MIXED CONSTRAINED INFINITE HORIZON LINEAR QUADRATIC OPTIMAL CONTROL

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ABSTRACT

For a given initial state, a constrained infinite horizon linear quadratic optimal control problem can be reduced to a finite dimensional problem [12]. To find a conservative estimate of the size of the reduced problem, the existing algorithms require the on-line solutions of quadratic programs [10] or a linear program [2]. In this paper, we first show based on the Lyapunov theorem that the closed-loop system with a mixed constrained infinite horizon linear quadratic optimal control is exponentially stable on proper sets. Then the exponentially converging envelop of the closed-loop trajectory that can be computed off-line is employed to obtain a finite dimensional quadratic program equivalent to the mixed constrained infinite horizon linear quadratic optimal control problem without any on-line optimization. The example considered in [2] showed that the proposed algorithm identifies less conservative size estimate of the reduced problem with much less computation.

KeyWords: Linear quadratic optimal control, constrained systems, linear discrete time systems.

I. INTRODUCTION

In 1960, employing state space approach, Kalman provided a complete Riccati equation based solution strategy for infinite horizon linear quadratic optimal control problems [8]. However, constraints are always present in any practical control problems. For instance, physical restriction of actuator limits the value the input can assume. Moreover due to safety, environmental regulation and so on, states of plant are desired to lie within a designated area in state space. Under presence of these constraints, closed-loop system becomes nonlinear and the analysis by Kalman is no longer valid. However due to difficulty associated with the nonlinearity, constrained infinite horizon linear quadratic optimal control problems remained unsolved until Sznaier and Damborg [12] pioneered the area in 1987 by realizing the fact that a constrained infinite horizon linear quadratic optimal control problem can be reduced to a finite dimensional quadratic program. This reduction was possible because, for the infinite horizon quadratic cost to be finite, the constraints are only active over a finite time horizon and thus the remaining infinite horizon unconstrained part of the problem beyond the finite time horizon can be analytically solved by the Kalman’s technique. Based on this fact, they proposed a solution strategy with a set of on-line quadratic programs. Their results were further extended and refined by Scokaert and Rawlings [10]. Recently, Chmielewski and Manousiouthakis provided a computationally less demanding technique where only an on-line linear program is required to find a finite dimensional quadratic program equivalent to the constrained infinite horizon linear quadratic optimal control problem [2].

In this paper, exponential stability of mixed constrained linear quadratic optimal control on proper sets is first established based on the Lyapunov theory. Exponential stability for all feasible trajectory is also claimed in [2] and [12] from the fact that each feasible trajectory converges exponentially. However, as shown in [4], exponential stability for all feasible trajectories cannot be achieved with any control strategy if the plant is marginal or unstable. From the exponential stability proofs, we can find an exponentially converging envelop associated with the exponential stability on proper sets. Since the envelop is exponentially converging to zero, we can...
find an estimate of the time for which the trajectory enters the maximal output admissible set \([7]\) on which constraints are never active. Clearly the proposed reduction does not require any on-line optimization. The proposed technique is tested with the example considered in \([2]\). The example shows that the proposed technique not only identifies a reduced problem much faster but also identifies a smaller size problem.

Since the main idea of this paper is first presented in 1998 \([6]\), the closed form solution of linear quadratic optimal control problems has been found \([1,11]\). However, these approaches are limited to finite horizon problems. Moreover, as mentioned in the conclusion section of \([1]\), these techniques cannot be applied to large size problems due to computational complexity. Hence the technique proposed in this paper is still valuable.

II. EXPONENTIAL STABILITY

Consider the following mixed constrained infinite horizon linear quadratic optimal control problem:

\[
J(x(k)) = \min_{z,v} \sum_{j=0}^\infty z(j)^T R z(j) + \sum_{j=0}^\infty v(j)^T S v(j) + \epsilon^T Q \epsilon (P)
\]

subject to

\[
\begin{align*}
  &z(j+1) = A z(j) + B u(j), \quad z(0) = x(k) \\
  &v^\min \leq v(j) \leq v^\max, \quad j = 0,1,\cdots \\
  &G z(j) \leq g + \epsilon, \quad j = 1,2,\cdots,
\end{align*}
\]

where \(R > 0, S > 0, P > 0\) are symmetric matrices, \(Q > 0\) is a diagonal matrix, \(G \in \mathbb{R}^{m \times n}\). For well-posedness of the problem, it is assumed that \(g > 0\) for a neighborhood of the origin to be feasible and the system is stabilizable and detectable.

The solution of this problem is in general an open loop control. An easy remedy for feedback control is to adopt the receding horizon idea; the control input at each \(k\)th sampling time is determined by the argument minimum of the quadratic optimization problem \(J(x(k))\). The feedback implemented mixed constrained infinite horizon linear quadratic optimal control is a nonlinear static state feedback control law.

Clearly the closed-loop system is exponentially stable on the associated maximal output admissible set \(O_+ \) \([7]\) on which no constraints are active.

Before we proceed, let \(J_*(x(k)) = x(k)^T P x(k)\) be the optimal cost of the unconstrained linear quadratic optimal control where \(P\) is the solution of

\[
P = A^T (P - PB(B^T PB + S)^{-1} B^T P) A + R.
\]

2.1. Exponentially stable plants

Notice that

\[
J(x(k)) \geq J_*(x(k)) \geq \lambda_{\min}(P) |x(k)|^2, \quad \forall x(k) \in \mathbb{R}^n.
\]

We next establish that there exists \(b > 0\) such that \(J(x(k)) \leq b |x(k)|^2\) for all \(x(k) \in \mathbb{R}^n\).

Theorem 2.1. Suppose \(A\) is exponentially stable. Then, for all \(x(k) \in \mathbb{R}^n\),

\[
J(x(k)) \leq \left\{ \lambda_{\max}(R) + \lambda_{\max}(Q) p \left[ \max_{j \geq 1} \|GA'\| \right]^2 \right\} |x(k)|^2
\]

where \(R\) is a unique positive definite solution of the following Lyapunov equation:

\[
R = A^T RA + R.
\]

Proof. Let \(v(j) = 0\) for all \(j\). Then \(z(j) = A' x(k)\) for all \(j\). Hence,

\[
|Gz(j)| = \|GA' x(k)\| \leq \|GA'\| \|x(k)\| \leq \max_{j \geq 1} \|GA'\| |x(k)|
\]

Notice that, since \(A\) is exponentially stable, \(\max_{j \geq 1} \|GA'\| < \infty\) and the maximum is achieved with a finite \(j\). Hence, there exists \(\epsilon_i \geq 0\) such that

\[
\max_{j \geq 1} \{\|GA'\| |x(k)|\} \leq g_i + \epsilon_i, \quad i = 1,\cdots, p.
\]

Since \(g > 0\),

\[
(\nu(j), \epsilon) = (0, \max_{j \geq 1} \{\|GA'\|\} |x(k)| [1 \cdots 1]^T)
\]

is a feasible point of the problem \((P)\). Hence,

\[
J(x(k)) \leq x(k)^T \left[ \sum_{j=1}^\infty (A')^T RA' \right] x(k)
\]

\[
+ \lambda_{\max}(Q) p \left[ \max_{j \geq 1} \|GA'\| \right]^2 |x(k)|^2
\]

\[
= x(k)^T Rx(k) + \lambda_{\max}(Q) p \left[ \max_{j \geq 1} \|GA'\| \right]^2 |x(k)|^2
\]

\[
\leq \lambda_{\max}(R) |x(k)|^2 + \lambda_{\max}(Q) p \left[ \max_{j \geq 1} \|GA'\| \right]^2 |x(k)|^2.
\]

After implementing \(u(k-1) = \nu(0)\) of \(J(x(k-1))\) at the time \(k - 1\), the rest of the optimal solution is a feasible solution for the problem at the time \(k\). Let \(J'(x(k))\) be the cost function with the feasible solution. Since the feasible solution may not be optimal, \(J(x(k)) \leq J'(x(k))\) and thus for all \(x(k) \in \mathbb{R}^n\).
\[ \Delta J(x(k)) = J(x(k)) - J(x(k-1)) \leq J'(x(k)) - J(x(k-1)) = -z(0)^T Rz(0) - u(0)^T Su(0) \leq -x(k)^T Rx(k) \leq -\lambda_{\max}(R) |x(k)|^2. \]

To this end, we have the following theorem.

**Theorem 2.2.** Suppose the plant is exponentially stable. Then the closed-loop system with the mixed constrained infinite horizon linear quadratic optimal control is globally exponentially stable.

Similarly to mixed constrained model predictive control [5], it can be shown that the output feedback system composed of a mixed constrained linear quadratic optimal control and an asymptotic observer is also globally exponentially stable for stable plants.

### 2.2. Marginal plants

From the global exponential stability of the mixed constrained infinite horizon linear quadratic optimal control for stable plants and the fact that the constrained asymptotically stabilizable set \( C_\infty \) [3] is \( \mathbb{R}^n \) for marginal plants, one may conjecture that the global exponential stability of the mixed constrained infinite horizon linear quadratic optimal control is also attained for marginal plants. However, as shown in [4], constrained global exponential stabilization of a marginal plant is impossible. Nonetheless, as shown in [9], a marginally stable plant is semi-globally exponentially stabilizable using linear state feedback. Namely, for any \( N \)-step stabilizable set \( S_N \) [3], there exists a state feedback law \( u(k) = -\sigma(Kx(k)) \) such that the closed-loop system is exponentially stable on \( S_N \). We now employ this feedback law to prove exponential stability of the closed-loop system with a marginal plant and the mixed constrained infinite horizon linear quadratic optimal controller on any compact subset of \( S_N \).

It can be shown similarly to the exponentially stable plant case that there exist \( a, c > 0 \) such that for all \( x(k) \in \mathbb{R}^n \),

\[ a |x(k)|^2 \leq J(x(k)), \quad \Delta J(x(k)) \leq -c |x(k)|^2. \]

We now show that there exists \( b > 0 \) such that for all \( x(k) \in S_N \),

\[ J(x(k)) \leq b |x(k)|^2. \]

**Theorem 2.3.** Suppose \( A \) is marginal. Let \( u(j) = -\sigma(Kz(j)) \) the control such that the closed-loop system is exponentially stable on \( S_N \). Let \( \alpha > 0 \) and \( 0 < \rho < 1 \) be such that for all \( x(k) \in S_N \),

\[ |z(j)| \leq \alpha |x(k)| \rho^j \]

where

\[ z(j + 1) = (A + B_\sigma \circ K) (z(j)). \]

Then, for all \( x(k) \in S_N \),

\[ J(x(k)) \leq \frac{1}{1 - \rho^2} \lambda_{\max}(R) \alpha^2 + \frac{1}{1 - \rho^2} \lambda_{\max}(Q) \alpha^2 + \frac{1}{1 - \rho^2} \lambda_{\max}(Q) \rho \|G\| \alpha^2 \rho^2 \]

Moreover, \( S_N \to \mathbb{R}^n \) [3]. Hence there exists \( S_m \) that contains all the closed-loop trajectories starting from a state in \( S_N \cap B(\gamma) \) where \( B(\gamma) \) is the
ball centered at the origin with radius \( \gamma \). Notice that a trajectory starting in \( S_N \cap B(\gamma) \) doesn’t have to be completely contained in \( S_N \cap B(\gamma) \). To this end, we have established the following theorem.

**Theorem 2.4.** Suppose the plant is marginal. Then the closed-loop system with the mixed constrained infinite horizon linear quadratic optimal control is exponentially stable on any \( S_N \cap B(\gamma) \).

**Corollary 2.1.** Suppose the plant is marginal. Then the closed-loop system with the mixed constrained infinite horizon linear quadratic optimal control is globally asymptotically stable.

Notice that the constrained stabilization of a marginal plant achieved by the mixed constrained infinite horizon linear quadratic optimal control is stronger than the semi-global stabilization where a different controller must be constructed for a different \( S_N \cap B(\gamma) \). Indeed, this is the strongest possible constrained stabilization of a marginal plant.

Let \( D_M(\gamma) \) be the set of all estimated state and estimation error pairs such that the estimated state trajectory starting from them lies entirely in \( S_N \cap B(\gamma) \). Then, similarly to the results in [5], the output feedback system consisting of the mixed constrained linear quadratic optimal control and an asymptotic observer is exponentially stable on \( D_M(\gamma) \).

### 2.3. Exponentially unstable plants

In [4], it was shown that constrained global exponential stabilization of a marginal plant is impossible. Nonetheless, as shown in [3], exponentially unstable plants subject to input constraints are exponentially stabilizable by linear periodic variable structure state feedback control on any \( S_N \). Similarly to the marginal case, this feedback law can be employed to prove the following theorem.

**Theorem 2.5.** Suppose the plant is exponentially unstable. Let \( K^j_p \) be the linear periodic variable structure feedback gain that exponentially stabilizes the plant on \( S_N \). Let \( \alpha > 0 \) and \( 0 < \rho < 1 \) be such that for all \( x(k) \in S_N \), \n
\[
|z(j)| \leq \alpha |x(k)| \rho^j
\]

where \( z(j+1) = (A + B\sigma \circ K^j_p)(z(j)) \) \n
Then for all \( x(k) \in S_N \), \n
\[
J(x(k)) \leq \left[ \frac{1}{1-\rho^2} \lambda_{\text{max}}(R)\alpha^2 + \frac{1}{1-\rho^2} \lambda_{\text{max}}(R)ight.

\[
\cdot \left( \max_{j \in J} \|K^j_p\| \alpha^2 + \lambda_{\text{max}}(Q) \eta \|G\| \alpha^2 \rho^2 \right) \|x(k)\|^2.
\]

Similarly to the exponentially stable case, there exist \( a, c > 0 \) such that for all \( x(k) \in C_{\infty} \), \n
\[
a \|x(k)\|^2 \leq J(x(k)), \quad \Delta J(x(k)) \leq -c \|x(k)\|^2.
\]

Again \( J(\cdot) \) is strictly convex and continuous on \( C_{\infty} \) and, as shown in [3], \( S_N \rightarrow C_{\infty} \). Hence, there exists \( S_M \) that contains all the closed-loop trajectories starting from a state in \( S_N \cap B(\gamma) \). To this end, we proved the exponential stability of the closed-loop system on any \( S_N \cap B(\gamma) \).

**Theorem 2.6.** Suppose the plant is exponentially unstable. Then, the closed-loop system is exponentially stable on any \( S_N \cap B(\gamma) \).

**Corollary 2.2.** Suppose the plant is exponentially unstable. Then, the closed-loop system is asymptotically stable on \( C_{\infty} \).

Hence, the mixed constrained linear quadratic optimal control achieves the strongest possible constrained stabilization of an exponentially unstable plant.

Similarly to the marginal case, the output feedback system consisting of the mixed constrained linear quadratic optimal control and an asymptotic observer is exponentially stable on \( D_M(\gamma) \).

### III. FEEDBACK IMPLEMENTATION

Consider the following truncated mixed constrained infinite horizon linear quadratic optimal control problem:

\[
J_N(x(k)) = \min_{\epsilon, \eta} \sum_{j=0}^{N-1} z(j)^T R z(j) \quad + \sum_{j=0}^{N-1} u(j)^T S u(j) + \epsilon^T \eta \quad \text{(P_N)}
\]

subject to (1),

\[
u_{\min} \leq u(j) \leq u_{\max}, \quad j = 0, 1, \ldots, N-1,
\]

\[
G z(j) \leq g + \epsilon, \quad j = 1, 2, \ldots, N-1.
\]

As shown in [12], the problem \( (P_N) \) can be transformed into a finite dimensional problem:

\[
J_N(x(k)) = \min_{\epsilon, \eta} \sum_{j=0}^{N-1} z(j)^T R z(j) \quad + \sum_{j=0}^{N-1} u(j)^T S u(j) + z(N)^T P z(N) + \epsilon^T \eta
\]
subject to (1),

\[ \nu^\text{min} \leq \nu(j) \leq \nu^\text{max}, \quad j = 0, 1, \ldots, N - 1, \]

\[ Gz(j) \leq g + \epsilon, \quad j = 1, 2, \ldots, N - 1, \]

where \( P \) is the solution of (2).

Moreover, the following fact has also been established in [12].

**Fact 3.1.** \((P)\) and \((P_{N})\) are equivalent iff \( z(N) \in O_{w} \).

For feedback implementation of the mixed constrained infinite horizon linear quadratic optimal control, we need to find \( N^{*} \) from a given \( x(k) \) such that \((P)\) and \((P_{N})\) are equivalent. For this, let \( B \) be the largest ball contained in \( O_{w} \). As shown in [2], the radius \( r \) of this ball can be found solving the following problem that can be decomposed into convex quadratic programming problems:

\[ r^{2} = \inf_{x \in \Omega_{w}} x^{T} x = \inf_{x \in \Omega_{w}} x^{T} x = \min_{i} \left\{ \inf_{x \in \Omega_{w}} x^{T} x \right\} \]

where \( \Omega_{w} \) is the complement of the set defined by an inequality for \( O_{w} \).

### 3.1. Exponentially stable plants

As shown in the previous section, for exponentially stable plants, the mixed constrained infinite horizon linear quadratic optimal control is shown to be globally exponentially stable and there exist \( a, b, c > 0 \) such that for any \( x(k) \in \mathbb{R}^{n} \),

\[ a \| x(k) \|^2 \leq J(x(k)) \leq b \| x(k) \|^2, \]

\[ \Delta J(x(k)) \leq -c \| x(k) \|^2. \]

Notice that \( a, b, c \) depend only on the problem parameters and can be computed off-line. Then, for any \( x(k) \in \mathbb{R}^{n} \),

\[ \| z(j) \|^2 \leq \frac{b}{a} \| x(k) \|^2 \left( 1 - \frac{c}{b} \right)^{j}. \]

Since \( 1 - \frac{c}{b} \in (0, 1) \), there exists \( N^{*} \) such that

\[ \frac{b}{a} \| x(k) \|^2 \left( 1 - \frac{c}{b} \right)^{N^{*}} < r^{2}. \]

Then \((P)\) and \((P_{N})\) are equivalent.

We now summarize the feedback implementation for stable plants. We need the off-line computations of \( O_{\infty} \) [7], \( r \) [2] and \( a, b, c \) from the problem data.

**On-line Computation:**

1. If \( x(k) \in O_{\infty} \), implement \( u(k) = -Kx(k) \).
2. Otherwise, find \( N^{*} \), solve \((P_{N})\) and implement the first control input.

### 3.2. Marginal or unstable plants

Since the closed-loop system is not exponentially stable on \( C_{w} \) for marginal or unstable cases, the technique for stable plant case should be applied with the exponential envelop of \( S_{N} \) for sufficiently large \( N \). However, this results in a very conservative result since the exponential envelop diverges as \( N \to \infty \). Hence, for marginal or unstable plants, a set of exponential envelopes for \( S_{N} \) is employed to reduce the conservatism. Although \( x(k) \in S_{N} \), the entire trajectory starting from it doesn’t have to stay within \( S_{N} \) and thus the exponential envelop \( S_{N} \) obtained similarly to the exponentially stable case may not guarantee that \( x(N) \in O_{\infty} \). However, in most cases, such problems do not occur since only the first few states in the closed-loop state trajectory can usually be outside \( S_{N} \) and \( N^{*} \) obtained from the exponential envelop is conservative. In the case that such a problem occurs, there can be many different ways to cope with it. A straightforward way is to use the largest \( N \) we have. However, the solution of the quadratic programming problem for the largest \( N \) is computationally rather demanding. A possibly better strategy is to increase \( N^{*} \) by a certain amount each time until \( x(N^{*}) \in O_{\infty} \) is achieved.

We now summarize the implementation for marginal and unstable plants with the straightforward remedy for \( x(N) \notin S_{N} \) case. The off-line computations are the same as the stable plant case except \( b \) must be computed for each \( S_{N} \).

**On-line Computation:**

1. If \( x(k) \in O_{\infty} \), implement \( u(k) = -Kx(k) \).
2. Otherwise, find the smallest \( N \) such that \( x(k) \in S_{N} \).
3. Using \( b \) associated with \( S_{N} \), find \( N^{*} \).
4. Solve \((P_{N})\). If \( x(N) \notin S_{N} \), implement the first control input. Otherwise solve \((P_{N_{\max}})\) where \( N_{\max} \) is the largest \( N \) available, and implement the first control input.

For smaller \( N \), we have faster decaying exponential envelop. Hence, to further reduce the conservatism, one may modify the step 3) so that a couple of exponential envelopes are used simultaneously when \( N \) is large. For instance, instead of finding \( N^{*} \) directly, find \( N_{1} \) with the exponential envelop for \( S_{N} \) such that \( x(N_{1}) \in S_{N-M} \) and then find \( N_{2} \) with the exponential envelop for \( S_{N-M} \) such that \( x(N_{1} + N_{2}) \in O_{\infty} \). Clearly many alternative modifications are possible. Notice that, for this modified algo-
IV. EXAMPLE

Consider the example in [2]:

\[
\min \sum_{j=0}^{\infty} z^2(j) + \sum_{j=0}^{\infty} u_j^2(j)
\]

subject to

\[
z(j+1) = 2z(j) + u(j),
\]

\[-1 \leq u(j) \leq 1.
\]

As shown in [2], the solution to the Riccati equation associated with the unconstrained problem is

\[P = 4.236\]

and the optimal gain is

\[F = 1.618.\]

Moreover,

\[O_\infty = [-0.618, 0.618], \quad r = 0.618.\]

Notice that \(a = P = 4.236, c = R = 1\) and

\[S_N = C_N = C^{\infty}_N = \left[-1 + \frac{1}{2^N}, 1 - \frac{1}{2^N} \right], \quad C_\infty = (-1,1).\]

Associated with \(C_N\), consider the polyhedral sector \([0.1 - \frac{1}{2^N}]\). Then the vertex is \(x^N = 1 - \frac{1}{2^N}\) and the control that drives \(x^N\) to 0 in \(N\)-steps is \(u^N(j) = -1\) for \(j = 0, \ldots, N-1\). Hence, the exponentially stabilizing linear periodic variable structure state feedback control on \(S_N\) [3] is

\[u_j(x) = \frac{1}{1 - \frac{1}{2^N}} x = \frac{2^N x}{2^N - 1}\]

and thus

\[u(j) = u_j(x) = -\frac{2^N}{2^N - 1} x, \quad 0 \leq j \leq N-1,
\]

\[z(j) = A^j x(k) + \sum_{i=0}^{j-1} A^{j-i-1} u_i(x(k))
\
= \left[2^j - \sum_{i=0}^{j-1} 2^{j-i-1} \frac{2^N}{2^N - 1} \right] x(k)
\]

To this end, \(b\) associated with \(S_N\) is

\[b = \sum_{j=0}^{N-1} \left[2^j - \frac{(2^j - 1)2^N}{2^N - 1}\right]^2 + \sum_{j=0}^{N-1} \left[\frac{2^N}{2^N - 1}\right]^2.
\]

The computed \(N\)'s for several \(x(k)\)'s are summarized together with those predicted by the on-line linear program based algorithm [2] in the following table:

<table>
<thead>
<tr>
<th>(x_0)</th>
<th>Smallest Possible</th>
<th>Algorithm in [2]</th>
<th>Proposed Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>0.62</td>
<td>1</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>0.8</td>
<td>2</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>0.9</td>
<td>3</td>
<td>13</td>
<td>9</td>
</tr>
<tr>
<td>0.95</td>
<td>4</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

Notice that the proposed algorithm identifies less conservative size estimate of the reduced problem with much less computation.

REFERENCES


8. Kalman, R. E., “Contributions to the Theory of Op-

