce a larger control input.

**Example 1.** Consider a two-dimensional system:
\[ \dot{x}(t) = \begin{pmatrix} 2 & 1 \\ -2 & -4 \end{pmatrix} x(t) + \begin{pmatrix} -1.5 & -1 \\ 1 & 1 \end{pmatrix} x(t) u(t). \] (9)

We choose \( P \) as the two-dimensional identity matrix. The state on the unit circle can be written as \( x = (\cos \theta, \sin \theta) \), and \( \theta \) varies from 0 to \( 2\pi \). In Fig. 1, the solid and dashed lines represent the values of \( x^T(A^T P + PA)x \) and \( x^T(N^T P + PN)x \), respectively, for all the states on the unit circle. Notice that the values of \( x^T(A^T P + PA)x \) are negative for those \( x \) with \( x^T(N^T P + PN)x \) close to zero. When the value of \( x^T(A^T P + PA)x \) is equal to zero, the values of \( x^T(N^T P + PN)x \) are about \(-1.6\) and \(-0.95\). Therefore, when \( \epsilon_\alpha \) is chosen as 0.9, \( S_\alpha(P, \epsilon_\alpha) \) will be a subset of \( S_\alpha(P) \), and \( \beta \) will become 9.9. Figure 2 shows that the value of \( V \) is always negative when \( \theta \) varies from 0 to \( 2\pi \). During the simulation, the initial condition \( x_0 \) was set as \([-1, 1]^T\). The state responses and the control input are shown as solid lines in Figs. 3 and 4, respectively. As shown in Fig. 5, \( V \) decreases to zero as time increases.

Notice that the statement in Theorem 1 is a sufficient condition for the closed-loop system to be exponentially stable. Simulation experience indicates that \( \beta \) can be chosen smaller than the value obtained from Equation (6).
It is found that $u_{\text{max}}$ reduces to 3 when $\beta$ is chosen to be 1, while $V$ is still negative for all the states on the unit circle. For the same initial condition, the results of simulation for $\beta = 1$ are shown as dashed lines in Figs. 3, 4 and 5. Notice that when $\beta = 1$, the control input is much smaller than that in the previous case at the cost of a slower decreasing rate of $V$.

**Corollary 1.** If all the eigenvalues of $N$ are in either the left half plane or the right half plane, then there always exists $P$ and $\epsilon_N$ such that $\mathcal{S}_N(P, \epsilon_N) \subseteq \mathcal{S}_N(P)$ is always true. Therefore, the system (1) can always be stabilized by the control (3).

**Proof.** First, we consider the case where all the eigenvalues of $N$ are in the left half plane. For the same initial condition, the results of simulation for $\beta = 1$ are shown as dashed lines in Figs. 3, 4 and 5. Notice that when $\beta = 1$, the control input is much smaller than that in the previous case at the cost of a slower decreasing rate of $V$.

**Example 2.** Consider a system:

$$\begin{align*}
\dot{x}(t) &= 
\begin{pmatrix}
0 & 2 & -1 \\
2 & 1 & 4 \\
1 & 1 & 3
\end{pmatrix}
x(t) + 
\begin{pmatrix}
6 & 2 & 2 \\
1 & 2 & 1 \\
-5 & -2 & -1
\end{pmatrix}
x(t)u(t).
\end{align*}
$$

In this example, the eigenvalues of the $N$-matrix are 1, 2 and 4. Hence,

$$\mathcal{N} = \begin{pmatrix}
-6 & -2 & -2 \\
-1 & -2 & -1 \\
5 & 2 & 1
\end{pmatrix}.$$  

Choose $Q_N$ to be

$$Q_N = \begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  

Then,

$$P = \begin{pmatrix}
0.700 & 0.050 & 0.650 \\
0.050 & 0.375 & 0.175 \\
0.650 & 0.175 & 0.975
\end{pmatrix}.$$  

In this case, $\sigma_{AP}$ is equal to 10.39. If we select $\beta = 10.5$, then the control input (3) will stabilize the above system exponentially. Figure 6 shows the states’ response for the initial condition $(1, 1, -1)^T$ and Fig. 7 shows the corresponding control input.

**Theorem 1** can be easily extended to bilinear time varying systems with a small modification. Consider a time varying bilinear system:

$$\dot{x}(t) = A(t)x(t) + N(t)x(t)u(t).$$

with bounded $A(t)$ and $N(t)$. We can define two time varying sets for a positive definite matrix $P$ and a selected positive number $\epsilon_N$ as follows:

$$\mathcal{S}_N(P, \epsilon_N, t) = \left\{ x \mid x \neq 0 \text{ and } x^T(N(t)P + PN(t))x \leq \epsilon_N \left\{ x \right\} \right\}.$$
Theorem 2. If there exists a positive definite matrix $P$ and a positive function $\epsilon_N$ such that $S_N(P, \epsilon_N, t)$ is a subset of $S_A(P, t)$, then the closed-loop system (11) with the control

$$u(t) = \begin{cases} \beta(t) x(T)/(N(T)P + PN(t)) & \text{if } x(t) \neq 0 \\ 0 & \text{if } x(t) = 0 \end{cases}$$

(12)
can made exponentially stable by selecting

$$\beta(t) > \frac{\sigma_{AP}(t)}{\epsilon_N}$$

(13)

where $\sigma_{AP}(t) = |A^T(t)P + PA(t)|$. 

The proof of this theorem can easily be obtained by following the same reasoning in Theorem 1 and, hence, is omitted.

Example 3. Consider a two-dimensional system (11) with

$$A(t) = \begin{pmatrix} -9(2 + \cos 2\pi t) & 2(1 + \sin 2\pi t) \\ 2 & 6(2 + \cos 2\pi t) \end{pmatrix},$$

$$N(t) = \begin{pmatrix} 6(2 + \sin 2\pi t) & 0 \\ 0 & -9(3 + 2\sin 2\pi t) \end{pmatrix}.$$ 

In this example, $P$ is chosen as the two-dimensional identity matrix. Since the periods of $A(t)$ and $N(t)$ are both 1, the statement in Theorem 2 needs to be verified from 0 to 1. At each moment, the values of $x^T(A^T(t)P + PA(t))x$ and $x^T(N^T(t)P + PN(t))x$ need to be evaluated to find a suitable $\epsilon_N$. Figure 8 shows the maximum allowable values of $\epsilon_N$ that make the statement in Theorem 2 valid for $t \in [0, 1]$. Therefore, we can choose $\epsilon_N$ equal to 1.5 to make $S_N(P, \epsilon_N, t)$ a subset of $S_A(P, t)$ for all $t$. Figure 9 shows the state responses for the initial condition $x_0 = [-1, 1]^T$ when $\epsilon_N$ equals 1.5 and $\beta$ is selected as $(\sigma_{AP}(t)/\epsilon_N^2) + 0.1$. The corresponding $V$ shown in Fig. 10 is exponentially convergent to zero.

IV. CONCLUSION

In this paper, a nonlinear controller has been proposed for open-loop unstable bilinear systems. The closed-loop system is exponentially stable if the assumption in Theorem 1 is satisfied. This control algorithm can be extended to time-varying system as stated in Theorem 2. Rigorous stability analysis along with simulations based on three examples have been presented in this paper.
Lemma 1. The set $\Omega^-$ is compact.

Proof. Note that $\Omega^-$ is a subset of $\Omega$, which is a compact set, When $x \in S_N(P, \epsilon_N)$,

$$\left| x^T(N^TP + PN)x \right| \leq \epsilon_N \left| x \right|^2$$

or

$$\left| \frac{x^T(N^TP + PN)x}{ \left| x \right|^2 } \right| \leq \epsilon_N.$$

This implies that

$$\left| y^T(N^TP + PN)y \right| \leq \epsilon_N,$$

and $\Omega^-$ becomes

$$\Omega^- = \{ y \mid y \in \Omega \text{ and } - \epsilon_N \leq y^T(N^TP + PN)y \leq \epsilon_N \}. $$

Therefore, $\Omega^-$ is closed and bounded.

REFERENCES