DYNAMICAL OUTPUT FEEDBACK STABILIZATION OF MIMO BILINEAR SYSTEMS WITH UNDAMPED NATURAL RESPONSE

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ABSTRACT

This paper considers the globally asymptotic stabilization problem of multi-input multi-output bilinear systems with undamped natural response. Under the conditions for asymptotic stabilization by static state feedback control and system detectability, two output dynamic feedback controllers with saturation bounded control are constructed. The global asymptotic stability of the closed-loop system is verified by Lyapunov stability theory and LaSalle’s Lemma. An example is given to demonstrate the obtained results.

KeyWords: Bilinear systems, detectability, Lyapunov function, asymptotic stability, state feedback, output feedback.

I. INTRODUCTION

Bilinear systems as a special family of nonlinear systems is an area of considerable interests in both theory and applications, see for example [1-8,10-13]. And there exist many results on stabilization of bilinear systems in the literature ([1,4,3,5,8,13,14]).

The bilinear systems stabilization problem was studied in [4,3,8,13] for single-input systems and only the state feedback control is applied. In [10], affine nonlinear systems, which is a more general class of nonlinear systems, were considered and the controller design involved solution of a set of nonlinear algebraic equations. Recent results on asymptotic stabilization of bilinear systems with undamped natural response were presented [5,13], where the eigenvalues of the linear state matrix have zero real part. In [13], the state feedback globally asymptotic stabilization for the single-input bilinear systems is discussed, where the system state matrix $A \in \mathbb{R}^{2n \times 2n}$ has $2n$ pure imaginary eigenvalues. In [5], the stabilization of a class of single-input single-output (SISO) bilinear systems with dissipative drift, with the state matrix $A$ satisfying $x^T A x \leq 0$, was discussed. The bilinear systems considered in [5,13] are of very special structure and the design is based on the state observer and the separation principle.

The main result of this paper is on dynamical output feedback stabilization of multi-input multi-output (MIMO) bilinear systems with undamped natural response. To obtain this result, this paper firstly extends the assumption in [13] to introduce two different sufficient conditions. Each sufficient condition can be used to the design of global stabilization controllers with state feedback. Based on this and motivated by the Luenberger full-order and reduced-order observers for linear systems, the design of full-order and reduced order dynamic output feedback controllers for globally asymptotic stabilization of the bilinear systems is presented. Taking into account the practical constraints on the control actuator, the proposed bilinear controllers are subject to input saturation bounds. The global asymptotic stability of the closed-loop system is established by using Lyapunov stability theory, LaSalle’s Lemma. Such a result has not been studied in the existing literature.

This paper is organized as follows. Section 2 presents some assumptions and preliminary results on state feedback asymptotic stabilization of nonlinear systems. Two sufficient conditions for state feedback asymptotic stabilization of bilinear systems are further derived to prepare the design of dynamic output feedback controllers. Section 3 presents the main results of the paper on the full order and reduced order dynamic output controllers. A numerical example for demonstrating the design
of the proposed output feedback controller is given in Section 4. Section 5 is the conclusion of the paper.

II. STATE FEEDBACK STABILIZATION OF MIMO BILINEAR SYSTEMS

This section presents preliminary results of the state feedback stabilization of MIMO bilinear systems, which are useful for the main results of the paper on dynamic output feedback stabilization of bilinear systems. Consider an MIMO nth order bilinear system as follows.

\[ \dot{x} = Ax + \sum_{i=1}^{n} B_i x u_i, \]
\[ y = C x, \]  
(1)

where \( x \in \mathbb{R}^n, u = [u_1, u_2, \ldots, u_n] \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \) are the system state, control input and output, respectively, \( A, B_i \in \mathbb{R}^{m \times m} \) for \( 1 \leq i \leq m \) and \( C \in \mathbb{R}^{n \times m} \) are constant matrices, and \( B(x) = [B_1 x, B_2 x, \ldots, B_m x] \).

Let \( \lambda_i \) and \( q_i, 1 \leq k \leq n, \) be the \( n \) eigenvalues and \( n \) eigenvectors of \( A, \) respectively, where each \( q_i \) is associated with \( \lambda_i. \) In this paper we consider that the bilinear system (1) has undamped natural response under the zero input, \( u = 0. \) This is specified in terms of the eigenvalues of the state matrix \( A \) as follows.

**Assumption 2.1** The state matrix \( A \in \mathbb{R}^{n \times n} \) has \( n \) distinct imaginary eigenvalues, i.e., \( \text{Re}(\lambda_i) = 0 \) for \( 1 \leq k \leq n \) and \( \lambda_i \neq \lambda_j \) for \( i \neq j. \)

We note that Assumption 2.1 allows that there is a single eigenvalue of \( A \) at \( \lambda = 0. \) Define a set of matrices \( \Sigma \) such that

\[ \Sigma = \{ P : P \in \mathbb{R}^{n \times n}, \; P^* = P, \; P > 0, \; A^T P + PA = 0 \}. \]

Under Assumption 2.1 it is easy to verify that the matrix set \( \Sigma \) is not empty, and some \( P \in \Sigma \) can be easily found. Without loss of generality, suppose that \( \lambda = 0 \) is an eigenvalue of \( A, \) then there exists a positive integer \( l \) such that \( 2l + 1 = n. \) For convenience, let \( \lambda_l = 0, \lambda_k = j \omega_k, \lambda_k = -j \omega_k, (2 \leq k \leq l + 1), \) be the \( n \) eigenvalues of \( A \) and their eigenvector matrix be \( M = [q_1, q_2, \ldots, q_{2l}, q_{2l+1}, q_{2l+2}, \ldots, q_{2n}]. \) Let

\[ P = (TT^*)^{-1}, \]  
(2)

where \( T = (q_1, \text{Re}(q_2), \ldots, \text{Re}(q_{2l}), \text{Im}(q_{2l+1}), \ldots, \text{Im}(q_{2n})). \) In this case, \( AT = TA, \) where \( A \) is a real constant skew symmetry matrix. Then,

\[ PA + A^T P = P(A(TT^T + TT^T)A^T)P, \]
\[ = P(TAT^T + T^T A^T)P = 0, \]  
(3) which implies that \( P \in \Sigma. \)

We now state the well-known LaSalle’s Lemma, (see e.g. [15]) in the following.

**Lemma 2.1** (LaSalle’s Lemma) Consider an \( nth \) order nonlinear system

\[ \dot{x} = f(x) \]  
(4)

where \( f(x) : \mathbb{R}^n \to \mathbb{R}^n \) is smooth vector field in \( \mathbb{R}^n. \) If there exists a Lyapunov function \( V(x) \) such that the nonlinear system (4) satisfies \( \dot{V}(x) \leq 0, \) then any trajectory of system (4) tends to the largest positive invariant set included in set \( M = \{ x \in \mathbb{R}^n : V(x) = 0 \} \) when \( t \to +\infty. \)

For system (1) \( \{ Ax, B_i x, i \in m \} \) are \( m + 1 \) analytical vector fields. Define a distribution

\[ \Omega(x) := \text{span} \{ \text{ad}^i(A, B_i)x, k = 0, 1, 2, \ldots, n; \]  
\[ i = 1, 2, \ldots, m \}, \]  
(5)

where \( \text{ad}^i(A, B_i), k = 0, 1, 2, \ldots, n; \) \( i = 1, 2, \ldots, m, \) are defined as follows:

\[ \text{ad}^i(A, B_i) = B_i, \]
\[ \text{ad}^i(A, B_i) = AB_i - B_i A, \]  
(6)

\[ \text{ad}^{i+1}(A, B_i) = A \text{ad}^i(A, B_i) - \text{ad}^i(A, B_i) A, \]
\[ k = 0, 1, \ldots, n, \]  
\[ i = 1, 2, \ldots, m. \]

**Assumption 2.2** For systems (1) \( \dim \Omega(x) = n \) for any \( x \neq 0. \)

When \( \dim \Omega(x) = n \) for any \( x \neq 0, \) then the bilinear control system (1) satisfies the Lie-rank condition of controllability. Assumption 2.2 implies that the bilinear systems (1) is controllable. We construct a static state feedback control for the bilinear system (1) by

\[ u = -c B^T(x)Px, \]  
(7)

where \( c > 0 \) is a real constant and \( P \in \Sigma. \) The matrix \( P \) can be defined, for example, by (2).

**Theorem 2.1** Under Assumptions 2.1 and 2.2 the static state feedback control in the form of (7) globally asymptotically stabilizes the bilinear system (1).

**Proof.** Let \( V(x) = x^T P x \) be a Lyapunov function candidate. We have

\[ \dot{x} = Ax - c B(x) B^T(x) P x, \]  
(8)

\[ \dot{V}_1 = x^T (A^T P + PA)x - 2c x^T P B(x) B^T(x) P x. \]  
(9)

Due to \( P \in \Sigma \) we have \( \dot{V}_1 \leq 0. \) It is apparent from (9)
that any solution \( x(t) \) of (8), which satisfies that
\[ \dot{x} = Ax, \quad x^T P B(x) = 0. \] (10)

We now aim to find the solution of the set of equations (10). Under Assumption 2.1 it follows from (10) that for any initial state \( x(0) = x_0, x(t) = e^{At}x_0 \). Thus, we have
\[ x_k^T e^{At} P B_k e^{At} x_0 = 0, \quad i = 1, 2, \ldots, m. \] (11)

Taking consecutively derivatives of (11) at \( t = 0 \) and noticing that \( P \equiv -A^T P \), we have
\[ x_k^T P A \text{ad}^i(A, B)x_k = x_k^T P A \text{ad}^{i-1}(A, B)x_k, \ldots \]
\[ = x_k^T P \text{ad}^0(A, B)x_k = 0, \] (12)

which implies that vector \( P x_k \) is orthogonal to the tangent space \( \mathbb{Q}(x_k) \). Since \( \dim \mathbb{Q}(x_k) = n \), then \( P x_k = 0 \). Furthermore, \( P > 0 \) implies that \( x_0 = 0 \). Thus, \( x(t) = 0 \) is the unique solution for \( \dot{x} = Ax \) and \( \dot{y}(t) = 0 \). With this result, the globally asymptotic stability of the closed-loop bilinear system (1) is established by applying Lemma 2.1.

We now give a second sufficient condition for asymptotic stabilization of bilinear system (1), which is a generalization of the main result of [13] into multi-input case of system (1). This condition depends on properly choosing a definite positive matrix \( P \in \Sigma \).

We make new assumption on the bilinear system (1) as follows.

**Assumption 2.3** There exists a \( P \in \Sigma \), such that the input matrix \( B_k \) satisfies
\[ q_k^T P B_k q_k \neq 0 \] (13)
for each eigenvector \( q_k, 1 \leq k \leq n \).

**Theorem 2.2** If Assumptions 2.1 and 2.3 with \( P \in \Sigma \) are satisfied, the static state feedback control in (7) globally asymptotically stabilizes the bilinear system (1).

**Proof.** Let \( V(x) = x^T P x \) be a Lyapunov function candidate. Under feedback control (7) we obtain (8)-(9) and \( A^T P + P A = 0 \), as well as \( \dot{V} \leq 0 \). Under Assumption 2.1 and for any initial state \( x(0) \), the solution for \( \dot{x} = Ax \) can be written as
\[ x(t) = e^{At}x(0) = \sum_{k=1}^{n} a_k e^{\lambda_k t} q_k, \] (14)
where \( a_k \in \mathbb{C}, 1 \leq k \leq n \), are complex constants determined by \( x(0) \). Without loss of generality we assume that the \( n \) distinct eigenvalues of \( A \) with zero real part satisfies
\[ |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n| \] (15)

Substituting the solution (14) into \( x^T P B(x) = 0 \) yields
\[ \left( \sum_{k=1}^{n} a_k e^{\lambda_k t} q_k \right) P B \left( \sum_{k=1}^{n} a_k e^{\lambda_k t} q_k \right) = a_k^2 \lambda_k e^{\lambda_k t} \] (16)
where
\[ \lambda(t) = a_k^2 \lambda_k e^{\lambda_k t} \] (17)

Since \( \lambda_k \) satisfies (15) and \( \lambda_k, 1 \leq k \leq n \) are distinct eigenvalues, the exponential term \( e^{\lambda_k t} \) cannot be linearly represented by the terms \( e^{\lambda_i t} e^{\lambda_j t} \) with \( i \leq n \) and \( j < n \) in the function \( \lambda(t) \) of (16). Thus \( x^T P B(x) = 0 \) implies that \( a_k^2 \lambda_k P B(q_k) = 0 \) in (16). Since \( q_k^T P B(q_k) \neq 0 \) by Assumption 2.2. We obtain \( a_k = 0 \). With \( a_k = 0 \), the solution (14) is rewritten as
\[ x(t) = e^{At}x(0) = \sum_{k=1}^{n} a_k e^{\lambda_k t} q_k. \] (17)

We continue to substitute the solution (17) into \( x^T P B(x) = 0 \) to obtain \( a_k = 0 \) by using the procedure for obtaining \( a_0 = 0 \). Further, \( a_k = 0 \) for all \( k = n - 2, n - 3, \ldots, 2, 1 \) can be obtained by consecutively using this procedure.

Thus \( x = 0 \) is the unique solution for the equations in (10). It follows that for any \( x = 0 \) is the unique solution for \( \dot{x} = Ax \) and \( \dot{y} = 0 \). With this result, the globally asymptotic stability of the closed-loop bilinear control system (1) is established by applying Lemma 2.1.

The choice of the matrix \( P \), which satisfies Assumption 2.3, may be not unique if it exists. In fact, when the system matrix \( A \) satisfies Assumption 2.1, \( A \) can be given in a canonical diagonal-block form. Thus, we may find several different \( P \), which belong to \( \Sigma \), and check if some of them satisfy Assumption 2.3. (Refer to Section 4).

Comparing the two sufficient conditions for static state feedback stabilization of systems (1), we find that Theorem 2.2 provides less conservative condition than Theorem 2.1. Actually, we can show that under Assumption 2.1 the validity of Assumption 2.2 implies that Assumption 2.3 holds for any \( P \in \Sigma \).
Assume that Assumption 2.3 does not hold under Assumption 2.2 and exists \( k \) such that \( q_i^1 B(q_i) = 0 \), that is, for all \( i (1 \leq i \leq m) \), \( q_i^1 B(q_i) = 0 \).

Let \( x = q_i \), then \( Aq_i = \lambda q_i \).

Therefore,

\[
\Omega(q_i) = \text{span}\{B(q_i), (A-\lambda I)B(q_i), \ldots, (A-\lambda I)^{i-1}B(q_i)\};
\]

\( i = 1, 2, \ldots, m \).

We now show that \( q_i^1 P(A-\lambda I)^{i-1}B(q_i) = 0 \) for \( 1 \leq i \leq m \), \( 0 \leq s \leq n \).

Then we have that

\[
(19)
\]

We further require that \( q_i^1 P \) is orthogonal to space \( \Omega(q_i) \). Since \( \dim(\Omega(q_i)) = n \), \( q_i^1 P = 0 \), this is a contradiction.

Remark 2.1

(1) By the controllability condition of bilinear control system (see [7]), Assumption 2.2, i.e. \( \dim(\Omega(x)) = n \) for any \( x \neq 0 \), implies that the bilinear systems (1) is controllable. Assumption 2.2 is, in general, not a necessary condition for controllability of system (1).

(2) Assumption 2.3 is introduced leading to the result of Theorem 2.2. The validity of Theorem 2.2 depends on a proper choice of the \( P \in \Sigma \). A conservative yet feasible choice is to let \( P = S^* S \) or \( P \) being diagonal positive-definite matrix. Then (13) in Assumption 2.3 can be replaced by the following form

\[
q_i^1 S^* S B(q_i) \neq 0,
\]

which is easy to check.

Before we end this section, we consider globally asymptotic stabilization of the bilinear system (1) using a bounded state feedback control. Let \( \text{sat}\{\cdot\} \) be a saturation function defined as

\[
\text{sat}\{u\} = \begin{cases} u, & \text{for } |u| \leq 1 \\ -1, & \text{for } u > 1 \\ 1, & \text{for } u < -1. \end{cases}
\]

This definition can be extended to an \( m \)-dimensional vector \( u = [u_1, u_2, \ldots, u_m]^T \) as

\[
\text{sat}\{u\} = (\text{sat}\{u_1\}, \text{sat}\{u_2\}, \ldots, \text{sat}\{u_m\})^T.
\]

We now consider the following saturation bounded state feedback control to the bilinear system (1).

\[
u = -c_s \text{sat}\{B^T(x)Pw\},
\]

where \( c_s \) is a real constant with \( c_s > 0 \). It is straightforward to extend the result of Theorem 2.1 to the closed-loop bilinear system (1) and (19) to obtain the following stability result.

**Corollary 2.1** If Assumptions 2.1 and 2.2 (or 2.3) are satisfied, the saturation bounded state feedback control (19) with the \( P \in \Sigma \) globally asymptotically stabilizes the bilinear system (1).

**III. DYNAMIC OUTPUT FEEDBACK STABILIZATION OF MIMO BILINEAR SYSTEMS**

In this section, we present the main results on globally asymptotic stabilization of bilinear system (1) using saturation bounded dynamic output controllers. It is assumed that the system satisfies the Assumption 2.1 and 2.2 (or 2.3) with a \( P \in \Sigma \) and is globally asymptotically stabilizable by state feedback control in the form (7). We further require that the bilinear system (1) satisfies the following assumption.

**Assumption 3.1** The pair \((C, A)\) of the bilinear system (1) is detectable, and the output matrix \( C \) is of full rank.

**3.1 An nth order dynamic output feedback controller**

We firstly consider an \( n \)-th order saturation bounded dynamic output feedback controller. Under Assumption 3.1 and for a given real constant \( \gamma_c > 0 \), there exist matrices \( L \in \mathbb{R}^{m \times n} \) and \( Q \in \mathbb{R}^{n \times n} \) with \( Q > 0 \) which satisfy

\[
Q(A + LC) + (A + LC)^T Q = -\gamma_c I.
\]

Then, for the given \( \gamma_c \) and \( P \), we can choose a real constant \( c_s > 0 \) to satisfy

\[
c_s \sum_{i=1}^m \| (P + Q) B_i \|_2 < \gamma_c, \tag{21}
\]

where \( \| \cdot \|_2 \) denotes the Euclidean norm of a square matrix. We construct the following \( n \)-th order dynamic output feedback controller.

\[
w = Aw - L(y - Cw) + Q^{-1} T(w) u + B_i w u.
\]

\[
u = -c_s \text{sat}\{B^T(w) Pw\},
\]

where \( w \in \mathbb{R}^n \) is the state vector of the dynamic controller and

\[
T(w) := (PB_1 w PB_2 w \cdots PB_n w)
\]

\[
+ (B_1^T Pw B_2^T Pw \cdots B_n^T Pw).
\]
We present the following asymptotic stability result for the closed-loop bilinear system (1) and (22).

**Theorem 3.1** If Assumptions 2.1, 3.1 and 2.2 (or 2.3) are satisfied and with \( P \in \Sigma_r \) (7) asymptotically stabilizes system (1), then the saturation bounded nth order dynamic output feedback control in (22) globally asymptotically stabilizes the bilinear system (1).

**Proof.** Consider the Lyapunov function candidate

\[
V_v(x, w) = x^T P x + (x - w)^T Q(x - w),
\]

(23)

Noticing that

\[
\dot{x} - \dot{w} = (A + LC)(x - w) + B(x - w)u - Q^{-1}T(w(u)
\]

(24)

determine the derivative of \( V_v \) along the trajectory of \( x(t) \) and \( w(t) \) of the closed-loop systems (1) and (22) is

\[
\dot{V}_v = x^T (PA + A^T P)x + 2x^T PB(x)u
+ (x - w)^T \left[ (Q(A + LC) + (A + LC)^T Q)(x - w)
\right. \\
+ \left. 2(x - w)^T \left[ (QB(x - w)u - T(w(u)) \right] \right.
\]

\[
= x^T (PA + A^T P)x + 2x^T PB(x)u
+ (x - w)^T \left[ (Q(A + LC) + (A + LC)^T Q)(x - w)
\right. \\
+ \left. 2(x - w)^T \sum_{i=1}^{n} \tau_i \tau_i \B_i \B_i (x - w) - 2(x - w)^T T(w(u)) \right]
\]

It follow from

\[
2(x - w)^T PB(x - w)u = 2(x - w)^T \sum_{i=1}^{n} PB_{wi}(x - w)
\]

and

\[
2w^T PB(x - w)u + 2(x - w)^T PB(w)u
\]

\[
= 2w^T \sum_{i=1}^{n} B_i (x - w)u_i + 2(x - w)^T \sum_{i=1}^{n} B_i w_i
\]

\[
= 2 \sum_{i=1}^{n} \left[ (x - w)^T B_i P w + (x - w)^T P B_i(w)u_i \right]
\]

\[
= 2 \sum_{i=1}^{n} \left[ (x - w)^T \sum_{i=1}^{n} \B_i P w + PB_i w \right]
\]

\[
= 2(x - w)^T \sum_{i=1}^{n} \B_i P w + PB_i w \]

\[
= 2(x - w)^T \left[ (P B_w + B_1 P B_1 w + ... + B_n P w) \right]
\]

\[
= 2(x - w)^T \left[ PB_w(P B_w + B_1 P B_1 w + ... + B_n P w) \right]
\]

\[
= 2(x - w)^T T(w(u))
\]

that

\[
2x^T PB(x)u = 2(x - w)^T PB(x - w + w)u
\]

\[
+ 2(x - w)^T PB(x - w)u + 2w^T PB(x - w)u
\]

\[
= 2(x - w)^T \sum_{i=1}^{n} PB_{wi}(x - w)
\]

\[
+ 2(x - w)^T T(w(u)) + 2w^T PB(x - w)u.
\]

Substituting (26) and \( u = -c_n sat \{ B'(w)Pw \} \) into (25) yields

\[
\dot{V}_v = x^T (PA + A^T P)x - 2c_n w^T PB(w) sat \{ B'(w)Pw \}
\]

\[
+ (x - w)^T \left[ (Q(A + LC) + (A + LC)^T Q)(x - w)
\right. \\
+ \left. 2(x - w)^T PB(x - w)u - T(w(u)) \right]
\]

\[
= -[P + A^T P] \sum_{i=1}^{n} \tau_i \tau_i \B_i (x - w)
\]

\[
\leq \sum_{i=1}^{n} \tau_i \tau_i \B_i \B_i (x - w)
\]

\[
\leq 0.
\]

If \( \dot{V}_v = 0 \) is satisfied, then

\[
x(t) - w(t) = 0, \quad w'(t)PB(w(t)) = 0,
\]

and consequently

\[
u = -c_n sat \{ B'(w)Pw \} = 0, \quad B'(x)Px = 0.
\]

Thus, the closed-loop system equation of (1) becomes

\[
\dot{x} = Ax. \quad \text{This together with } B'(x)Px = B'(w)Pw = 0 \quad \text{leads to that } x(t) \quad \text{is the unique solution for the}
\]

closed-loop system. Hence, the closed-loop system is globally asymptotically stable by Lemma 2.1. ■

### 3.2 An \((n - p)\)th order output feedback dynamic controller

In order to reduce the order of the controller, the information of system state \( x \), which can be obtained directly from the measured output \( y \), is used. That is, some part of the system state \( x \) can be directly observed and used, rather than by full-order observer. Therefore, the order of output dynamic controller can be reduced. We now proceed to consider globally asymptotic stabilization of bilinear systems by a saturation bounded reduced \((n - p)\)th order dynamic output feedback controller.

Since \( C \) is a full rank matrix, there exists a constant matrix \( D \in \mathbb{R}^{n \times p} \) such that \( T = \begin{bmatrix} C \\ D \end{bmatrix} \in \mathbb{R}^{n+p} \) is a full
rank matrix. We use this to obtain the following partitioned matrices.

\[
\begin{align*}
TAT^{-1} & = \begin{pmatrix} A_1 & A_2 \\ A_1 & A_2 \end{pmatrix}, \\
TB^T & = \begin{pmatrix} B_{21} & B_{22} \\ B_{21} & B_{22} \end{pmatrix},
\end{align*}
\]

where \(A_1, B_{21} \in \mathbb{R}^{n \times p}, A_2, B_{22} \in \mathbb{R}^{m \times (n-p)}\), \(A_1, B_{21} \in \mathbb{R}^{(n-p) \times p}, A_2, B_{22} \in \mathbb{R}^{m \times (n-p)}\) and \(i = 1, 2, \ldots, m\). Noting that \((C, A)\) is detectable and \(CT^{-1} = (I_p \ 0)\), we have that for all \(\lambda \in \mathbb{C}^+\),

\[
\begin{align*}
\text{rank} \left( \lambda I_n - A \right) & = \text{rank} \left( \lambda I_n - TAT^{-1} \right) \\
& = \text{rank} \left( \lambda I_n - A \right) = n.
\end{align*}
\]

Hence, (29) implies

\[
\text{rank} \left( \lambda I_n - A \right) = n - p, \quad \forall \lambda \in \mathbb{C}^+.
\]

Thus \((A_2, A_2)\) is also detectable. Then for a given constant \(\gamma > 0\) there exist matrices \(K \in \mathbb{R}^{n \times p}\) and \(M \in \mathbb{R}^{m \times m}\) with \(M > 0\) such that

\[
\begin{align*}
M(A_2 + KA_2) + (A_2 + KA_2)M & = -p I \\
& \text{Noting that } \left( C \ D + KC \right) \begin{pmatrix} I_p \ 0 \\ K \ I_{n-p} \end{pmatrix} \begin{pmatrix} C \\ D + KC \end{pmatrix} = \text{non-singular, let}
\end{align*}
\]

\[
(E \ F) = \begin{pmatrix} C \\ D + KC \end{pmatrix},
\]

where \(E \in \mathbb{R}^{n \times p}\) and \(F \in \mathbb{R}^{m \times (n-p)}\). Using the matrices in (28), (31)-(32) and a \(P \in \Sigma\), we construct the following saturation bounded \((n - p)\)th order dynamic output feedback controller.

\[
\begin{align*}
\dot{z} & = (A_2 + KA_2)z + \sum_{i=1}^{n} (B_{22} + KB_{22})u_i \\
& = M^{-1}F^T \dot{z} u_i + [(KA_1 + A_1 - KA_2 - A_2)K \gamma] + \\
& \sum_{i=1}^{n} (KB_{21} + B_{21} - KB_{22} - B_{22})u_i y_i \\
u & = c_1 \text{sat}(B^T(2)P^2)z,
\end{align*}
\]

where \(z \in \mathbb{R}^{n \times p}\) is the state vector of the dynamic controller and \(\dot{z} = \begin{pmatrix} C \\ KC + D \end{pmatrix} y \) and \(c_1 > 0\) is a real constant chosen to satisfy

\[
c_1 \sum_{i=1}^{n} 2 \| MB_{22} + MK B_{22} + F' PB_F \| < \gamma z.
\]

We establish the globally asymptotic stability of the closed-loop bilinear system (1) and (33) in the following theorem.

**Theorem 3.2** If Assumptions 2.1, 3.1 and 2.2 (or 2.3) are satisfied and with \(P \in \Sigma\), (7) asymptotically stabilizes system (1), then the saturation bounded \((n - p)\)th order dynamic output feedback controller (33) globally asymptotically stabilizes the bilinear system (1).

**Proof.** Let

\[
x = (KC + D)y .
\]

Based on the foregoing statement we consider the following Lyapunov function candidate

\[
V(x, z) = x^T P x + (x_i - z)^T M(x_i - z),
\]

then (35) implies that

\[
\dot{x} = (K \ I) \begin{pmatrix} C \\ D \end{pmatrix} x + B(x)u
\]

\[
\begin{align*}
& = (K \ I) \begin{pmatrix} A_1 & A_2 \\ A_1 & A_2 \end{pmatrix} C x + D x \\
& \quad + (K \ I) \sum_{i=1}^{n} \begin{pmatrix} B_{21} & B_{22} \\ B_{21} & B_{22} \end{pmatrix} C x_i \\
& = (A_2 + KA_2)x + \sum_{i=1}^{n} (B_{22} + KB_{22})x_i u_i \\
& \quad + [(KA_1 + A_1 - KA_2 - A_2)K \gamma] + \\
& \quad \sum_{i=1}^{n} (KB_{21} + B_{21} - KB_{22} - B_{22})u_i y_i.
\end{align*}
\]

Thus (37) together with (33) yields

\[
\dot{x} - \dot{z} = \begin{pmatrix} A_2 + KA_2 \sum_{i=1}^{n} (B_{22} + KB_{22})u_i \\ \gamma z \end{pmatrix} (x_i - z) - M^{-1}F^T \dot{z} u.
\]

Using (38), the derivative of \(V\) along the trajectory of \(x(t)\) and \(z(t)\) of the resulting closed-loop systems is obtained as

\[
\dot{V} = x^T (P A + A^T P) x + 2x^T P B x u - 2(x_i - z)^T F^T \dot{z} u + (x_i - z)^T [M(A_2 + KA_2) + (A_2 + KA_2)^T M] (x_i - z) + \\
2M \sum_{i=1}^{n} (B_{22} + KB_{22})u_i (x_i - z).
\]

Noting

\[
x - z = \begin{pmatrix} C \\ KC + D \end{pmatrix} y = (E \ F) \begin{pmatrix} C \\ KC + D \end{pmatrix} (x - z)
\]

\[
= (E \ F) \begin{pmatrix} 0 \\ x_i - z \end{pmatrix} = F(x_i - z),
\]
we have
\[ 2x^TPB(x)u = 2(x - \hat{z})^TPB(x - \hat{z})u \]
\[ = 2(x - \hat{z})^TPB(x - \hat{z})u + 2(x - \hat{z})^TPB(\hat{z})u \]
\[ + 2\hat{z}^TPB(x - \hat{z})u + 2\hat{z}^TPB(\hat{z})u \]
\[ = 2(x - \hat{z})^TPB(x - \hat{z})u + 2\hat{z}^TPB(\hat{z})u. \] (40)

Substituting this into (39) yields
\[
\dot{V}_2 = x^T(PA + AP)x + 2\hat{z}^TPB(\hat{z})u
+ (x - \hat{z})^T[M(A_{21} + K_AK) + (A_{22} + K_AK)^T]M
+ (x - \hat{z})^T2\hat{z}^TPB(\hat{z})u
+ 2\hat{z}^TPB(\hat{z})u. \] (41)

Thus from (31), (34), \( P4 + A^TP \leq 0 \) and \( u = -c, \text{satisfy } B(\hat{z})P_2 \) we obtain
\[
\dot{V}_2 \leq -(x - \hat{z})^T(\gamma I - 2\sum_{i=1}^m(MB_{i2} + MK_{i2})\gamma + F^TPB(\hat{z})u \]
\[ + 2\hat{z}^TPB(\hat{z})u \leq 0. \]

It follows from \( |u_i| \leq c_i, 1 \leq i \leq m \) and (34) that if \( \dot{V}_2 = 0 \) holds, then \( x_2 = z \) and \( u = -c, \text{satisfy } B(\hat{z})P_2 \) must be satisfied. Thus, the closed-loop system of (1) becomes \( \dot{x}_2 = Ax, \) which can be equivalently written as
\[
\begin{bmatrix} \dot{y} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} C \\ KC + D \end{bmatrix} \begin{bmatrix} C \\ KC + D \end{bmatrix}^T \begin{bmatrix} y \\ \hat{z} \end{bmatrix}. \] (42)

Using \( \hat{z} = \begin{bmatrix} C \\ KC + D \end{bmatrix}^T(\hat{z})P_2 \) we get \( x_2 = z, \) we obtain
\[
B\begin{bmatrix} y \\ \hat{z} \end{bmatrix}P + B\begin{bmatrix} y \\ \hat{z} \end{bmatrix} = 0. \] (43)

Thus, \( \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = 0 \) is the only solution for (42) and (43) by using the result of Theorem 2.1 (or Theorem 2.2). Consequently, \( \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = 0 \) is the only solution for the close loop system equations (1), (33) and \( \dot{V}_1 = 0. \) Hence, the globally asymptotic stability of the system is established following from Lemma 2.1.

### IV. NUMERICAL EXAMPLE

Consider the following MIMO bilinear systems:
\[
\begin{align*}
x_1 &= x_2 + x_3u_1 - x_1u_1, \\
x_2 &= x_3u_1 + x_2u_2, \\
x_3 &= -2x_4 + (x_1 + 2x_2 + x_1)u_1 + (-2x_3 + x_1)u_2, \\
x_4 &= 2x_5 + (2x_1 + x_2 - x_4)u_1 + (x_1 - 2x_2 + x_1)u_2, \\
y_1 &= x_1, \\
y_2 &= x_2.
\end{align*} \] (44)

That is,
\[
A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
B_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

First we check if the condition of Theorem 2.1 is satisfied for this example.
Let \( x = (0 \ 0 \ 0 \ 0)^T, \) then it is easy to get
\[
\Omega(x) = \text{span } \{(0 \ 0 \ 0 \ 0)^T, (0 \ 0 \ 0 \ 1)^T\},
\]
that is, \( \text{dim}(\Omega(x)) = 2 < 4. \) Therefore, Theorem 2.1 is not valid for this numerical example.

Then, we check if the condition of Theorem 2.2 is valid.
The eigenvalues are \( \pm i, \pm 2i, \) and eigenvectors are
\[
q_{1,2} = (0.7071 \pm 0.7071i, 0 \ 0 \ 0 \ 0)^T, \\
q_{3,4} = (0 \ 0 \ 0 \ 0)^T \pm 0.7071i)^T.
\]

Choose \( P = I_n, \) then Assumptions 2.1 and 2.3 hold. It is easy to find the \( P \) is not uniquely chosen.
By Theorem 2.2, we can obtain the following static state feedback controller:
\[
u_i = -c_i\begin{bmatrix} x_1(x_1 + x_2 + 2x_4) + x_2(x_2 + 2x_3 + x_4) + x_3^2 - x_1^2 \end{bmatrix}, \\
\]
\[
u_2 = -c_i\begin{bmatrix} x_3(-x_1 - 2x_2 + x_4) + x_4(x_1 - x_3 - 2x_4) + x_1^2 + x_2^2 \end{bmatrix}, \\
c > 0. \] (45)
To verify the globally asymptotic stability of the closed-loop systems (44) and (45), take $c = 1$ and let, for example, $x_1(0) = 10$, $x_2(0) = 20$, $x_3(0) = 30$, and $x_4(0) = 40$, the state response of the resulting closed-loop systems can be shown in Fig. 1.

Secondly, we present the full-order dynamic output feedback design: Choose $L = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -4 & -4 & -4 \end{bmatrix}$, and $\gamma_w = 2$, then $Q = \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 1.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$. When we choose $\gamma_u = 0.08$, after some algebraic manipulations, we can obtain a full-order dynamic output feedback stabilization controller as follows:

\[
\begin{align*}
\dot{w}_1 &= -w_2 + 2(y_1 - w_1) + 4w_1 + w_2 + 2.5w_3 + 3.5w_4u_1 \\
& \quad + (4w_1 + w_2 - 3.4w_3 + 0.5w_4)u_2, \\
\dot{w}_2 &= w_1 + (w_1 + 2w_2 + 1.5w_3 + 1.5w_4)u_1 \\
& \quad + (-w_1 + 2w_2 - 1.5w_3 + 0.5w_4)u_2, \\
\dot{w}_3 &= -2w_4 + (3w_2 + 3w_3 + 2w_4)u_1 \\
& \quad + (-5w_3 + 3w_4 - 2w_5)u_2, \\
\dot{w}_4 &= 2w_4 + (4y_2 - w_4) + (7w_1 + 2w_2 - 2w_3 - 7w_4)u_1 \\
& \quad + (6w_1 - 7w_2 - 2w_3 + 7w_4)u_2, \\
& \quad + 0.08\text{sat}\left[ (w_1 + w_2 + 2w_4)w_1 + (w_1 + 2w_2 + w_4)w_2 \\
& \quad + w_3^2 - w_4^2 \right], \\
& \quad + 0.08\text{sat}\left[ (-w_1 - 2w_3 + w_4)w_1 + (w_2 - w_3 - 2w_4)w_2 \\
& \quad + w_1^2 + w_2^2 \right].
\end{align*}
\]

Fig. 2. Global asymptotical stability for the closed-loop systems (44) and (46).

Choose arbitrarily an initial condition, for example, $x_1(0) = 10$, $x_2(0) = 20$, $x_3(0) = 30$, $x_4(0) = 40$, $w_1(0) = -10$, $w_2(0) = -20$, $w_3(0) = -30$ and $w_4(0) = -40$, then the state response of the resulting closed-loop systems (44) and (46) is shown in Fig. 2.

Finally, we present a reduced-order output feedback stabilization controller. Choose $D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then $TAT^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$, $TB^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$.

If choose $K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\gamma_z = 2$, then $M = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ and let $c_z = 0.19$. Therefore, a reduced-order output feedback stabilization controller can be given as follows:

\[
\begin{align*}
\dot{z}_1 &= -z_1 + 2y_1 + (-2y_1 + 3y_2 + 3z_1 + 2z_2)u_1 \\
& \quad + (-4y_1 - 3y_2 + 3z_1 - z_2)u_2, \\
\dot{z}_2 &= -2z_2 - 4y_2 + (-4y_1 + 6y_2 + 5z_1 + 5z_2)u_1 \\
& \quad + (-6y_1 + 4y_2 - z_1 + 5z_2)u_2, \\
\dot{u}_1 &= -0.19\text{sat} \left[ 2y_1^2 + z_1^2 + z_2^2 - y_1z_1 - 2y_1z_1 \\
& \quad + 3y_2z_1 + 2z_2z_2 + 2y_2z_2 \right],
\end{align*}
\]
Fig. 3. Global asymptotical stability for the closed-loop systems (44) and (47).

\[ u_2 = -0.19 \text{sat}(2y^2 + z^2 - y_1z_2 + 2y_1y_2 - 2y_2z_1 - z_1z_2 - 3y_2z_2 + 2y_1z_1). \]  

Similarly, let \( x_1(0) = 10, x_2(0) = 20, x_3(0) = 30, x_4(0) = 40, z_1(0) = -20 \) and \( z_2(0) = -30 \), then the state response of the resulting closed-loop systems (44) and (47) is shown in Fig. 3.

V. CONCLUSION

This paper works on globally asymptotic stabilization of a class of MIMO bilinear systems with undamped natural response. It first presents two sufficient conditions for the globally asymptotic stabilization by static state feedback. Under these conditions the full-order and reduced order dynamic output feedback controllers are constructed, respectively, for globally asymptotic stabilization of the MIMO bilinear systems. The design of the proposed controllers is demonstrated by a numerical example. How to extend the existing global stabilization results to more general class of bilinear systems by means of dynamic output feedback is under investigation, especially to a class of MIMO bilinear systems with unstable modes.

REFERENCES


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