DESIGN OF ROBUST POLE ASSIGNMENT BASED ON PARETO-OPTIMAL SOLUTIONS

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ABSTRACT

In this paper, a new design method for robust pole assignment based on Pareto-optimal solutions for an uncertain plant is proposed. The proposed design method is defined as a two-objective optimization problem in which optimization of the settling time and damping ratio is translated into a pole assignment problem. The uncertainties of the plant are represented as a polytope of polynomials, and the design cost is reduced by using the edge theorem. The genetic algorithm is applied to optimize this problem because of its multiple search property. In order to demonstrate the effectiveness of the proposed design method, we applied the proposed design method to a magnetic levitation system.

KeyWords: Robust pole assignment, polytope of polynomials, edge theorem, genetic algorithm, Pareto-optimal solutions.

I. INTRODUCTION

The PID controller [1] is effective enough to give satisfactory control performance in practical applications. However, a design problem of a PID-type control system is not always formulated as a concise convex problem. Moreover, when robustness to the uncertainties of actual plants is considered, the design problem becomes more complex. Although some design methods ($H_2$, $H_\infty$, etc.) based on postmodern theory can be considered powerful tools for producing PID-type controllers, they are not straightforward design methods for PID-type controllers because they require approximation of the full order controller. Consequently, a universal design method for PID-type control systems has not yet been established.

In practice, the settling time and damping ratio are often important factors for the design of PID-type control systems. Since there is a trade-off between the settling time and the damping ratio, a controller is designed by means of trial and error by observing both properties. Thus, a design method based on the optimization of the settling time and the damping ratio is required for practical applications.

In this paper, we will formulate this design problem as a two-objective optimization problem which optimizes both the settling time and the damping ratio. The solution of the two-objective optimization problem does not become a single solution, but a set of solutions known as the Pareto-optimal set because of the trade-off. If the Pareto-optimal set of this problem can be found efficiently, this will reduce the burden on the controller designers. In order to formulate this design problem as a tractable optimization problem, the optimization of the settling time and the damping ratio is considered as a pole assignment problem. To reduce the settling time, it is necessary to minimize the maximum value of the real part of all the poles. On the other hand, to increase the damping ratio, it is necessary is to minimize the maximum value of the angle between the real axis and the straight line connecting the pole with the origin.

The uncertainties of the plant are represented as a polytope of polynomials, and the design cost is reduced by using the edge theorem [3]. In the proposed design method, a genetic algorithm (GA) [4-6] is applied to the optimization problem, taking into account the advantages of its multiple search property. Conventional optimization techniques, such as the gradient-based method, simplex-based method and simulated annealing, are difficult to extend to the true multi-objective optimization case because they were not designed to obtain multiple...
solutions. In practical applications, controller designers can use the proposed design method to easily find a desirable solution from among the Pareto-optimal solutions. Hence, efficient designs of PID-type control systems can be obtained by using the proposed design method.

This paper is organized as follows. Section 2 describes a polytope of polynomials, the edge theorem and multi-objective optimization as preliminaries. Section 3 formulates the problem statement. Section 4 describes the approach to multi-objective optimization. Section 5 presents the formulation of the GA used in the proposed design method. Section 6 shows the designed results and the actual step responses obtained for a magnetic levitation system. Section 7 compares the proposed design method with the conventional design methods. Section 8 concludes the paper.

II. PRELIMINARIES

In this paper, the plant uncertainties are represented as a polytope of polynomials. For the purpose of reducing the design cost, the edge theorem is employed. In this section, the polytope of polynomials, the edge theorem and multi-objective optimization problem are briefly explained.

2.1 Polytope of polynomials

By identifying both of the coefficients in \((n + 1)\)-dimensional vector space and the \(n\)-dimensional polynomials \(A_i(s)\), we can define a polytope of polynomials as a convex hull containing polynomials \(A_i(s)\)'s as follows:

\[
\Pi = \{ A(s) \mid A(s) = \sum_{i=1}^{N} \lambda_i A_i(s), \lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda_N \},
\]

where \(\Lambda_N\) and \(\lambda\) are defined as

\[
\Lambda_N = \{ \lambda = (\lambda_1, \ldots, \lambda_n) | \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \}.
\]

Let \(G\) be a set of polynomials. Each element of \(G\) is a polytope of polynomials. If a convex hull of any proper subset of \(G\) differs from the convex hull of \(G\), then \(G\) is said to be a generator if a convex hull of any proper subset of \(G\) differs from the convex hull of \(G\).

2.2 Edge theorem

The polynomial considered in the edge theorem is described as follows:

\[
f(s, \delta) = s^m + a_m(\delta)s^{m-1} + \ldots + a_1(\delta)s + a_0(\delta),
\]

where \(\delta\) denotes the \(m\)-dimensional vector,

\[
\delta = (\delta_1, \ldots, \delta_m)^T,
\]

the coefficient \(a_i(\delta)\) indicates a real affine function of \(\delta\), and the range of \(\delta\) is the \(m\)-dimensional polytope \(\Delta \subset \mathbb{R}^m\). A polytope of polynomials \(\Pi\) is produced by using Eq. (3). The exposed set is defined as a common set between a nontrivial supporting hyperplane and a polynomial family in coefficient space. Generally, one-dimensional exposed sets are called exposed edges. The edge theorem provides a condition for the root locations of the polytope of polynomials in the region \(D\), which is an open subset of the complex plane.

Edge Theorem \([3]\). Let \(\delta \in \Delta\) be an arbitrary vector; all the roots of the polytope of polynomials \(\Pi\) are contained in \(D\) if and only if the root space of all the exposed edges is contained in \(D\).

2.3 Multi-objective optimization problem

A multi-objective optimization problem is defined as a vector minimization problem as follows:

\[
\min_{x \in \Omega} F(x) = (F_1(x), \ldots, F_l(x))^T,
\]

where \(x\) is a variable vector, \(\Omega\) is a feasible solution space, and \(F_1(x), F_2(x), \ldots, F_l(x)\) denote real valued objective functions.

Pareto-optimal solutions represent one of the major approaches to multi-objective optimization. For any two points \(x_1\) and \(x_2\) in \(\Omega\), if the following conditions hold:

\[
F_i(x_1) \leq F_i(x_2) \quad \forall i \in \{1, \ldots, l\},
\]

\[
F_j(x_1) < F_j(x_2) \quad \exists j \in \{1, \ldots, l\},
\]

then \(x_1\) is said to be superior to \(x_2\), vector \(x_1\) is at least as good as \(x_2\) with respect to all \(l\) objectives, and \(x_1\) is strictly better than \(x_2\) with respect to at least one objective. If no other solution is superior to \(x_1\), then \(x_1\) is called a Pareto-optimal solution. The boundary consisting of the set of Pareto-optimal solutions is called a trade-off surface, or a Pareto frontier.

III. PROBLEM STATEMENT

3.1 I-PD control system

Suppose a plant can be modeled by an all-pole transfer function as follows:

\[
g(s) = \frac{1}{a_0 + a_1s + a_2s^2 + \ldots + a_ms^m},
\]

where \(s\) is the Laplace operator.
Consider the control system with the I-PD controller [7] shown in Fig. 1.

From Fig. 1, the I-PD control system is described by

\[ e(t) = r(t) - y(t), \]  
\[ u(t) = \frac{k_I}{s} e(t) - (k_P + k_D s) y(t), \]  
\[ y(t) = g(s) u(t), \]

where \( r(t) \) is the reference input, \( e(t) \) is the error signal, \( u(t) \) is the controller output, and \( y(t) \) is the plant output. The variables \( k_I, k_P, \) and \( k_D \) indicate the I-PD parameters.

Hence, the transfer function of the closed-loop system is described as follows:

\[ H(s) = \frac{g(s)k_I}{s + g(s)(k_I + k_P s + k_D s^2)} \]  
(10)

By applying the I-PD control system to the plant that is modeled by an all-pole transfer function, we can also model the transfer function of the closed-loop system by means of an all-pole transfer function.

### 3.2 Representation of uncertainties by polytope of polynomials

We assume that the plant uncertainties lie in the coefficients of the denominator polynomials. Each system identification trial will produce a distinct plant model. Thus, the plant can be represented by a set of models. The set of models is defined as a polytope of denominator polynomials as follows:

\[ \Pi = \{ A_i(s) \mid \lambda \in \Lambda_i \}, \]  
(11)

where \( A_i(s) = \lambda_i A_0(s) + \lambda_{i1} A_1(s) + \ldots + \lambda_{in} A_n(s) \)  
(12)

\[ = a_0 + \alpha_{i0} s + \alpha_{i1} s^2 + \ldots + \alpha_{in} s^n. \]  
(13)

The element \( A_i(s), i = 1, \ldots, N \) denotes a denominator polynomial of a plant model. The parameter \( \lambda_i \), \( \Sigma \lambda_i = 1, 0 \leq \lambda_i \leq 1, i = 1, \ldots, N \), represents the plant uncertainties, and \( \alpha_{ij}, j = 0, \ldots, n \), is a coefficient parameter of polynomials with plant uncertainties.

By using Eq. (13), we can transform the closed-loop system (10) via

\[ H_{cl}(s) = \frac{1}{1 + \frac{\alpha_{i0} + k_P}{k_I} s + \frac{\alpha_{i1} + k_D}{k_I} s^2 + \ldots + \frac{\alpha_{in}}{k_I} s^n}. \]  
(14)

The uncertainties of the characteristic polynomial of the closed-loop system are represented by a polytope of polynomials if the uncertainties of the denominator polynomial of the all-pole plant models are represented by a polytope of polynomials. The generator of the plant denominator polynomials corresponds to the generator of the polytope of characteristic polynomials.

For computational purposes, it is convenient to define a "discrete" exposed edge. Let \( \Lambda_N^D \) be as follows:

\[ \Lambda_N^D = \{ \lambda \mid (\lambda_1, \ldots, \lambda_N) \in \Lambda, \lambda \geq 0, \sum \lambda_i = 1, \lambda_i \in \{ 0, \frac{1}{N_N}, \ldots, 1 \} \}. \]  
(15)

Then, the discrete exposed edge is defined as a set of points, each of which is on an exposed edge, and each \( \lambda \) is in \( \Lambda_N^D \).

### 3.3 Objective function

In order to apply the edge theorem to this design problem, we will introduce a subset of \( \Lambda_0 \) corresponding to a subset of \( \Pi \). Let \( E \) be a subset of \( \Pi \). Then, \( \Lambda(E) \) is defined as follows:

\[ \Lambda(E) = \{ \lambda \mid \lambda(s) \in E \}. \]  
(16)

Since \( E \) is a subset of \( \Pi \), \( \Lambda(E) \) becomes a subset of \( \Lambda_0 = \Lambda(\Pi) \).

The purpose of the design problem is to find the robust I-PD parameters that minimize the following objective functions simultaneously:

\[ \text{minimize } F(E) = (F_1(E), F_2(E))^T, \]  
(17)

\[ F_1(E) = \max_{\lambda \in \Lambda_0} \{ \text{Re}(s_i(\lambda; k_I, k_P, k_D)) \}, \]  
(18)
Fig. 2. The objective functions $F_1(E)$ and $F_2(E)$ for a typical pole placement of an uncertain system.  

$$F_i(E) = \max_{\lambda \in \Lambda} \left[ \theta_i \left( s_i, \lambda \right) \right],$$  

(19) 

where $s_i$ indicates the pole of the closed-loop transfer function obtained from Eq. (14), and $\theta_i$ indicates the angle between the real axis and the straight line that connects the pole $s_i$ with the origin. The objective function $F_1(E)$ is the maximum value of the real part of all the poles, which is related to the settling time. On the other hand, the objective function $F_2(E)$ is the maximum value of the angle between the real axis and the straight line that connects the pole $s_i$ with the origin, which is associated with the damping ratio. In Eqs. (18) and (19), $E$ is a set of exposed edges.  

Figure 2 illustrates the objective functions $F_1(E)$ and $F_2(E)$ for a typical pole placement of an uncertain system.  

Using the edge theorem, we can clearly see that $F_i(\Pi) = F_i(E)$, $i = 1, 2$. Thus, a robust controller for $\Pi$ is obtained by minimizing $F(E)$ instead of $F(\Pi)$ as in Eq. (17).  

IV. DESIGN ALGORITHM 

In practical use, it is likely that the design of a robust controller for $E$ will involve a computational cost even if the latest computer is used. On the other hand, a worst point which acquires the maximum values of the objective functions as in Eqs. (18) and (19) usually becomes a generator. Thus, we will propose a design algorithm to calculate robust I-PD parameters optimized over a set of generators, and then check the obtained I-PD parameters for $E$.  

The design algorithm for a robust I-PD controller based on the Pareto-optimal solutions can be summarized as follows: 

**Step 1.** Generate a set of plant denominators from system identifications. Then, calculate the generators and the exposed edges based on the results obtained.  

**Step 2.** Let $V$ be a set consisting of all the generators, and let $E_d$ be a set consisting of all the discrete exposed edges.  

**Step 3.** Compute the triplets of the I-PD parameters that minimize the objective functions $F_1(V)$ and $F_2(V)$ in Eqs. (18) and (19) by using the GA explained in the following section.  

**Step 4.** By using all the triplets of the I-PD parameters obtained in Step 3, update $V$ as follows: 

$$V := V \cup \{ e \in E_d \mid (F_1(e) > F_1(V)) \lor (F_2(e) > F_2(V)) \}.$$  

(20) 

If $V$ is updated, go to Step 3. Otherwise, go to Step 5.  

**Step 5.** According to the controller designer's preference, select a desirable triplet of the I-PD parameters by observing both the settling time and the damping ratio of the actual step response. Use the binary search method to select a desirable triplet of the I-PD parameters.  

**Remark 1.** The procedure for selecting a desirable triplet of the I-PD parameters can be described as follows. To begin with, the medium triplet of the I-PD parameters is selected as an initial search point. Then by using the binary search method, the designer chooses a desirable solution by observing both the settling time and the damping ratio of the actual step response. Thus, the controller designer can easily obtain a desirable triplet of the I-PD parameters in only a few trials.  

V. GA FORMULATION 

To set up a GA for the design of I-PD controller, certain problem-dependent algorithm elements must be defined. These elements affect both the efficiency of the algorithm and the quality of its results.  

5.1 Representation of individuals 

The real-valued coding method is employed for candidate solutions. Each of the I-PD parameters $k_I$, $k_P$, and $k_D$ is defined as a real number. Each individual $p_i$ based on the real-valued coding method is expressed as follows: 

$$p_i = (b_{i1}, b_{i2}, b_{i3}),$$  

(21) 

where $b_{i1}$, $b_{i2}$, and $b_{i3}$ correspond to the real numbers of $k_I$, $k_P$, and $k_D$, that is, each population member consists of
three real numbers. In this way, each individual represents a candidate solution. In the GA, a population \( P(\tau) = \{p_1, p_2, \ldots, p_M\} \) that consists of \( M \) individuals is used to solve a given problem. The index \( \tau \) indicates the generation number, and the index \( M \) indicates the population size.

5.2 Fitness function

Since the problem to be optimized has two objective functions, (18) and (19), the following fitness function based on Pareto-optimal solutions is employed. In order to obtain the Pareto-optimal solutions, the multi-objective ranking method proposed by Fonseca et al. [8] is adopted as the evaluation criterion for individuals. In the multi-objective ranking method, the rank of an individual is defined based on the superiority of the individuals. To avoid partial convergence of the individuals in the objective function space, the sharing method [9] is also used to design the fitness function.

Let \( q_i \) be the number of individuals that are superior to the individual \( p_i \) in the population \( P(\tau) \). The rank \( r_i \) of the individual \( p_i \) is defined as follows:

\[
    r_i = 1 + q_i.
\]  

(22)

Thus, the Pareto-optimal individual has a rank of 1. Generally, the more desirable the individual is, the higher the fitness value is. From Eq. (22), the rank of a good individual is small. Thus, by using the following equation, we can transform the ascending order rank \( r_i \) of the individual \( p_i \) into the descending order value \( f_i \):

\[
    f_i = \frac{r_{\text{max}} - r_i + 1}{r_{\text{max}} - r_{\text{min}} + 1}.
\]  

(23)

where \( r_{\text{max}} \) and \( r_{\text{min}} \) denote the maximum rank and the minimum rank. This transformation does not affect the GA search performance.

Next, the fitness of an individual is defined by using both the sharing method and the value obtained from Eq. (23). The distance between the individuals is needed to use the sharing method. The distance must return a value of zero or higher, where zero means that the two individuals are identical. The distance \( d(p_i, p_j) \) between the individuals \( p_i \) and \( p_j \) is defined as the Euclidean distance in the objective function space as follows:

\[
    d(p_i, p_j) = \sqrt{(F_1(p_i) - F_1(p_j))^2 + (F_2(p_i) - F_2(p_j))^2}.
\]  

(24)

The sharing function \( S_d(d) \) is defined by

\[
    S_d(d) = \begin{cases} 
        1 - \frac{d}{\sigma_{\text{share}}} & \text{if } d < \sigma_{\text{share}} \\
        0 & \text{otherwise},
    \end{cases}
\]  

(25)

where \( \sigma_{\text{share}} \) denotes a sharing parameter.

Thus, the fitness \( f_i^* \) of the individual \( p_i \) is calculated as follows:

\[
    f_i^* = \frac{f_i}{\sum_{j=1}^{M} S_d(d(p_i, p_j))},
\]  

(26)

where \( M \) is the number of individuals in the population. In this way, when the population density is high in the objective function space, the fitness value changes to give a better chance for relatively unfit individuals to survive selection. The purpose of applying the sharing method is to avoid the partial convergence of individuals, and to maintain population diversity. The sharing parameter \( \sigma_{\text{share}} \) is set as an average Euclidean distance of the Pareto-optimal individuals.

5.3 Procedure of GA

The continuous generation model with a Pareto-optimal preservation strategy is employed. In this model, Pareto-optimal solutions are always forced to appear in the following generation. The proposed procedure for the GA consists of the following steps.

Step 1. Set a generation number \( \tau = 0 \). Randomly generate an initial population \( P(0) \) of \( M \) individuals.

Step 2. Calculate the fitness of each individual in the current population \( P(\tau) \) according to the fitness function as in Eq. (26).

Step 3. Generate a new population \( P'(\tau) \) from \( P(\tau) \) as follows: select individuals from \( P(\tau) \) using fitness; recombine them using BLX-\( \alpha \) crossover [10].

Step 4. Apply non-uniform mutation [5] to the newly generated population \( P' (\tau) \) according to the mutation rate \( \rho \).

Step 5. Calculate the fitness of both \( P(\tau) \) and \( P'(\tau) \). Select the top \( M \) individuals from all the population members on the basis of their fitness.

Step 6. If a terminal condition is satisfied, stop and return the superior individuals. Otherwise, set \( \tau = \tau + 1 \) and go to Step 2.

In this procedure, the current population size is always constant \( M \). Here, to avoid the rapid loss of population diversity, multiple equivalent individuals are eliminated from the current population.

Remark 2. In order to execute the proposed design method using the GA, some GA parameters must be defined in advance, namely, each range of the I-PD parameters \( k_I, k_P \) and \( k_P \), a population size \( M \) and a mutation rate \( \rho \).
VI. APPLICATION TO MAGNETIC LEVITATION SYSTEM

In order to investigate the effectiveness of the proposed design method, it was applied to a magnetic levitation system.

6.1 Magnetic levitation system

The experimental system is shown in Fig. 3. The position of the steel ball was measured by a laser gap sensor and sent to a computer through an A/D converter. The signal from the computer was sent to the voltage control circuit through the D/A converter, and it became the voltage of the coil edge. An I-PD controller was implemented as a program on the computer.

The control objective was to keep the steel ball at the specified position. Since the magnetic levitation system was modeled by a 3rd order nonlinear system, changes of the equilibrium point caused uncertainties of the plant. Moreover, heat fluctuations and hysteresis of the magnetic materials also caused the plant uncertainties.

6.2 Plant model

Suppose the plant can be modeled as an all-pole 3rd order transfer function. The transfer function of the plant is expressed as follows:

\[ g(s) = \frac{1}{a_0 + a_1 s + a_2 s^2 + a_3 s^3} \]  

(27)

Table 1 shows the values of the plant parameters, which were obtained by means of system identification. These parameters were used as the generator of the polytope of polynomials.

6.3 Design of the controller

First, we established the GA parameters as described in Remark 2. In the design of the I-PD controller, each I-PD parameter \( k_I \), \( k_P \), and \( k_D \) was subject to an interval constraint. We assumed that the constraints of the I-PD parameters were those given below:

\[
\begin{align*}
0.0 & \leq k_I \leq 1000.0 \\
0.0 & \leq k_P \leq 1000.0 \\
0.0 & \leq k_D \leq 10000.0
\end{align*}
\]  

(28)

Thus, the population size \( M \) was 128, and the mutation rate \( \rho \) was 0.1. The terminal condition was either that the generation number was greater than 500, or that the number of Pareto-optimal solutions did not update during 100 generations.

Figure 4 shows the designed results obtained by

![Fig. 3. Experimental system.](image)

![Fig. 4. Distribution of solutions obtained using the proposed design method.](image)

Table 1. Values of the plant parameters.

<table>
<thead>
<tr>
<th>( g_1 )</th>
<th>( a_0 )</th>
<th>( a_1 \times 10^{-8} )</th>
<th>( a_2 \times 10^{-4} )</th>
<th>( a_3 \times 10^{-4} )</th>
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<tbody>
<tr>
<td>( g_1 )</td>
<td>-1.9574</td>
<td>-1.4343</td>
<td>4.4111</td>
<td>4.6881</td>
</tr>
<tr>
<td>( g_2 )</td>
<td>-1.9461</td>
<td>-1.2193</td>
<td>7.1521</td>
<td>6.3381</td>
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<td>( g_3 )</td>
<td>-2.1854</td>
<td>-1.2269</td>
<td>4.0168</td>
<td>13.8616</td>
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<tr>
<td>( g_4 )</td>
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<td>6.2101</td>
<td>7.7753</td>
</tr>
<tr>
<td>( g_5 )</td>
<td>-2.2852</td>
<td>-1.0942</td>
<td>10.1078</td>
<td>15.0714</td>
</tr>
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<td>( g_6 )</td>
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<td>-1.3289</td>
<td>5.3064</td>
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<td>( g_7 )</td>
<td>-1.8959</td>
<td>-1.3791</td>
<td>4.6511</td>
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<td>( g_8 )</td>
<td>-1.8340</td>
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<td>3.7883</td>
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<td>( g_{11} )</td>
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<td>( g_{12} )</td>
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<td>( g_{13} )</td>
<td>-1.8322</td>
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<td>( g_{14} )</td>
<td>-1.7475</td>
<td>-1.3379</td>
<td>4.8904</td>
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</table>
using the proposed design method. In the figure, the axis of the abscissa indicates the value of the objective function $F_1$, and the axis of the ordinate indicates the value of the objective function $F_2$. The dots indicate the solutions obtained from the final population. The solutions $p_i$ were sorted according to increasing values of the objective function $F_1$. It is clear that the proposed design method could seek widely distributed solutions on the trade-off surface. From the computational point of view, the design result was obtained in 40.42 seconds (198 generations) on a Pentium III 866MHz personal computer.

Table 2 shows the I-PD parameter values of the representative points shown in Fig. 4. The following subsection investigates the property of the actual step responses by applying these I-PD parameter values to the experimental system.

### 6.4 Experimental results

Figure 5 shows the pole placement obtained by using each triplet of the I-PD parameters $p_{64}$, $p_{32}$, $p_{48}$, and $p_{40}$. Figure 6 shows the actual step responses obtained by applying each triplet of the I-PD parameters to the actual experimental system. These figures exhibit the procedure for selecting unique I-PD parameters from among the Pareto-optimal solutions. The selection procedure based on the binary search method is described as follows. To start with, since solution $p_{64}$ was located in the middle of

**Table 2. Values of the designed I-PD parameters.**

<table>
<thead>
<tr>
<th>$p_i$</th>
<th>$k_I$</th>
<th>$k_P$</th>
<th>$k_D$</th>
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</thead>
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<td>$p_{64}$</td>
<td>325.1046</td>
<td>63.6331</td>
<td>4.3528</td>
</tr>
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<td>$p_{32}$</td>
<td>8.7655</td>
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<td>0.2041</td>
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<td>$p_{48}$</td>
<td>6.6381</td>
<td>3.2106</td>
<td>0.1943</td>
</tr>
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<td>$p_{40}$</td>
<td>5.7801</td>
<td>3.0975</td>
<td>0.1917</td>
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<td>$p_{128}$</td>
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<td>$p_{16}$</td>
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</tbody>
</table>
the order of the solutions, it was selected as an initial search point. Then, both the settling time and the damping ratio were observed. Because of the large settling time, solution \( p_{32} \) was chosen. Then, the actual step response was observed. Since the step response obtained using solution \( p_{32} \) was under damped, solution \( p_{48} \) was chosen. By iterating these steps, the controller designer could easily choose a desirable solution. From among these actual step responses, solution \( p_{40} \) was regarded as the desirable solution.

VII. COMPARISONS

In order to compare the proposed design method with other methods, a weighted-sum method and an \( H_{\infty} \) robust I-PD controller [2] were applied to this design problem.

7.1 Weighted-sum method

The weighted-sum method has a concise structure and is easy to apply to a multi-objective optimization problem. The optimization problem based on the weighted-sum method is defined as follows:

\[
\text{minimize } F_w(E) = c F_1(E) + (1-c) F_2(E),
\]

where \( F_1(E) \) and \( F_2(E) \) are the objective functions defined by Eqs. (18) and (19), and \( c \) is a weight in \([0, 1]\).

With the weighted-sum method, the objective function \( F_w(E) \) defined by Eq. (30) was minimized by using the GA. The representations of individuals, genetic operators and control parameters of the GA were identical to those in the proposed design method except for the objective function.

Figure 7 shows the designed results obtained by using the approach with the weighted-sum method. The weight \( c \) ran over the interval \([0, 1]\) with a step-size of 0.01. Figure 7 shows the results that were designed 101 times using the approach with the weighted-sum method at each of 101 weights. As expected, there were no solutions in the concave regions. This absence of solutions
was due to the existence of a duality gap. This meant that a desirable controller could not be found even if the search was conducted over all the weights. In comparison with this design approach, the proposed design method has an advantage in that it can obtain Pareto-optimal solutions not only in the convex regions, but also in the concave regions.

7.2 $H_\infty$ robust I-PD controller

The $H_\infty$ I-PD controller was defined as a reduced order controller of a full order $H_\infty$ robust controller, which was designed as a mixed sensitivity minimization problem. Since the uncertainties of the plant were considered to lie in the denominator of the transfer function, the generalized plant was that shown in Fig. 8. In this figure, $g(s)$ indicates a nominal plant, which is composed of the medium value of the identified results.

The uncertainties of the plant were represented by a weight function $W_2(s)$. The weight function $W_2(s)$ was defined as follows:

$$W_2(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3,$$

where $(a_0, a_1, a_2, a_3) = (-0.1942, -0.0286, 6.44 \times 10^{-4}, 3.978 \times 10^{-6})$.

Since the design objective was to track a constant input, the weight function $W_1(s)$ was defined as follows:

$$W_1(s) = \frac{w}{s + \varepsilon},$$

where $\varepsilon = 2\pi \times 10^{-3}$, and $w$ denotes a weight parameter. Figure 9 shows a Bode plot of the full order $H_\infty$ controller and its approximated I-PD controller. The approximated I-PD controller was calculated by using the method of least squares to fit the frequency between $10^6$ and $10^1$ of the full order $H_\infty$ controller. Figure 10 shows the actual step response obtained using the $H_\infty$ robust I-PD controller.

Although the design of the $H_\infty$ robust I-PD controller based on a mixed sensitivity minimization problem has a concise procedure and requires little prior information, it has several problems as follows:
1) This design method requires human assistance to approximate the full order controller. Furthermore, a slight change of the approximating procedure sometimes results in serious deterioration of the attainable performance. The resulting I-PD parameters shown in Fig. 10 can be obtained through several approximation trials.

2) It is difficult to guarantee the prescribed $H_{\infty}$ performance by taking into account the deterioration owing to the approximation.

3) When the generated I-PD controller does not achieve the expected performance, recomposition of the generalized plant and refinement of the weight functions require the application of expert knowledge of controller designers.

4) The more severe the sensitivity property, the more difficult it is to approximate the PID-type controller to obtain the full order controller. In other words, it is difficult to design weight functions that are suitable for the I-PD approximation. In the actual experiments, we could not design a stable $H_{\infty}$ robust I-PD controller when the weight parameter $w$ in Eq. (32) became large.

In comparison with this $H_{\infty}$ design approach, the proposed design method can produce a set of I-PD parameters that guarantee the expected performance. Moreover, the primary advantage of the proposed design method is that the desirable triplet of the I-PD parameters can be obtained automatically by simply adjusting the GA parameters described in Remark 2.

VIII. CONCLUSION

In this paper, we have proposed a new design method for robust pole assignment based on Pareto-optimal solutions for an uncertain plant. The proposed design method has been defined as a two-objective optimization problem based on both the settling time and the damping ratio. A genetic algorithm has been applied to optimize this problem because of its multiple search property. A magnetic levitation system has been used as an example of an uncertain plant.

We have compared the performance of the proposed design method with that of the weighted-sum method and an $H_{\infty}$ robust I-PD controller. Based on the designed results, the proposed design method was able to find widely distributed solutions on the trade-off surface. The main reason for this is that the multiple search based on the GA effectively solved the concave and multi-objective optimization problem. When the proposed design method is used in practical applications, the controller designer can easily find a desirable solution from among the Pareto-optimal solutions. Hence, an efficient design of an I-PD control system can be obtained by using the proposed design method.

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