IMPLICIT TRIANGULAR OBSERVER FORM DEDICATED TO A SLIDING MODE OBSERVER FOR SYSTEMS WITH UNKNOWN INPUTS

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ABSTRACT

It has been presented in previous works that every uniformly observable single-output system can be put on a triangular observation form. For this structure a special kind of sliding mode observer has been designed by authors, which ensures a finite-time state reconstruction using a step by step observation algorithm. In this paper, we show that the multi-output case is more delicate to study especially when the system has some unknown inputs. Thus, in order to generalize the triangular observer form, from single to multi-output case, we define an Implicit Triangular Observer (ITO) form. For such a form, two results are given. Firstly, we design a finite time converging observer for all values of the unknown inputs. Secondly, we give the necessary and sufficient condition, including a matching condition, for the existence of a coordinate change to put the system into this form. It is also shown that this class of systems is a subset of the uniform observable class of systems.

KeyWords: Sliding mode observer, multi-output systems, unknown inputs, Implicit Triangular observer form.

I. INTRODUCTION

The techniques employed for the observation of the systems with unknown inputs are applicable to many problems ([33]) such as the fault detection and disturbance rejection. We can distinguish two approaches: An approach which consists in observing the unknown inputs or at least in determining the thresholds such as in the diagnosis as described in [8,9,13,15,17,18,24,29] for the linear and time variant case and [16,20,35] for the bilinear case. The other approach consists in observing the states of the system in the presence of unknown inputs which can be regarded as disturbances.

The present work deals with the problem of the observation of systems with unknown inputs in the multi-output case. The objective is to obtain an observer converging for all possible values of bounded unknown inputs. We treat on the case where the system can be put on the “ITO(Implicit Triangular Observer) form”. For such a form we construct an observer which carries out the objective of convergence defined above and we give the matching condition.

To solve this problem, we use a step by step sliding mode observation algorithm which generalize the idea of the geometrical linearization of the observation error dynamics. In fact, the idea to linearize the observation error dynamics presented in [21,22], is developed by the authors for the output and output derivative injection form ([7]), and a step by step sliding mode observer with linear error dynamics was introduced. The works [31,32], gave the conditions under which a nonlinear system can be put with a coordinate change on the so-called “output injection form”.

Always for a single output, and using sliding mode method to obtain a finite time convergence instead of an exponential one as in [28], in [2] the authors gave a triangular observation form which is a generalization of the output injection form [21]. This work will be recalled in section 2 of the present paper. Thus, in this paper a more general form is introduced : the ITO form.

The paper is organised as follows : In the 2nd section we present the single output triangular form, the matching condition is given for this case, and the diffi-
ulty to generalize it to the multi-output case on the basis of an example. Section 3 is devoted to the triangular implicit observer form. In the next Section we construct a sliding mode observer for such a form.

This observer converges in finite time independently from the input. In Section 5, we set the necessary and sufficient conditions under which we obtain the ITO form. In Section 6, we give the matching condition in the multi-output case, i.e. we give the geometrical conditions which must verify a nonlinear system with unknown inputs for the existence of a local coordinate change which put it on the triangular implicit form with a total rejection of the unknown inputs.

II. SOME RECALLS ON TRIANGULAR FORM

2.1 Single output case

In [3,6], the authors show that all uniformly observable system can be put on a special form called the single-output triangular form:

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \bar{g}_1(\xi_1, u) \\
\dot{\xi}_2 &= \xi_3 + \bar{g}_2(\xi_1, \xi_2, u) \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n + \bar{g}_{n-1}(\xi_1, \ldots, \xi_{n-1}, u) \\
\dot{\xi}_n &= \bar{f}_n(\xi) + \bar{g}_n(\xi, u)
\end{align*}
\]

(1)

where \( y = \xi_1 \) and with \( \bar{g}_i(., u = 0) = 0 \) for any \( i \in \{1, \ldots, n\} \). We remark that this form is close to the parametric strict-feedback form introduced in the adaptive control context ([19,23]) and it is especially similar to the triangular form used in [4].

Moreover, it is proved in [14] that a single output system

\[
\begin{align*}
x' &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

(2)

can be put on the triangular form (1) if and only if it is uniformly observable.

The particularity of such a form is that it is possible to define a step by step observation algorithm allowing the reconstruction of all the state with finite-time convergence.

Therefore, using the sliding mode properties of finite time convergence, in [3,6], on the basis of Drakunov and Utkin previous work [10], we introduce a sliding mode observer dedicated to this triangular single-output system. This sliding mode observer consists in a step by step convergence algorithm.

Thus, if the system is uniformly observable, in \( (n - 1) \)-steps we obtain a convergence of all the observer state components.

More precisely in [2], we have presented a new type of sliding mode observer under the BIBS assumption in finite time\(^2\).

This observer allows to obtain a uniform stable observable error dynamics for all the systems of the form (1). To illustrate our purpose, we use a step by step sliding mode observer for the system (1) which authorizes a finite time convergence of the observation error at each step.

Applying the results of [2] of those of Drakunov et al. in [10], we have the following observer for the system (1):

\[
\begin{align*}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \bar{g}_1(\hat{\xi}_1, u) + \lambda_1 \text{sign}(\hat{\xi}_1 - \xi_1) \\
\dot{\hat{\xi}}_2 &= E_1 \left[ \hat{\xi}_3 + \bar{g}_2(\hat{\xi}_1, \hat{\xi}_2, u) + \lambda_2 \text{sign}(\hat{\xi}_2 - \xi_2) \right] \\
&\vdots \\
\dot{\hat{\xi}}_{n-1} &= E_{n-2} \left[ \hat{\xi}_n + \bar{g}_{n-1}(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_{n-1}, u) + \lambda_{n-1} \text{sign}(\hat{\xi}_{n-1} - \xi_{n-1}) \right] \\
\dot{\hat{\xi}}_n &= E_{n-1} \left[ \bar{f}_n(\hat{\xi}) + \bar{g}_n(\hat{\xi}, u) \right] + \lambda_n \text{sign}(\hat{\xi}_n - \xi_n)
\end{align*}
\]

(3)

where \( \hat{\xi}_i = \hat{\xi}_i + E_{i-1} \lambda_{i-1} \text{sign}(\hat{\xi}_{i-1} - \xi_{i-1}) \) for \( i = 2, \ldots, n - 1 \) (by convention \( \hat{\xi}_1 = \xi_1 \)), with \( \text{sign}_{av} \) is an averaging value of the discontinuous \( \text{sign} \) function which can be obtained using a low pass filter whose high cutoff frequency preserves the slow component of the motion but is small enough to eliminate the high frequency well known under the name "chattering phenomenon" on the sliding surface. The need of a low pass filter is due to the fact that in the Filippov’s resolution method, the discontinuous \( \text{sign} \) function, which is not defined at zero, generates an infinite frequency chattering around its equivalent value \( \text{sign}_{av} \). Thus, to eliminate such a phenomenon, which becomes in practice unacceptable, a low pass filter is introduced (see also [10]). The \( E_i \) are used for an anti-peaking strategy\(^3\).

This anti-peaking strategy comes from the idea that we do not inject the observation error information before reaching the sliding manifold linked with this information. More precisely \( E_i = 0 \) if there exists \( j \in \{1, i\} \) such that \( \hat{\xi}_j - \xi_j \neq 0 \), else \( E_i = 1 \). In the observer structure, these functions allow \( \hat{\xi}_j - \xi_j \) to converge to zero if all the \( \hat{\xi}_j - \xi_j \) with \( j < i \) have converged to zero before.

---

\( ^1 \) Bounded Input Bounded State

\( ^2 \) We assume that the BIBS condition is verified by the nonlinear system, the aim of the paper is not to characterize the systems which are BIBS.
Moreover, at each step, only a sub-dynamic of dimension one is assigned. Let us explain briefly this step by step convergence.

In fact, at step 1, the sliding term is only injected in the dynamics of $\hat{\xi}_1$, thus if $\lambda_1 > |\hat{\xi}_2 - \hat{\xi}_2^\ast|$, we have convergence in finite time of $\hat{\xi}_1$ to $\xi_1$. Therefore, using the equivalent vector [30], we obtain $\hat{\xi}_2 = \hat{\xi}_2^\ast + \lambda_1 \text{sign}(\hat{\xi}_1 - \hat{\xi}_1^\ast) \equiv \hat{\xi}_2^\ast$. After this first step, we obtain the real value of $\xi_2$. Iterating this argumentation, we obtain at the step 2, $\hat{\xi}_3 = \hat{\xi}_3^\ast$, and finally, at the step $n - 1$, $\hat{\xi}_n = \hat{\xi}_n^\ast$. The following theorem summarizes this result:

**Theorem 1.** ([2]) Considering the BIBS system (1) and the observer (3), for any bounded initial state $\xi(0)$, $\hat{\xi}(0)$ and any bounded input $u$, there exists $\lambda_i$ such that the observer state $\hat{\xi}$ converges in finite time to $\xi$.

**Proof.** See ([2]).

**Example 1.** Let us consider the following system $\Sigma$ which is in the triangular observer form

$$
\begin{align*}
\dot{x}_1 &= x_2 - x_1^3 u \\
\dot{x}_2 &= x_3 + x_2 x_1 u \\
\dot{x}_3 &= -3x_3 - 3x_2 - x_1 - x_3^3 - u \\
y &= x_1
\end{align*}
$$

(4)

For this system the observer (3) takes the form

$$
\begin{align*}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 - x_1^3 u + \lambda_1 \text{sign}(x_1 - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \hat{\xi}_2 x_1 u + \lambda_2 \text{sign}_1(\hat{\xi}_2 - \hat{\xi}_2) \\
\dot{\hat{\xi}}_3 &= -3\hat{\xi}_3 - 3\hat{\xi}_2 - x_1 - x_3^3 - u + \lambda_3 \text{sign}_2(\hat{\xi}_3 - \hat{\xi}_3) \\
y &= \hat{x}_1
\end{align*}
$$

(5)

with

$$
\hat{\xi}_2 = \hat{\xi}_2 + \lambda_1 \text{sign}(x_1 - \hat{\xi}_1)
$$

and

$$
\hat{\xi}_3 = \hat{\xi}_3 + \lambda_2 \text{sign}_1(\hat{\xi}_2 - \hat{\xi}_2),
$$

where $\text{sign}_i$ functions are designed as noted in section 3.

This approach has been tested by simulation with the following initial conditions $x = (1, 0.5, 0.5)^T$ and $\hat{x} = (0, 0, 0)^T$. Moreover, we have chosen a first-order low pass filter with a cut frequency equal to 100Hz and observation gain $\lambda_1, \lambda_2$ and $\lambda_3$ respectively equal to 4, 2 and 2. Moreover the function “sign” is approximated by a saturation function with a slow rate equal to $10^4$.

- In Fig. 1, $\hat{x}_1$ reaches $x_1$ in finite time $= 0.25s$.
- In Fig. 2, $\hat{x}_2$ also reaches $x_2$ in finite time $= 0.75s$.

But $\hat{x}_2$ will only reach $x_2$ when $\hat{x}_1$ will be equal to $x_1$.
- In Fig. 3, $\hat{x}_3$ reaches $x_3$ in finite time $= 1s$.

Now, starting from the same initial conditions, we add an output noise in order to show the behavior of the
observer. Following the work of Yaz and Azemi ([34]), in [3] we have proposed to use a saturation function with a dead zone for the observer in the case of the extended injection form. This reduces the observer sensitivity to the noise, but we were obliged to change the observer gain as follows $\lambda_1 = \lambda_2 = \lambda_3 = 4$ in order to recover a time response quite similar to the previous simulation.

- In Figs. 4, 5, 6 and 7, we see that the observer state $\hat{x}$ reaches the neighborhood of the system state $x$ in finite time. But we also see that the noise is not totally suppressed in the observer.

We can reduce this noise with some minor modifications by introducing an asymptotic gain or a sign function modified with respect to the noise output knowledge [34] for example.

2.2. Observer matching condition (OMC)

An immediate consequence of the previous step by step observation algorithm is the convergence of the proposed sliding mode observer independently from the $\xi$ dynamics.

If in this dynamics, some unknown terms are present, it does not change anything in the observation results if the states $\xi_i (i = 1, \ldots, n)$ stay bounded. So we have ([1,33]):

**Proposition 1.** Let us consider the system

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\vdots \\
\dot{\xi}_n \\
\end{pmatrix} = 
\begin{pmatrix}
\xi_2 + g_1(\xi_1, u) \\
\xi_3 + g_2(\xi_1, \xi_2, u) \\
\vdots \\
g_{n-1}(\xi_1, \xi_2, \ldots, \xi_{n-1}, u) \\
g_n(\xi_1, \xi_2, \ldots, \xi_n, u) \\
\end{pmatrix} 
\]

where $y = \xi$, $g_i (\cdot, 0) = 0$ for any $i \in \{1, \ldots, n\}$ and $w$ is an unknown bounded input. Then, for any bounded initial state $\xi(0)$, $\xi(0)$ and any bounded inputs $u$ and $w$, there exists $\lambda$, so that the observer state $\hat{\xi}(3)$ converges in finite time to $\xi$. 

\[
\begin{align*}
0 & = \xi_1 \\
0 & = \xi_2 \\
0 & = \xi_3 \\
\vdots \\
0 & = \xi_n \\
\end{align*}
\]
In [3], conditions were given to obtain the form (6) starting from nonlinear single input single output affine system using a local diffeomorphism.

**Proposition 2.** Let us consider the BIBS system

\[
\dot{x} = f(x) + g(x)u + l(x)w \\
y = h(x)
\]

where \(x\) is the state, \(y\) the output, \(w\) is the unknown input and \(u\) the known input, can be put on the form (6) using a local coordinate change if and only if:

- The unperturbed system ((2) with \(w = 0\)) is uniformly observable with respect to the known input \(u\),
- The relative degree of the unknown input \(w\) is equal to \(n - 1\),

**Proof.** The proof of this proposition is obvious using the results of [14] on the relationship between the uniform observability and the triangular form and the Theorem 1. \[\Box\]

**Remark 1.** The second item is called the observer matching condition. This condition is completely different from the well known matching condition for the control case given in [11].

The generalization of the form (6) to the multi-output case is one of the main purpose of this paper. For this, it is necessary to define a new step by step sliding mode observation algorithm.

The main problem in the multi-output case is the definition of the triangular observer form. This is due to the fact that, firstly, states information are not all the same derivatives outputs. Secondly, we do not want to use the input derivatives in our observer, thus, it is necessary to define a new step by step sliding mode observation algorithm.

**2.3. Multi-output case introduction example**

A trivial idea is to define an explicit triangular observer form as the simplest generalization of the single-output triangular form defined in (1). This gives

\[
\begin{align*}
\dot{x}_{i,1} & = x_{i,2} + g_1(x_{i,1})u \\
\dot{x}_{i,2} & = x_{i,3} + g_2(x_{i,1}, x_{i,2})u \\
\vdots & \vspace{2em} \\
\dot{x}_{i,p-1} & = x_{i,p} + g_{p-1}(x_{i,1}, \ldots, x_{i,p-1})u \\
\dot{x}_{i,p} & = f_u(x) + g_{p}(x)u \\
y_i & = x_{i,1}, \quad i = 1, \ldots, p \quad j = 1, \ldots, p
\end{align*}
\]

where \(x \triangleq (X_1^T, X_2^T, \ldots, X_p^T)^T \in \mathbb{R}^n\), \(X_i \triangleq (x_{i,1}, x_{i,2}, \ldots, x_{i,p})^T \in \mathbb{R}^p\) is the state, \(u \in \mathbb{R}^m\) is the input, \(y \in \mathbb{R}^p\) is the output, \(f, g\) are analytic function vectors of appropriate dimensions and the numbers \(\mu\) are the observability indices and are defined for example in [21]. Unfortunately this form is not sufficiently general.

Moreover, the system (8) affected by the perturbation \(w\) verify the OMC\(^3\) can be written as:

\[
\begin{align*}
\dot{x}_{i,1} & = x_{i,2} + g_1(x_{i,1})u \\
\dot{x}_{i,2} & = x_{i,3} + g_2(x_{i,1}, x_{i,2})u \\
\vdots & \vspace{2em} \\
\dot{x}_{i,p-1} & = x_{i,p} + g_{p-1}(x_{i,1}, \ldots, x_{i,p-1})u \\
\dot{x}_{i,p} & = f_u(x) + g_{p}(x)u + l(x)w \\
y_i & = x_{i,1}, \quad i = 1, \ldots, p \quad j = 1, \ldots, p
\end{align*}
\]

where \(w\) is the unknown input and \(l\) is analytic function vectors of appropriate dimensions.

Unfortunately this form is not sufficiently general again as it is shown in the example hereafter.

**Example 2.** Let us consider the uniformly observable system

\[
\begin{align*}
\dot{x}_1 & = x_2 + \phi_1(x_1, x_2)u \\
\dot{x}_2 & = x_3 + \phi_2(x_1, x_2, x_3)u \\
\dot{x}_3 & = \phi_3(x)u + l_3(x)w \\
\dot{x}_4 & = \phi_4(x)u + l_4(x)w \\
y_i & = x_i, \quad i = 1, \ldots, p
\end{align*}
\]

where \(x = (x_1, x_2, x_3, x_4, x_5)^T\). Using a step by step algorithm, we obtain:

\[
\begin{align*}
\dot{x}_2 & = \dot{x}_3 + \phi_1(x_1, x_2, x_3, x_4)u + \lambda_1 \text{sign}(x_1 - \hat{x}_1) \\
\dot{x}_3 & = \dot{x}_4 + \phi_2(x_1, \hat{x}_2, x_3, x_4)u + \lambda_2 \text{sign}(x_3 - \hat{x}_3) \\
\dot{x}_4 & = \dot{x}_5 + \phi_3(x_1, \hat{x}_2, \hat{x}_3, x_4)u + \lambda_3 \text{sign}(x_4 - \hat{x}_4) \\
\dot{x}_5 & = \dot{x}_6 + \phi_4(x_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, x_5)u + \lambda_4 \text{sign}(x_5 - \hat{x}_5)
\end{align*}
\]

The observer parameters are assumed to be well defined as in [3], where the auxiliary states are defined by:

\[
\tilde{x}_2 = \tilde{x}_2 + E_{\lambda} \hat{\lambda}_1 \text{sign}(x_1 - \hat{x}_1), \\
\tilde{x}_3 = \tilde{x}_3 + E_{\lambda} \hat{\lambda}_2 \text{sign}(x_3 - \hat{x}_3)
\]

\(^3\) Observer Matching Condition.
and
\[ \tilde{x}_i = \dot{x}_i + E_i A_0 \text{sign}(x_i - \tilde{x}_i), \]
while the commutation parameters \( E_i \) are defined such as:
\[ E_1 = 1 \text{ if } x_1 = \hat{x}_1 \text{ otherwise } E_1 = 0; \text{ and } E_2 = 1 \text{ if } (\tilde{x}_2 = \hat{x}_2, \ x_3 = \bar{x}_3 \text{ and } E_1 = 1), \text{ otherwise } E_1 = 0. \]

At step 1, from \( \dot{x}_1 \) we can obtain \( x_2 \) as \( x_2 = \hat{x}_1 - \phi_1(x_1, x_4)u \); the nonlinear terms in \( \hat{x}_1 \) are function of \( x_1, x_3 \) and \( u \) which are known. Nevertheless, we obtain nothing from the equation of \( \dot{x}_4 \) as the nonlinear term \( \phi_4 \) is function of \( x_2, x_3 \) and \( x_4 \), which are not obtained yet.

At step 2, from \( \dot{x}_2 \) we obtain \( x_3 \) as \( \phi_2 \) depends only on \( x_1, x_2 \) and \( x_4 \).

At step 3, from \( \dot{x}_4 \) we also obtain the state \( x_5 \).

After three steps we obtain all the state variables. Moreover, we can see that the direction of \( x \) only on \( \dot{x}_1 \) are function of \( x_1, x_4 \) and \( u \) which are known. Nevertheless, we obtain nothing from the equation of \( \dot{x}_4 \) as the nonlinear term \( \phi_4 \) is function of \( x_2, x_3 \) and \( x_4 \), which are not obtained yet.

At step 2, from \( \dot{x}_2 \) we obtain \( x_3 \) as \( \phi_2 \) depends only on \( x_1, x_2 \) and \( x_4 \).

At step 3, from \( \dot{x}_4 \) we also obtain the state \( x_5 \).

After three steps we obtain all the state variables. Moreover, we can see that the direction of \( y_1 \) is used twice before \( y_2 \)’s one. From the steps previously described, whatever the values of the input \( w \), we can deduce, using trivial generalization of [2], a sliding mode observer for such systems. It is important to note that this observer converges whatever the values of the input \( w \).

It is easy to verify that we can not transform the system (10) into the triangular explicit form (8) but the step by step observation algorithm works. Nevertheless we will see later in a other example : it won’t be the more general form.

III. IMPLICIT TRIANGULAR OBSERVER (ITO) FORM

For the sake of simplicity, we will first study the case without unknown inputs. Let us consider the following multi-output system:
\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x) \]
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the input and \( y \in \mathbb{R}^n \) is the output. the functions \( f, g \) and \( h \) are assumed sufficiently smooth. Then we defined a more general class of system in (8) for which the step by step observation algorithm works.

**Definition 1.** The ITO form for the multi-output system (12) is given by
\[ \alpha_i^0(\zeta^0) = \zeta^{i+1} + \Phi_i^0(\zeta^0)u \]
\[ \alpha_i^0(\zeta^0) = \zeta^{i+1} + \Phi_i^0(\zeta^0)u \]

\[ \alpha_i^0(\zeta^0) = \zeta^{i+1} + \Phi_i^0(\zeta^0)u \]

where \( \alpha_i^0 \) is the \( i \)-th implicit function used at the step \( i \) of the observation algorithm. At the first step \( i = 0 \), the functions \( \alpha_i^0 \) for \( j = 1, \ldots, n \) represent combination of the outputs. The goal of such a combination is to eliminate the input and the unknown state variables from the nonlinear terms of each equation. For the ensuing steps \( i = 1 \ldots k_0 \), the function \( \alpha_i^0 \), for \( j = 1 \ldots v_i \) is a combination of all the known state variables until the step \( i – 1 \).

At each step \( i \), \( v_i \) new directions of the state vector are observed. The \( \beta_i \) functions are used to complete the observer dynamics dimension to \( n \), but they are not useful for the observer convergence.

**Example 3.** Let us consider the following system:
\[ \dot{x}_1 = -x_1 + x_2(x_4 + 1) + ux_4 \]
\[ \dot{x}_2 = -x_2 + x_3(x_4 - 1) + ux_4 \]
\[ \dot{x}_3 = -x_3 + x_4u \]
\[ \dot{x}_4 = \Phi_1(x_1, x_2, x_3, x_4)u \]
\[ y_1 = x_1 \quad y_2 = x_2 \]

The observability index of \( x_3 \) are \( r_1 = 1 \) if \( x_4 \neq -1 \) and \( r_2^2 = 1 \) if \( x_4 = 1 \). For the state \( x_4 \), we obtain \( x_1 = x_2 = 1 \) if \( u \neq 0 \). As we see, the observability index are not constant and depends on the input and the state vector.

Let us now rewrite the system into the ITO form:
We take the following local coordinate change:

\[
\zeta^0_1 = x_1, \zeta^0_2 = x_2, \zeta^1_1 = x_3 \text{ and } \zeta^2_1 = x_4(x_4 + 1).
\]

The functions \( \alpha \) are:

\[
\alpha^0_1 = \frac{(\xi^0_1 - \xi^0_2)}{2} \text{ and } \alpha^1_1 = \frac{(\xi^0_1)^2}{2} - \xi^1_1.
\]

We obtain the following system:

\[
\begin{align*}
\dot{\alpha}^0_1 &= \frac{\xi^0_1 - \xi^0_2}{2} + \xi^1_1 \\
\dot{\alpha}^1_1 &= -\frac{(\xi^0_1)^2}{2} + \xi^1_1 - \xi^2_1.
\end{align*}
\]

We deduce from the example that, contrary to the observability index, \( v_1 = v - 2 = 1 \). We do not specify the values of \( \beta_1 \) and \( \beta_2 \) for sake of simplicity. We can also see through this example, that the functions \( \alpha \) are a combination of the output only for the first step \( \alpha^0 \) but are not for next steps. This fact justifies the implicit triangular form denomination.

The ITO form, as we see is a triangular form which does not use the input derivatives as in the triangular forms of [5,25,26].

Now, we prove the observability for the ITO form introduced in the definition 1.

For this, from (13), we define the following pseudo observability matrix:

\[
\mathcal{O}_o = \begin{pmatrix}
d y_1 \\
\vdots \\
d y_p \\
d \alpha_1^0(\xi^0) \\
\vdots \\
d \alpha_1^0(\xi^0) \\
\vdots \\
d \alpha_1^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_1^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^0(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^0(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^0(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^0(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_i^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{i+1}^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{i+1}^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{k-1}^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{k-1}^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{k-1}^i(\xi^0,\ldots,\xi^i) \\
\vdots \\
d \alpha_{k-1}^i(\xi^0,\ldots,\xi^i) \\
\end{pmatrix}
\]

then we obtain

\[
\begin{pmatrix}
d \xi^0_1 \\
\vdots \\
d \xi^0_p \\
d \xi^1_1 \\
\vdots \\
d \xi^1_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\end{pmatrix}
\]

\[
\mathcal{O}_o = \begin{pmatrix}
d \xi^0_1 \\
\vdots \\
d \xi^0_p \\
d \xi^1_1 \\
\vdots \\
d \xi^1_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\vdots \\
d \xi^i_1 \\
\end{pmatrix}
\]

which leads to a triangular matrix with

\[
\text{rank}(\mathcal{O}_o) = n
\]

Thus, we can conclude to the observability.

Practically, and to illustrate the efficiency of the ITO form, we construct in the next section a sliding mode observer converging in finite time.

**IV. SLIDING MODE OBSERVER FOR ITO FORM**

Using the ITO form (13), a sliding mode observer can be written so that at each step we obtain the states \( \zeta^i \) and consequently at the step \( k_0 \) the whole state is obtained. Let us assume that at step \( i \) we know all the states \( \zeta^j, j = 1, \ldots, i \) and then from the equations

\[
\begin{align*}
\dot{\alpha}_1^0(\zeta^0,\zeta^1,\ldots,\zeta^i) &= \xi^0_{i+1} + \phi^0_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u \\
\dot{\alpha}_1^i(\zeta^0,\zeta^1,\ldots,\zeta^i) &= \xi^i_{i+1} + \phi^i_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u \\
\vdots &\quad \vdots \\
\dot{\alpha}_i^i(\zeta^0,\zeta^1,\ldots,\zeta^i) &= \xi^i_{i+1} + \phi^i_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u \\
\dot{\alpha}_{i+1}^i(\zeta^0,\zeta^1,\ldots,\zeta^i) &= \xi^i_{i+1} + \phi^i_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u \\
\vdots &\quad \vdots \\
\dot{\alpha}_k^i(\zeta^0,\zeta^1,\ldots,\zeta^i) &= \xi^i_{i+1} + \phi^i_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u \\
\end{align*}
\]

we construct the observer

\[
\dot{\alpha}_i(\zeta^0,\zeta^1,\ldots,\zeta^i) = E_{i+1}[\xi^0_{i+1} + \phi^0_{i+1}(\zeta^0,\zeta^1,\ldots,\zeta^i)u + \lambda_i \text{sign}(\alpha_1^0(\zeta^0,\zeta^1,\ldots,\zeta^i))]
\]

\[
- \dot{\alpha}_i(\zeta^0,\zeta^1,\ldots,\zeta^i))]
\]
The observer (16) is consequently an open loop iterative argument to explain it. The observer (16) is a step by step observer and we use an iterative argument to explain it.

**Step 1.** We have at any time the information on $\hat{\xi} = y$. Let us consider the first sub-dynamics,

$$\alpha_0^0(\xi^0) = \xi_1^0 + \phi_1(\xi^0)u$$

$$\alpha_1^0(\xi^0) = \xi_2^0 + \phi_2(\xi^0)u$$

$$\vdots$$

$$\alpha_v^0(\xi^0) = \xi_v^1 + \phi_v(\xi^0)u$$

and we consider the related observer sub-dynamics

$$\alpha_0^v(\xi^v) = \xi_1^v + \phi_1(\xi^v)u + \lambda_v^v \text{sign}(\alpha_v^v(\xi^v) - \alpha_v^0(\xi^0))$$

$$\alpha_1^v(\xi^v) = \xi_2^v + \phi_2(\xi^v)u + \lambda_v^v \text{sign}(\alpha_v^v(\xi^v) - \alpha_v^0(\xi^0))$$

$$\vdots$$

$$\alpha_v^v(\xi^v) = \xi_v^1 + \phi_v(\xi^v)u + \lambda_v^v \text{sign}(\alpha_v^v(\xi^v) - \alpha_v^0(\xi^0))$$

then the observer error

$$e_{i,j}^v = \alpha_i^v(\xi^v) - \gamma^j$$

dynamics is for $i \in \{1, \ldots, v\}$

$$e_{i,j}^v = (\xi_i^v - \gamma^j) - \lambda^v_i \text{sign}(e_{i,j}^v)$$

Thus, the condition in order to guarantee the stability of $e_{i,j}^0$ is that for any $i \in \{1, \ldots, v\}$ and $\forall t > 0$

$$\lambda^v_i > \left| \xi_i^v(t) - \gamma^j \right| \equiv |\xi_i|$$

The existence of such a $\lambda^v_i$ is given by the fact that firstly $\xi_1^v$ is bounded and secondly, before the convergence of $\hat{\xi}^v$ to $\xi^v$ the observer is in an open loop with respect to the observer gain, so $\hat{\xi}^v$ is consequently bounded from the BIBS in finite time assumption.

Then, after a finite time $t_i$ all $\xi_i^v$ goes to $\xi_i^0$. Thus, we have after $t_i$ in the meaning of the differential inclusions [12]

$$0 = e_{i,j}^1 - \lambda^0_i \text{sign}(e_{i,j}^0)$$

and this gives [30]

$$\tilde{\xi}^1_i = \xi_i^1 + \lambda^0_i \text{sign}(e_{i,j}^0)$$

where the $\text{sign}$ function is low pass filtered with an arbitrary large frequency band width in order to preserve the right and side function convexity at the next step and obtain $\lambda^v_i$ equal to $\lambda^v_i$ which implies $\xi_i^v = \xi_i^0$

For step $i$ (for $i \in \{2, \ldots, k_0 - 1\}$), we assume that we know all the states $\xi_j^i$ (i.e. $\xi_j^i = \xi_j^i$) for $j = 1, \ldots, i$. Thus we have

$$\alpha_i^i(\xi^0, \xi^1, \ldots, \xi^i) = \xi_{i+1}^i + \phi_{i+1}(\xi^0, \xi^1, \ldots, \xi^i)u$$

$$\alpha_1^i(\xi^0, \xi^1, \ldots, \xi^i) = \xi_2^i + \phi_2(\xi^0, \xi^1, \ldots, \xi^i)u$$

$$\vdots$$

$$\alpha_v^i(\xi^0, \xi^1, \ldots, \xi^i) = \xi_v^1 + \phi_v(\xi^0, \xi^1, \ldots, \xi^i)u$$

where only $\xi_{i+1}^i$ is unknown and appears linearly in the subdynamics (17). Therefore, as $E_i = 1$ we have the following observer sub-dynamics

$$\alpha_i^i(\xi^v) = \xi_{i+1}^i + \phi_{i+1}(\xi^0, \xi^1, \ldots, \xi^i)u$$

$$\alpha_1^i(\xi^v) = \xi_2^i + \phi_2(\xi^0, \xi^1, \ldots, \xi^i)u$$

$$\vdots$$

$$\alpha_v^i(\xi^v) = \xi_v^1 + \phi_v(\xi^0, \xi^1, \ldots, \xi^i)u$$

thus the observation error dynamics in $e_{i,j}^i$ is

$$e_{i,j}^i = \alpha_i^i(\xi^v) - \gamma^j$$

Consequently, from (19) we see that $e_{i,j}^i$ converges to zero in a finite time. Thus, $\exists \ t_i > 0$ such that

$$\forall t > t_i \quad e_{i,j}^i = 0 \Rightarrow e_{i,j}^i - \lambda^i_j \text{sign}(e_{i,j}^i) = 0$$

then

$$\forall t > t_i \quad e_{i,j}^i = \xi_{i+1}^i + \lambda^i_j \text{sign}(e_{i,j}^i) = \xi_{i+1}^i$$

and the step $i$ is finished after the time.
\[ t_i \triangleq \max_{j \in \{1, \ldots, n\}} (t_{i,j}) . \]

Finally, at step \( k_0 \) as all \( \xi^i_j \) have converged to \( \xi^i_j \), the last sub-dynamics observation error
\[ e_{\beta^i_j}^o = (\beta^i_j(\xi^o_j, \xi^i_j, \ldots, \xi^{i+p-1}_j) - \beta^i_j(\xi^o_j, \xi^i_j, \ldots, \xi^{i+p-1}_j, \xi^{i+p}_j)) \]
is equal to
\[ e_{\beta^i_j}^o = -\lambda^i_j \text{sign}(e_{\beta^i_j}^o) \]
which is stable if \( \lambda^i_j > 0 \) and without any other condition, this last sub-dynamics is just given in order to ensure that \( e^i_j \) is bounded but we know the whole state at the step \( k_0 - 1 \) thank to \( \tilde{\zeta} \). Consequently, let us state

**Theorem 2.** For any system in ITO form (13), there exists a step by step sliding mode observer of the form (16) such that the observation error converges to zero, independently of the input, step by step in finite time.

**Proof.** The proof is explicitly detailed in the paragraph before the theorem statement. ■

**Remark 2.** The \( \tilde{\zeta}^{i+1}_j \) dynamics is implicitly obtained from the \( \tilde{\alpha}^i_k \), \( k = i + 1, \ldots, k_0 \) and \( \tilde{\beta}^i_k = 1, \ldots, p \) dynamics.

**Remark 3.** We will note that in the multi-input case, the characterization of BIBS systems is not usual. However, we make this assumption work on systems not being unstable in finite time for reasons of stability of the observer, the aim of this paper being not the geometrical characterization of BIBS systems, but the observability forms associated with a certain type of sliding mode observer.

As it can be seen in the last step, the knowledge of the functions \( \beta^i_j \) is not necessary to the convergence of the \( \zeta \) to \( \zeta^i_j \) in finite time. As a consequence of this fact, we can state that some unknown terms as unknown inputs can occur in all \( \zeta \) functions.

An outline of the proof of stability and conditions of convergence of the observer presented above can be exposed briefly as being similar to those stated in paper [2]. For this, it is necessary to use a particular function \( \text{sign}(\cdot) \) is equal to zero if there exists \( j \in \{1, i\} \) such as \( \tilde{\zeta}^i_j \neq 0 \), else \( \text{sign}(\cdot) \) is equal to the usual \( \text{sign}(\cdot) \) function.

V. CHARACTERIZATION OF THE ITO FORM

Previously, we have highlighted the efficiency of a sliding mode observer to reconstruct in finite time the whole state of a multi-output system in the ITO form. Now, we want to characterize the necessary and sufficient conditions in order to obtain this form starting from (20) using a local diffeomorphism. Roughly speaking, we will “algebraise” the structured implicit triangular observer form. For this, let us begin by giving some definitions

**Definition 2.** Considering the system
\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x) \]
where \( f(x) \) and \( g(x) \) are nonlinear analytic functions, \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the input and \( y \in \mathbb{R}^p \) is the output. Let us define the following sets
\[ E^0 = F^0 = \text{span} \{d_{h_i}, i = 1, \ldots, p\} \]
Moreover, for \( k = 1, \ldots, n - p \), we define: \( S^k \) is the greater set of analytic function so that
\[ L_S(S^k) \in F^{k-1} \]
\[ F^k \triangleq L_k[S^k \cap (F^{k-1})^\perp] \]
\[ F^k = F^{k-1} \cup E^k \]
Roughly speaking, \( S^k \) represents the invariant part of \( F^{k-1} \) under the transformation \( L_k \), \( E^k \) the new state variables obtained by step \( k \) of the observation algorithm and \( F^k \) is the state variable part obtained since step \( k \). The constraint imposed here is not to use the input derivatives.

Let us take the following notations: \( \zeta^i_j \) is a basis of \( E^i \) and \( v_i \triangleq \text{dim}(E^i) \).

**Lemma 1.** For the system (20), if there exists \( k_0 \) so that \( S^{k_0} = \emptyset \), then \( \forall k \geq k_0, S^k = \emptyset \) and \( \text{dim}(F^k) = \text{dim}(F^{k_0}) \).

**Proof.** From the definition of \( E^k \), we can state that \( S^{k_0} = \emptyset \Rightarrow E^{k_0 \perp} = \emptyset \) and then it is clear that \( S^{k_0 \perp} = \emptyset \). Thus for any \( k \geq k_0, E^k = \emptyset \) and \( F^k = F^{k_0} \).

**Lemma 2.** For the system (20), \( \forall i, j \in \{0, n - p\} \) such as \( i < j \) we have \( E^i \cap E^j = \emptyset \) and \( S^i \subseteq S_j \).

**Proof.** Starting from the definition of \( E^i \), it is clear that \( E^i \cap F^{k-1} = \emptyset \) and we deduce from the definition of \( F^{k-1} \), that \( E^i \cap E^j = \emptyset \), and this for any \( j < k \). Moreover, if \( L_k(S^i) \in F^{k-1} \), as \( F^{k-1} \subseteq F^k \), then \( L_k(S^i) \subseteq F^k \).

**Lemma 3.** For the system (20), if there exists \( k_0 \) so that \( E^{k_0} = \emptyset \), then \( \forall k \geq k_0, E^k = \emptyset \) and \( F^k = F^{k_0} \).

**Proof.** If \( E^{k_0} = \emptyset \), we have \( F^{k_0 \perp} = F^{k_0} \), then \( S^{k_0 \perp} = S^{k_0} \) and for any \( k > k_0 \) we have \( F^k = F^{k_0} \) and \( S^k = S^{k_0} \). ■
Remark 4. In this case the observation algorithm does not give a new variable. In the case where \( \dim(F^{n_0}) \neq n \), this can possibly lead to an unobservability of the system or at least a system which is not directly observable, i.e. without the use of \( u \) knowledge.

Let us take the following notations: \( \zeta_j \triangleq (y_j, j = 1, \ldots, p) \) and \( \zeta^l \triangleq (\zeta^l_j, j = 1, \ldots, p) \) such as \( d\zeta^l \) is a basis for \( E^l \) for \( l = 1, \ldots, k_0 \) where \( k_0 \) is the least integer so that \( \dim(F^{n_0}) = n \). Now, we can give the necessary and sufficient conditions to put the system in the ITO form.

Remark 5. We see that if at step \( i \) any new state variable is obtained, then no state variable will be obtained for any step \( j > i \). Consequently, at each step we must obtain at least one new state, so the maximal step number is \( n - p \).

Theorem 3. The system (20) can be put in an ITO form (13) using a local diffeomorphism if and only if

\[
\dim(F^{n-p}) = n
\]

Proof.

Sufficiency If \( \dim(F^{n-p}) = n \) then, after \( n - p \) steps, we obtain \( n \) independent variables \( \zeta_j^l, i = 0, \ldots, n - p, j = 1, \ldots, v_l \). As there exists a non singular coordinate change \( \Psi \) such that \( \zeta = (\zeta^l_j, i = 0, \ldots, n - p, j = 1, \ldots, v_l)^T = \Psi(x) \) because \( \Psi(\chi) \)'s Jacobian is equal to \( \dim(F^{n-p}) \). In this case, we can define the functions \( \alpha_i \) of the form (13).

From the definition of the sets \( E^l \), \( F^l \) and \( S^l \), let the subset \( T' \) of \( T \) for \( i = 1, \ldots, k_0 \) so that \( L_0 T' = E^l \). We also know that \( T' \subset F^{i-1} \). Let us consider the states \( \zeta^l \) such that \( d\zeta^l \) is a basis of \( E^l \), which means\(^2\) that \( \zeta^l \) can be written as \( L_t \chi^l \) where \( \chi^l \in T' \). Thus, \( \chi^l \) can be written in the basis of \( F^{i-1} \) i.e. \( \chi^l = \alpha(\zeta^l_0, \ldots, \zeta^l_{i-1}) \). This gives

\[
\dot{\chi}^l = \alpha(\zeta^l_0, \ldots, \zeta^l_{i-1}) L_t \chi^l + L_f \chi^l u
\]

with

\[
\dot{\phi}(\zeta^l_0, \ldots, \zeta^l_{i-1}) \triangleq L_f \chi^l.
\]

Necessity Suppose that \( \dim(F^{n-p}) < n \), then there exists \( k_0 < n - p \) such that \( \dim(F^{n_0}) = 0 \). Thus, from Lemma 2 we find that \( \dim(F^{n-p}) = \dim(F^{n_0}) \). This means that \( \forall d \Xi \in F^{n_0}, L_d(\Xi) \) depends on the unobserved state part at the step \( k_0 - 1 \). Thus, \( F^{n_0} \) depends on the unobservable states and consequently, and we do not have the form (13).

Example 4. Let us consider the following locally observable system

\[
\begin{align*}
\dot{x}_1 &= x_2 x_1 - x_3 u \\
\dot{x}_2 &= x_3 + g(x_1, x_2, x_4) u \\
\dot{x}_4 &= \phi(x, u) \\
\dot{y}_1 &= x_1 \\
\dot{y}_2 &= x_4
\end{align*}
\]

For (21), the transformation to explicit triangular form is impossible. In fact, if we take \( y_1 = x_1 \) and \( y_2 = \frac{x_2^2}{2} - x_4 \), we have

\[
\begin{align*}
\dot{y}_1 &= x_2 x_1 - x_3 u \\
\dot{y}_2' &= x_2 (x_1^2 + 1) \triangleq \xi_{2,1}
\end{align*}
\]

But

\[
\xi_{2,1} = (x_1^2 + 1) u(x_1^2 + 1)
\]

\[+ 2x_2 x_1 (x_1 + 1) u(x_1 + 1)\]

has a nonlinear term depending on \( x_3 \) which is not yet known. Thus, using only the derivation operation on the two directions we can not obtain a new variable \( \xi_{2,2} \) or \( \xi_{1,2} \). Therefore, this system must be transformed into the ITO form. In order to design a sliding mode observer, we compute the sets \( E^l \) and \( F^l \), we will obtain \( E^0 = span\{dx_1, dx_4\} \), thus, \( S^0 = span\{dx_1, dx_4\} \), i.e. at step 1 we can only obtain the variable \( \frac{dx_1}{dt} = x_2 (x_4 - 1) \) because of the \( g(x) = -x_3, g_2(x) = -x_4, dx_3 \notin F^0 \). Then, \( E^1 = span\{dx_1(x_4^2 + 1)\} \) and a basis of \( F^1 \) is \( \{dx_1, dx_4\} \). At the second and last step, we deduce \( x_3 \) from the \( x_3 \) dynamics. In two steps we obtain the whole state, but using in the first step outputs combination and we can write the system in the ITO form with \( \xi_0 = x_1 \), \( \xi_1 = x_2^2 + x_3 \), \( \xi_2 = x_3 \), \( \xi_3 = x_4 \) and

\[
\begin{align*}
\dot{\alpha}_1^0(\xi_1^0, \xi_2^0) &= \xi_1^0 \\
\dot{\beta}_1^0(\xi_1^0, \xi_2^0) &= \dot{\phi}(\zeta^0, u) \\
\dot{\alpha}_2(\zeta^2_0, \zeta^2_1) &= \zeta_1^2 + g'(\zeta_0^0, \zeta_1^0) u \\
\dot{\beta}_2(\zeta^2_0, \zeta^2_1) &= -\zeta_1^2 - \zeta_0^0 \zeta_2^0 u \\
y_1 &= \xi_1^0 \\
y_2^0 &= \xi_1^0
\end{align*}
\]

\(^2\)Any one-form of \( F^{n-1} \) is generated by a function dependent only on \( y, \dot{y}, \ldots, y^{(p)} \).
with

\[ \alpha_i^0(\zeta^0, \zeta^1) = \frac{(\zeta^0_0)^2}{2} - \zeta^0_2, \quad \alpha_i^0(\zeta^0_1, \zeta^1) = \frac{\zeta^1_1}{1+(\zeta^0_1)^2}, \]

\[ \beta_i(\zeta^0_1) = \zeta^0_2 \]

and \( \beta_2(\zeta) = \zeta^2_2 \) and \( \phi^i, g^i \) denote respectively the functions \( \phi \) and \( g \) with respect to \( \zeta \) instead of \( x \). Due to the observability singularity given by \( -1 - \zeta^0_1 u = 0 \), we decide to use only information on \( y_1 \). This example shows that the ITO form is more general than the triangular form (8).

VI. GENERALIZED OBSERVER MATCHING CONDITION

In this section, we generalize the result of the proposition 2. Consider the following ITO form with unknown input \( w \in \mathbb{R}^q \):

\[
\begin{aligned}
\alpha^0_i(\zeta^0) &= \zeta^1_1 + \phi^i(\zeta^0)u \\
\alpha^0_i(\zeta^1) &= \zeta^1_1 + \phi^i(\zeta^1)u \\
\alpha^0_{i1}(\zeta^0) &= \zeta^1_{i1} + \phi^i_{11}(\zeta^0)u \\
\alpha^0_{i1}(\zeta^1) &= \zeta^1_{i1} + \phi^i_{11}(\zeta^1)u \\
\alpha^{i-1}_2(\zeta^0) &= \zeta^1_{i2} + \phi^{i-1}_2(\zeta^0)u \\
\alpha^{i-1}_2(\zeta^1) &= \zeta^1_{i2} + \phi^{i-1}_2(\zeta^1)u \\
\alpha^{i-1}_3(\zeta^0) &= \zeta^1_{i3} + \phi^{i-1}_3(\zeta^0)u \\
\alpha^{i-1}_3(\zeta^1) &= \zeta^1_{i3} + \phi^{i-1}_3(\zeta^1)u \\
\alpha^{i-1}_4(\zeta^0) &= \zeta^1_{i4} + \phi^{i-1}_4(\zeta^0)u \\
\alpha^{i-1}_4(\zeta^1) &= \zeta^1_{i4} + \phi^{i-1}_4(\zeta^1)u \\
\alpha^{i-1}_5(\zeta^0) &= \zeta^1_{i5} + \phi^{i-1}_5(\zeta^0)u \\
\alpha^{i-1}_5(\zeta^1) &= \zeta^1_{i5} + \phi^{i-1}_5(\zeta^1)u \\
\beta_i(\zeta) &= \psi_i(\zeta, u) + \phi_i(\zeta, w) \\
\beta_i(\zeta) &= \psi_i(\zeta, u) + \phi_i(\zeta, w) \\
\beta_i(\zeta) &= \psi_i(\zeta, u) + \phi_i(\zeta, w) \\
\beta_i(\zeta) &= \psi_i(\zeta, u) + \phi_i(\zeta, w) \\
\end{aligned}
\]

\[ (23) \]

As the unknown term depending on \( w \) occurs on the \( p \) last lines (in the \( \beta \) dynamics), the step by step observation algorithm works well and \( \zeta \) converges in finite time to \( \zeta \) in the observer (18). Thus, we have rejected the unknown input \( w \).

The objective now is to find what conditions must verify the system

\[
\begin{aligned}
\dot{x} &= f(x) + g(x)u + l(x)w \\
y &= h(x)
\end{aligned}
\]

(24)

to obtain the form (23) after a change of coordinates.

As in the previous section, let us use the following definition

**Definition 3.** Considering the system (24), we define the following sets

\[ E^0_w = F_w^0 \triangleq \text{span}\{d_{hi}, i = 1, \ldots, p\} \]

Moreover, for \( k = 1, \ldots, n - p \), we define: \( S^k \) is the greater set of analytic function such that

\[ Lg(S^k) \in F_w^{k-1} \quad \text{and} \quad L_i(S^k) = 0 \]

and

\[ E^k_w \triangleq L_j(S^k) \cap (F_w^{k-1})^\perp \]

\[ F_w^k \triangleq F_w^{k-1} \cup E^k_w \]

**Theorem 4.** The system (24) can be put in an ITO form (23), using a local diffeomorphism, if and only if

\[ \text{dim}(F_w^{n-p}) = n \]

**Proof.** The proof is similar to the theorem 3, where the condition \( L_i(S^k) = 0 \) ensures that there is no term depending on \( w \) in the \( \alpha^i \) dynamics.

**Example 5.** Let us consider the system:

\[ \begin{aligned}
\dot{x} &= f(x) + ug(x) \\
x &= x_1 + u_1 x_1 - u_2 x_2 \\
\end{aligned} \]

(25)

As the unknown term in the input \( u \) is not a linear function of the measurable state \( x \), this example cannot be put in the injection output or injection output and derivatives output form. Already, the derivative of the output depends not linearly on a part of the non measurable state.

Nevertheless, it checks the conditions of setting in the ITO form. Indeed, the calculation of subspaces \( E, F \) gives:
\[ E^0 \triangleq F^0 \triangleq \{dx_1, dx_2\} \]

As \( dL_N (\frac{x_1^2}{2} + x_2) \in F^0 \), one deduce that \( E^0 = L_N (\frac{x_1^2}{2} + x_2) = \text{span} \{d(x_1(1 + x_1^2))\} \). In this case \( F^1 = \text{span} \{dx_1, dx_2, dx_3\} \). Moreover, as \( dL_N (x_1 + x_2) \in F^1 \), then \( E^2 = L_N (x_1 + x_2) = \text{span} \{d(x_4 + x_5 x_1)\} \) and \( F^2 = \text{span} \{dx_1, dx_2, dx_3, dx_4\} \). Finally, \( dL_N (x_4) \in F^2 \), \( E^3 = L_N (x_4) = \text{span} \{d(x_5 (x_1^2 + 1))\} \) and \( F^3 = \text{span} \{dx_1, dx_2, dx_3, dx_4, dx_5\}, \dim (F^3) = 5 \), as the system is of dimension 5, according to the Theorem, the system (25) is put under ITO form.

Let us consider the following diffeomorphism: \( \zeta = \Psi(x) \)

\[
\begin{align*}
\zeta^0 &= (\zeta_1^0, \zeta_2^0)^T = (x_1, x_2)^T \\
\zeta^1 &= x_1 (1 + x_1^2) \\
\zeta^2 &= x_4 \\
\zeta^3 &= x_5 (x_4^2 + 1)
\end{align*}
\]

In these coordinates, the system (25) will be written as:

\[
\begin{align*}
\alpha^0_i &= \zeta^1 + u_i (\zeta_2^0)^2 \\
\alpha^1_i &= \zeta^2 + u_i (\zeta_1^0) / ((\zeta_2^0)^2 + 1) + \zeta_2^0 \\
&\quad + \zeta^0_i (\zeta_2^0 + 1) + \zeta_1^0 (\zeta_2^0 + 1) \\
\alpha^2_i &= \zeta^3 - \zeta^2 + u_i (\zeta_1^0) / ((\zeta_2^0)^2 + 1) + u_2 (\zeta_1^0) \\
\hat{\beta}_i &= \phi_i (\zeta, u_i, u_2) \\
\hat{\beta}_2 &= \phi_2 (\zeta, u_i, u_2)
\end{align*}
\]

with

\[
\begin{align*}
\alpha^0_i &\triangleq (\zeta_2^0)^2 + \zeta_2^0 \\
\alpha^1_i &\triangleq \zeta^1 / ((\zeta_2^0)^2 + 1) + \zeta^0_1 \alpha^1_i \triangleq \zeta^2
\end{align*}
\]

and to complete the coordinates change \( \beta_1 \triangleq \zeta^0 \) and \( \beta_2 \triangleq \zeta^1 \), which gives

\[
\phi_i (x, u_i, u_2) = x_i + u_2 x_i x_i \\
\phi_2 (x, u_i, u_2) = \phi(x, u_i, u_2)(x_i^2 + 1) + 2 x_i x_i (x_i^2 + 1) \\
+ u_i x_i x_i + u_2 x_i
\]

\( \phi_i (\zeta, u_i, u_2) \) and \( \phi_2 (\zeta, u_i, u_2) \) are obtained by substituting \( x = \Psi^{-1}(\zeta) \).

Starting from (27), we construct the sliding modes observer:

\[
\hat{\alpha}^0_i (\zeta_2^0) = \hat{\alpha}^1 + u_i (\zeta_2^0)^2 + \lambda_i^2 \text{sign}_n (\alpha^0_i (\zeta_2^0) - \alpha^0_i (\zeta_2^0))
\]

Figures (8,...,12) show simulations in \( \zeta \) coordinates. We can see the well step by step convergence. At step 3, all the states \( \hat{\zeta} \) converge to \( \zeta \). Gains are: \( \lambda_2^0 = 15 \), \( \lambda^1 = 4 \), \( \lambda_2 = 2 \), and \( \lambda_3 = 1 \).
can obtain in the multi-output case with unknown inputs. Furthermore, we give a geometric characterization of the class of systems which can be put into an ITO form using a diffeomorphism including a matching condition. These characterizations are constructive i.e. it allows to obtain the coordinate change. We show also that this class is a subset of the uniformly observable class systems. A step by step sliding mode observer was presented for such a class of ITO form and some illustrative examples were presented.

REFERENCES


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