STABILIZATION OF SWITCHED LINEAR SYSTEMS

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ABSTRACT

This paper addresses stabilization of switched linear control systems with unstabilizable individual switching models. First we consider the stabilization of switched systems via a controllable switching strategy. Then stabilization by both controls and switching laws is investigated. Certain sufficient conditions are obtained.

KeyWords: Switched linear systems, switching strategy, controllability, stabilization.

I. INTRODUCTION

In the last decade, the switched systems have been investigated by a number of researchers [1-3,6]. This problem is not only theoretically interesting but also practically important. Such control systems appear in robot manipulators [9], traffic management [19], power systems and power electronics [22,15]. In the meantime, the switching concerns stochastic process, adaptive stabilization of the switched systems was investigated [11,12].

One way to investigate the stability and stabilization problems is to find a common quadratic Lyapunov function (CQLF) to dominate all the switched models [4]. In addition, some other techniques such as multiple Lyapunov functions, piecewise linear Lyapunov functions etc. have also been investigated to solve the problems [14,20]. The stabilization is closely related to the controllability. In some recent works [17,18,23] and [5], the controllability and stabilizability of switched linear systems have been studied. The results obtained will be used for the latter discussion in this paper.

In this paper, we first investigate the stabilization of a switched linear system (without controls) by a switching strategy. The concepts and computation techniques of the stable region are introduced. Then the normalization of a system is proposed. Using them, we finally give some sufficient conditions for stabilizing a system via both feedback control and switching strategies.

The paper is organized as follows: Second section formulates the problem and presents preliminary analysis. A switching strategy is constructed in the third section to stabilize the system for a given quadratic Lyapunov function. Then a numerical solution for planer systems with general $P > 0$ is also presented. Fourth section investigates the problem of stabilization and provides a constructive design procedure. An illustrating example with simulation is given in fifth section. Sixth section is the conclusion.

II. NOTATIONS AND PRELIMINARIES

Consider a switched linear control system as follows [17].

$$\dot{x}(t) = A_{\alpha(x,t)}x(t) + B_{\alpha(x,t)}u_{\alpha(x,t)}$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ are the states; $\alpha(x,t): \mathbb{R}_n \times \mathbb{R}_+ \to \Lambda := \{1, 2, \ldots, N\}$ is a measurable mapping, it is the switching function to be designed; $u_{\alpha}(t) \in \mathbb{R}^{m_{\alpha}}$ are the inputs, and the matrix pair $(A_{\alpha}, B_{\alpha})$ for $i \in \Lambda$ are switching models of (1).

Given switched system (1), for any initial value $x_0$, the switching sequence can be defined as

$$t_{k+1} = \inf \{t > t_k : \alpha(x(t),t) \neq \alpha(x(t_k),t_k) \} \quad k = 0,1,2,\cdots.$$  \hspace{1cm} (2)

where $t_0 < t_1 < \cdots < t_k < \cdots < \infty$.

With $\delta_k = t_k - t_{k-1}$ denoting the dwell time on the duration $I_k = [t_k - 1, t_k]$, the switching sequence (2) can be uniquely determined by switching function $\alpha(x(t),t)$, and the equality $\alpha(x(t),t) = i$, for $t \in I_k$, $i \in \Lambda$.

Throughout the paper, we assume $(A_i, B_i), i \in \Lambda$ are uncontrollable, otherwise, the controllability and stabiliza-
tion become trivial as the switching law is controllable.

The following notations are necessary for further discussion.

Let $V_i = \langle A_i, B_i \rangle$ be the controllable subspace of the switching model $(A_i, B_i)$, and $\dim V_i = l_i$.

Denote $V = \langle A_1, A_2, \cdots, A_N, B_1, B_2, \cdots, B_N \rangle$ to be the smallest $A_i, i = 1, 2, \cdots, N$, invariant subspace containing all $B_i, i = 1, \cdots, N$, where $B_i$ is a space spanned by the columns of $B_i$.

**Theorem 2.1** For switched linear control system (1), the controllable subspace is the subspace $V$.

**Proof.** It is an immediate consequence of the algorithm (35) of [18].

In the rest of this paper, we consider only the $\dim V = l < n$, i.e., the controllable subspace of the switched system (1) is a subspace of $\mathbb{R}^n$.

We start with the stabilization via a switching strategy.

### III. STABILIZATION VIA SWITCHING STRATEGY

In this section, we consider the stabilization of switched systems via a controllable switching law. Only the state-feedback switching law is used. To begin with, we give the definition for the quadratic stabilization.

**Definition 3.1** Consider a dynamic system

$$
\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{3}
$$

1. (3) is said to be quadratically stable if there exists a positive definite matrix $P$ such that

$$
PA + A^T P < 0. \tag{4}
$$

2. For a given positive definite matrix $P$, the stable region $S_P$ is defined as

$$
S_P(A) = \{0\} \cup \{x \in \mathbb{R}^n \mid x^T (PA + A^T P)x < 0\}. \tag{5}
$$

When the stable region is considered, the topology of projective space $P^n-1(\mathbb{R})$ provides a suitable structure for it. Because if $x \in S_P(A)$, then for any real number $\lambda \neq 0$, $y = \lambda x \in S_P(A)$. Hence, we simply identify them as $y \sim x$. Under this equivalent relation, the quotient space is $P^n-1(\mathbb{R})$, i.e.,

$$
P^n-1(\mathbb{R}) = (\mathbb{R}^n \setminus \{0\}) / \sim.
$$

By definition it is clear that a matrix $A$ is stable iff there exists a $P > 0$, $S_P(A) = \mathbb{R}^n$.

For convenience, we technically remove zero from $S_P(A)$. That is, set

$$
S_P(A) = \{x \in \mathbb{R}^n \mid x^T (PA + A^T P)x < 0\}. \tag{6}
$$

The following example describes the stable region of two matrices.

**Example 3.2** Consider two matrices

$$
A = \begin{pmatrix}
1 & 2 \\
-1 & -3
\end{pmatrix},
B = \begin{pmatrix}
-2 & 1 \\
-0.5 & 3
\end{pmatrix}
$$

For a chosen $P > 0$ we denote

$$
PA + A^T P := \begin{pmatrix} a & b \\
b & c \end{pmatrix}.
$$

As discussed before, we can search stable region of $A$ over $P_1(\mathbb{R})$. It is done as follows: Let $x = (\cos(\theta), \sin(\theta))^T, \theta \in [-\pi/2, \pi/2]$. Then the region satisfies

$$
c \tan^2(\theta) + b \tan(\theta) + a < 0. \tag{7}
$$

The stable region can be obtained by (5) easily.

Choosing $P = I$, then for the above $A$ the solution of (5) is

$$
\theta \in U_A = [37.509551426248^\circ, 90^\circ] \cup [-90^\circ, -23.47330795373600^\circ].
$$

So the stable region of $A$ can be expressed in polar coordinate frame as

$$
S_I(A) = \{(r, \theta) \mid r \in \mathbb{R}, \theta \in U_A\}.
$$

Similarly, for $B$ we have

$$
\theta \in U_B = [-42.11583520926064^\circ, 36.40524207176099^\circ].
$$

Then

$$
S_I(B) = \{(r, \theta) \mid r \in \mathbb{R}, \theta \in U_B\}.
$$

Note that in $S_P(A)$ etc. we allow $r < 0$. We also want to point out that the eigenvalues of $A$ and $B$ are

$$
\sigma(A) = \{0.41421356237310, -2.41421356237309\},
\sigma(B) = \{2.89791576165636, -1.89791576165636\}.
$$

So both $A$ and $B$ are unstable.

The following result is an immediate consequence of the definition.

**Proposition 3.3** Let $A_i, i \in \Lambda := \{1, \ldots, N\}$ be a finite set.
of matrices. The switched system

\[ \dot{x} = A_{\sigma(t)} \]  

is quadratically stabilizable by a state feedback switching law if there exists a \( P > 0 \) such that

\[ \cup_{i \in \Lambda} S_P(A_i) = \mathbb{R}^n. \]

**Proof.** In fact we can choose the quadratic Lyapunov function, \( L(x) = x^TPx \). The state feedback switching law can be chosen as

\[ \sigma(x) = \arg\min \{ x^T(PA_i + A_i^TP)x < 0 \}. \]

(7)

Then it is easy to see that under such a switching law \( \dot{L}(x) \) is a continuous function. Note that since the system is not continuous we still have to show that the system is asymptotically stable. Given any \( \epsilon > 0 \) consider

\[ R = \{ x \mid L(x) \leq L(x_0) \} \]

\( R \) is a compact set, which is invariant with respect to (6). By continuity of \( L, \dot{L} \) can reach its maximum value \( \delta < 0 \). That is,

\[ \dot{L}(x) \leq \delta < 0, \quad x \in R. \]

Therefore, after a certain finite time \( T > 0 \) we have \( x(t) \in R_\epsilon := \{ x \mid L(x) < \epsilon \} \).

For a switched system a serious problem is the vibration. For instance, the switching law may be as: \( \sigma(t_j) = i \) and \( \sigma(t_{j+1}) = j \neq i \), and vise versa. That is, the system will go back and forth between these two models with 0 dwell time. If this kind of vibration occurs, even the existence of the solution is questionable. To avoid this, we have to modify the switching law (7). In the following we will design a new switching law, which will avoid this kind of vibration.

Consider, it is a compact topological space. (The topology on \( P^{n-1}(\mathbb{R}) \) is explained in the following Remark 3.4.) Let

\[ U_i := \{ S_P(A_i) \}, \quad i \in \Lambda. \]

Then \( \{ U_i \mid i \in \Lambda \} \) form an open covering of \( P^{n-1}(\mathbb{R}) \). Since \( P^{n-1}(\mathbb{R}) \) is a normal topological space [10], there exist open sets \( V_i \subset \overline{U_i} \subset \overline{U}_i \) such that

\[ \cup_{i \in \Lambda} V_i \supset P^{n-1}(\mathbb{R}). \]

Since \( \overline{V}_i \subset U_i \), we have

\[ x^T(PA_i + A_i^TP)x < 0, \quad x \in \overline{V}_i, \quad i \in \Lambda. \]

Note that \( \overline{V}_i \) is compact, we can find \( \epsilon_i < 0 \) such that

\[ \max_{i \in \mathbb{F}_i} x^T(PA_i + A_i^TP)x < \epsilon_i < 0, \quad i \in \Lambda. \]

Now we modify the switching law of (7) as follows:

\[ \sigma(x, t_i) = \arg\min \{ x^T(PA_i + A_i^TP)x, k \neq i; \}
\]

\[ x^T(PA_i + A_i^TP)x - \frac{\epsilon_i}{2}, \]

(8)

where \( i \) is the current model, i.e., \( \sigma(x, t) = i \).

Note that under the switching law (8) if \( \sigma(t_0) \) is a newly chosen model, then the system will stay in this model for a considerable time period to “consume” its \( \frac{\epsilon_i}{2} \) “privilege”. To see this, say at a moment \( t_0 \) we have

\[ x^T(t_0)(PA_i + A_i^TP)x(t_0) = x^T(t_0)(PA_i + A_i^TP)x(t_0), \]

and \( \sigma(x, t_0) = i \). Then the system will remain in model \( i \) until another moment \( t_1 \) when

\[ x^T(t_1)(PA_i + A_i^TP)x(t_1) = x^T(t_1)(PA_i + A_i^TP)x(t_1) - \frac{\epsilon_i}{2}, \]

and

\[ x^T((t_1), _i)(PA_i + A_i^TP)x((t_1), _i) < x^T((t_1), _i)(PA_i + A_i^TP)x((t_1), _i) - \frac{\epsilon_i}{2}. \]

This delay in switching avoids vibration.

To see that the switched system is still quadratically asymptotically stable, it is because that

\[ \dot{V} \leq \max_{i \in \mathbb{F}_i} \frac{\epsilon_i}{2} < 0. \]

(9)

**Remark 3.4** All the above arguments for modifying switching law (7) are based on the topology of the projective space \( P^{n-1}(\mathbb{R}) \). To understand it easily, we consider

\[ S^{n-1} = \{ x \in \mathbb{R}^n \mid \| x \| = 1 \} \]

with the inherent topology from \( \mathbb{R}^n \). Then \( S^{n-1} \) is homeomorphic to \( P^{n-1}(\mathbb{R}) \) in a natural way. So in all the above arguments we can simply consider that the points \( x \) are on the sphere \( S^{n-1} \), i.e., \( \| x \| = 1 \).

So for a set, say \( U_\epsilon \), even though we still use the same notation, it represents the quotient set as \( U_\epsilon \sim \), and considered as a subset of \( P^{n-1}(\mathbb{R}) \).

Particularly, if go back to \( \mathbb{R}^n \), (9) becomes
The following example shows the choice of matrix $P > 0$ is a key for switching law stabilization.

**Example 3.5** Consider switched system (6), with two switching models $A_1 = A$ and $A_2 = B$, where $A$ and $B$ are as in the Example 3.2. It is known that for $P = I$, $S(A) \cup S(B) \neq \mathbb{R}^2$. But if we choose

$$P = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

then we have

$$U_{A_1} = (0^\circ, 90^\circ) \cup [-90^\circ, -30.96375653207352^\circ),$$

and

$$U_{A_2} = (-38.56696195309441^\circ, 29.68630280257416^\circ).$$

Hence $S_p(A) \cup S_p(B) = \mathbb{R}^2$. According to Proposition 3.3 the system is stabilizable by state feedback switching law.

Note that we mentioned in the Example 3.2 that both $A$ and $B$ have a negative eigenvalue. A natural question is that if a set of matrices have only positive real part eigenvalues, is it possible to find a suitable switching law to stabilize the switched system with them as the switching models? The following example shows it is still possible.

**Example 3.6** Consider the switched system (6), with six switching models defined as follows. Let

$$A_0 = \begin{pmatrix} 0 & 1 \\ -10 & 11 \end{pmatrix}, \quad T = \begin{pmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$$

Then we can define $A_i$ as

$$A_i = T^{-i+1} A_0 T^{i-1}, \quad i = 1, 2, 3, 4, 5, 6.$$

Still choose

$$P = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

we can have

$$U_{A_{1i}} = (0^\circ, 39.28940686250036^\circ)$$

$$U_{A_{2i}} = (30^\circ, 69.28940686250036^\circ)$$

$$U_{A_{3i}} = (60^\circ, 90^\circ) \cup [90^\circ, -80.71059313749966^\circ)$$

$$U_{A_{4i}} = (-90^\circ, -58.18149998472269^\circ)$$

The eigenvalues of $A_i$ are 10 and 1, but since

$$\bigcup_{i=1}^6 S_p(A_i) = \mathbb{R}^2$$

the switched system is still state feedback switching law stabilizable.

In fact, it is very difficult to find the necessary condition for quadratic stabilization of switched systems. But for planar systems a numerical solution can be obtained by searching. Note that for any positive definite matrix $P > 0$, and $Q = cP$, $c > 0$ we have

$$S_p(A) \cup S_p(B) = \mathbb{R}^2.$$
Table 1. Feasible $P$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\theta$</th>
<th>$S_p(A_1)(\gamma)$</th>
<th>$S_p(A_2)(\gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.199</td>
<td>(44.73, 90) $\cup$ (−90, −49.44)</td>
<td>(−51.32, 65.40)</td>
</tr>
<tr>
<td>0.2</td>
<td>2.127</td>
<td>(59.80, 90) $\cup$ (−90, −45.57)</td>
<td>(−61.94, 65.46)</td>
</tr>
<tr>
<td>0.2</td>
<td>1.885</td>
<td>(60.46, 90) $\cup$ (−90, −51.24)</td>
<td>(−59.93, 65.21)</td>
</tr>
<tr>
<td>0.2</td>
<td>2.199</td>
<td>(42.88, 90) $\cup$ (−90, −45.99)</td>
<td>(−47.96, 65.53)</td>
</tr>
<tr>
<td>0.3</td>
<td>1.257</td>
<td>(53.26, 90) $\cup$ (−90, −43.53)</td>
<td>(−46.32, 65.90)</td>
</tr>
<tr>
<td>0.3</td>
<td>1.571</td>
<td>(56.12, 90) $\cup$ (−90, −46.25)</td>
<td>(−54.79, 65.51)</td>
</tr>
<tr>
<td>0.3</td>
<td>1.885</td>
<td>(52.00, 90) $\cup$ (−90, −46.41)</td>
<td>(−51.68, 65.50)</td>
</tr>
<tr>
<td>0.3</td>
<td>2.199</td>
<td>(41.41, 90) $\cup$ (−90, −43.27)</td>
<td>(−43.65, 65.68)</td>
</tr>
<tr>
<td>0.7</td>
<td>0</td>
<td>(56.12, 90) $\cup$ (−90, −46.25)</td>
<td>(−54.79, 65.51)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.314</td>
<td>(52.00, 90) $\cup$ (−90, −46.41)</td>
<td>(−51.70, 65.50)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.628</td>
<td>(41.41, 90) $\cup$ (−90, −43.27)</td>
<td>(−43.65, 65.68)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.827</td>
<td>(53.26, 90) $\cup$ (−90, −43.53)</td>
<td>(−46.32, 65.90)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.314</td>
<td>(60.46, 90) $\cup$ (−90, −51.24)</td>
<td>(−59.93, 65.21)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.628</td>
<td>(43.88, 90) $\cup$ (−90, −45.99)</td>
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<tr>
<td>0.9</td>
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<td>(59.80, 90) $\cup$ (−90, −45.57)</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.628</td>
<td>(44.73, 90) $\cup$ (−90, −49.44)</td>
<td>(−51.32, 65.40)</td>
</tr>
</tbody>
</table>

Next two assumptions are given for further investigation.

**H1.** The second part of system (12) is stabilizable by a state feedback switching law.

**H2.**

$$V = V_1 + V_2 + \cdots + V_n$$

where $V_i$ is the controllable subspace of $i$-th switching model.

For a given state $x$, if the $i$-th switching model is active, i.e., $\sigma(x) = i$, we can further split (12) into

$$\begin{align*}
\dot{x}_1 &= A_1x_1 + A_{12}x_2 + A_{13}x_3 + Bu_1, \quad x_1 \in \mathbb{R}^i \\
\dot{x}_2 &= A_2x_2 + A_{23}x_3, \quad x_2 \in \mathbb{R}^{n_i} \\
\dot{x}_3 &= C_3x_3, \quad x_3 \in \mathbb{R}^{n_3}
\end{align*}$$

where $x_1 \in V_1$, $(x_2', x_3') \in V_2$, and $x_3 = z_2$. We may regard (12) as a general form and (13) as a special decomposition form for $i$-th particular switching model.

**Remark 4.1** The expression (13) is particularly for $i$-th switching model. So the expression (13), or even the dimension of $x_1'$, is varying from model to model.

The following lemma is basically the squashing Lemma [13] with precise boundary estimation [8]. The proof is similar to the proof of Lemma 4.1 of [7].

**Lemma 4.2** Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be two matrices such that the pair $(A, B)$ is completely controllable. Then for any $\lambda > 0$, there always exists a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$\|e^{(A-B)K}\| \leq M \lambda^2 e^{-\lambda t}, \quad \forall t \geq 0$$

where $L = \frac{(n-1)(n+2)}{2}$ and $M > 0$ is a strictly computable constant depending only on $A, B$ and $n$, where and hereafter $\|\cdot\|$ denotes the Euclidean matrix norm on $\mathbb{R}^n$.

The Lemma shows that the convergence rate of linear controllable system to the origin can be dominated arbitrarily by properly designed control laws.

According to (14), for a given $t$, we have

$$\|e^{(A-B)K}\| \leq M \lambda^2 e^{-\lambda t} \leq \alpha e^{-\gamma t},$$

where $\gamma = \frac{\lambda}{2}, \alpha \geq M \left(\frac{2L}{t}\right)^{\frac{1}{2}}$ is independent of $\gamma$.

Under the assumptions H2, there exist state feedback controls

$$u_i(t) = F_i x_i(t), \quad i = 1, 2, \cdots, N,$$

such that (13) becomes

$$\begin{align*}
\dot{x}_1' &= (A_{11} + BF_1)x_1' + A_{12}x_2' + A_{13}x_3, \quad x_1' \in \mathbb{R}^i \\
\dot{x}_2' &= A_{22}x_2' + A_{23}x_3, \quad x_2' \in \mathbb{R}^{n_i} \\
\dot{x}_3' &= C_3x_3, \quad x_3' \in \mathbb{R}^{n_3}
\end{align*}$$

According to Lemma 4.2, there exists a negative real number $\gamma$ such that $\|e^{(A_{11}+BF_1)K}\| \leq \alpha e^{-\gamma t}$. Let $\sigma^i_2$ be the maximum value of the real part of eigenvalues of $A_{22}$, then we set $\sigma_2 = \max_{A_{22}} \{\sigma^i_2\}$, and also $\sigma_1$ is the maximum value of real part of $A_{11}$, then $\|e^{\sigma_1 K}\| \leq c e^{\sigma_1 T}$. Now let

$$\gamma = \min_{\alpha \in A} \{\gamma\},$$

$$c = \max\{a_i, c_i, d_{i, 1}, |i = 1, \ldots, N\},$$

$$p = \max\{\|A_2\|, \|A_1\|, \|A_3\|, 1, |i = 1, \ldots, N\}.$$

The idea to stabilize the system is as follows: First, according to the assumption H1, $x_3$ can be compressed as small as designed by suitable switching laws. Note that $x_1'$ can converge to the origin as rapidly as we wish, this is because $x_1'$ is controllable, then according to H2, $\cup_{i \in A} \{x_1' = \{z_1\}_i\}$, and while $z_1$ converges to the origin in a short time duration, $x_1 = z_2$ has only a slight change. Continue this procedure, finally we have $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$. Consequently $x(t)$ is stabilized.

The following is the main result.

**Theorem 4.3** Under the assumptions H1 and H2, system (12) is asymptotically stabilizable via both state feedbacks and switching laws.
Proof. Step 1. We prove that for any \( \varepsilon > 0 \) there exist controls, which drive the trajectory to \( \|z\| < \varepsilon \) a proper switching path.

Since \( z_i \) (i.e., \( x_i \)) is stabilizable by a switching law, choose a switching law such that at certain moment \( t_i \),
\[
\|x_i(t_i)\| \leq \frac{\varepsilon}{2N}. \quad \text{Choose } \delta_i > 0 \text{ such that } \|e^{\delta_i t_i}\| < 2, \forall i \in \Lambda \text{ as } t \leq \delta_i. \quad \text{Since } \|e^{\delta_i t_i}\| \rightarrow 1, \text{ the existence of } \delta_i \text{ is assured by continuity. Let } \delta_3 \text{ be universal dwell time, a sequence } t_0, i = 1, 2, \ldots, N + 1 \text{ is defined as } t_{i+1} - t_i = \delta_3. \quad \text{Then we have } \|x_i(t_{i+1})\| < 2\|x_i(t_i)\|. \quad i = 1, 2, \ldots, N. \quad \text{It follows that}
\[
\|x_i(t_{i+1})\| = \|x_i(t_i) + N\delta_3\| \leq \frac{\varepsilon}{2}. \quad (18)
\]

Next, we fix a switching law for \( t_1 \leq t \leq t_{N+1} \) as
\[
\sigma(t) = i, \quad t_i \leq t < t_{i+1}, \quad i = 1, \ldots, N. \quad (19)
\]

A straightforward computation for the second equation of the system (17) yields
\[
x_i'(t) = e^{\delta_i(t-t_i)}x_i(t_i) + \int_{t_i}^t e^{\delta_i(t-s)}A_{ii}x_i(s)ds,
\]
for \( t_i \leq t < t_{i+1}, \quad k \neq i. \)

Since \( \|x_i(t)\| \leq \frac{\varepsilon}{2}, \quad t \in [t_i, t_i + N\delta_3], \) we can find \( 0 < \delta_2 \leq \delta_3 \) such that \( \|x_i'(t_{i+1})\| \leq 2\|x_i'(t_i)\|, \quad k \neq i. \)

Then we have
\[
\|x_i'(t)\| \leq 2^{k-i+1}\|x_i'(t_i)\|, \quad i \geq k > s \text{ or } k > s \geq i+1. \quad (20)
\]

Finally, we consider \( x_i' \).

According to (18) and (20), we compute the first equation of system (17)
\[
\|x_i'(t)\| \leq c e^{\rho(t-t_i)}\|x_i'(t_i)\| + p_{ii}^{\frac{1}{2}} e^{2\rho(t-t_i)}(2N)\|x_i'(t_i)\| + \frac{\varepsilon}{2}dt
\]

By lemma 4.2, \( c \) is independent of \( i \), so we can choose a suitable \( F_i \) such that for small \( 0 < \delta_1 \leq \delta_2 \leq \delta_3 \), we have
\[
\|x_i'(t_{i+1})\| < \frac{\varepsilon}{2^{N-i+1}\sqrt{i}} \quad (21)
\]

where \( t_{i+1} - t_i = \delta_3, \quad i = 1, 2, \ldots, N. \) Note that as \( k > i \), we have \( x_i' \subset \{x_i^k, x_i^k\} \), it is easy to see that we still have
\[
\|x_i'(t_{i+1})\| \leq 2\|x_i'(t_i)\|, \quad k > i.
\]

Hence we conclude that
\[
\|x_i'(t_{N+1})\| \leq \frac{\varepsilon}{2^{N-i}}. \quad (22)
\]

Note that all the components of \( x_i' \) satisfies (22). Since \( V = V_1 + \ldots + V_N \), and \( V = \text{Span} \{z_i\} \), it follows that for any one component \( z_i' \), \( s \in \{1, \ldots, l\} \), there exists a \( 1 \leq i \leq N \), such that \( z_i' \) is a component of \( x_i' \). Hence \( z_i' \) satisfies (22).

Therefore,
\[
\|z_i(t_{N+1})\| = \sqrt{\sum_{i=1}^{N} (z_i')^2} \leq \frac{\varepsilon}{2}
\]

Combining it with (18), we have
\[
\|z(t_{N+1})\| < \varepsilon.
\]

Step 2. For the above \( \varepsilon \), we take \( \frac{\varepsilon}{2^q}, \quad q = 1, 2, \ldots \) respectively, repeat the above process, we obtain the corresponding switching sequences such that along them
\[
\|z(T_q)\| < \frac{\varepsilon}{2^q}, \quad q = 1, 2, \ldots
\]

where \( T_1 = t_{N+1} := t_i^1_{N+1} \) as obtained in Step 1. Correspondingly, we can find at iteration \( \sigma \) the \( t_i^N_{N+1} \), and set
\[
T_{q+1} - T_q = t_i^N_{N+1} > 0, \quad q = 1, 2, \ldots
\]

Consequently,
\[
\lim_{q \to \infty} \|z(t)\| = \lim_{q \to \infty} \|z(T_q)\| = 0.
\]

\[\square\]

Remark 4.4 In fact, the proof of this theorem provides a procedure for design of the controllers.

Remark 4.5 In practice, a fixed switching circle as (19) is not necessary. A more effective way is to choose \( i' \) for \( \|x_i'(t)\| = \max_i \|x_i'(t)\| \) at each switching moment.

V. ILLUSTRATING EXAMPLE

Consider a switched system
\[
\dot{x} = A_i x + B_i u_i, \quad i = 1, 2
\]
where matrices \( A_i, B_i \) are as follows:
\[
A_i = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix}, \quad B_i = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
Under a suitable coordinate frame, system (23) can be transformed into the form of system (12) as follows:

\[
\dot{x} = \bar{A}_i x + \bar{B}_i u_i, \quad i = 1, 2,
\]  

where \((\bar{A}_i, \bar{B}_i)\) and \((\bar{A}_i', \bar{B}_i')\) are as follows:

\[
\bar{A}_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & -1 & -3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix},
\]

\[
\bar{A}_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -0.5
\end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

For system (24), using the conclusion of Example 3.2, matrices

\[
\begin{bmatrix}
1 & 2 \\
-1 & -3
\end{bmatrix}, \quad \begin{bmatrix}
-2 & 1 \\
-0.5 & 3
\end{bmatrix}
\]

are stabilizable under a suitable switching law, i.e., assumption \(H1\) holds. In addition, it is easy to compute \(V = \text{Span}\{\bar{B}_1, \bar{B}_2\}\), which is \(\bar{A}_1, \bar{A}_2\) invariant subspace, and \(\dim (V) = 3\). Clearly, \(V_1 = \text{span}\{\bar{A}_1 | \bar{B}_1, \text{Span}\{\bar{B}_1\}\}, V_2 = \text{span}\{\bar{A}_2 | \bar{B}_2, \text{Span}\{\bar{B}_2\}\}\), then we have

\[V = V_1 + V_2.\]

Hence Assumption \(H2\) also holds. According to Theorem 4.3, this switched system is stabilizable.

Now we set initial state \(x(0) = (-1, -4, 2, -2, 3)\), stabilizing state feedbacks and switching laws can also be obtained accordingly.

The simulation results are illustrated as follows.

**VI. CONCLUSIONS**

In this paper the stabilization of switched linear systems was considered. First, the stabilization via state feedback switching law was investigated. Using the stability region of a given quadratic Lyapunov function, the design technique for switching law was presented. Then the stabilization via both switching law and controllers was considered. Based on a normal decomposition form, a two step stabilizer was designed. First step, the switching law is implemented to stabilize uncontrollable states, and in the second step controls are used stabilize the controllable states.

The method is basically a high gain control, so the robustness is a key point. It is left for further study.

**REFERENCES**