STABILITY AND $H_\infty$ DISTURBANCE ATTENUATION ANALYSIS FOR LTI CONTROL SYSTEMS WITH CONTROLLER FAILURES

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ABSTRACT

In this paper, we analyze stability and $H_\infty$ disturbance attenuation properties for linear time-invariant (LTI) systems controlled by a pre-designed dynamical output feedback controller which fails from time to time due to physical or purposeful reasons. Our aim is to find conditions concerning the controller failure time, under which the system’s stability and $H_\infty$ disturbance attenuation properties are preserved to a desired level. For stability, by using a piecewise Lyapunov function, we show that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures (ATBCF) is large enough, then the global exponential stability of the system is guaranteed. For $H_\infty$ disturbance attenuation, also by using a piecewise Lyapunov function, we show that if the unavailability rate of the controller is smaller than a specified constant, then a system with an ATBCF achieves a reasonable weighted $H_\infty$ disturbance attenuation level, and the weighted $H_\infty$ disturbance attenuation approaches normal $H_\infty$ disturbance attenuation when the ATBCF is sufficiently large.

KeyWords: Linear time-invariant (LTI) system, dynamical output feedback, exponential stability, (weighted) $H_\infty$ disturbance attenuation, controller failure, unavailability rate, average time between controller failures, piecewise Lyapunov function.

I. INTRODUCTION

In this paper, we consider some quantitative properties of linear time-invariant (LTI) control systems with controller failures. The motivation for studying such problem stems from the fact that controller failures always exist in real control systems due to various environmental factors. For example, for a feedback control system which is composed of a system and a feedback controller, controller failures occur when the signals are not transmitted perfectly between the system and the controller, or when the controller itself is not available sometimes due to known or unknown reasons. Another important motivation for studying controller failures stems from the fact that we can think about such “failure” in a positive way: that is, as “suspension”; i.e., in a situation where economic issues or system life considerations are important, we may desire to suspend the controller purposefully from time to time.

For feedback control systems, the problem of possessing integrity was considered in [1], where a state feedback controller was designed such that the closed-loop system remained stable even when some part of the controller failed. In [2], similar control systems were dealt with using unclear asynchronous dynamical systems (ADS), and two real systems, the control over asynchronous network and the parallelized algorithm, were discussed. In that context, a Lyapunov-based approach was proposed to construct the controller so that the system had the desired properties. In a recent paper [3], the authors examined similar control problems under the framework of networked control systems (NCS), where information (reference input, plant output, control input, etc.) is exchanged through a network of control system components (sensors, controller, actuators, etc.), and packet dropouts, thus, inevitably occur due to unreliable transmission paths that lead to controller failures.
Recently, we considered in [4] a controller failure time analysis problem for the exponential stability of LTI continuous-time systems. By using a piecewise Lyapunov function, we showed that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures is large enough, then the global exponential stability of the system is guaranteed. In [5], the result of [4] was extended to LTI discrete-time systems. Furthermore, the authors there extended the consideration of $L_2$ gain analysis to LTI continuous-time systems with controller failures as in [6]. However, the authors of [4-6] dealt with only state feedback. Although it is not difficult to extend the results in [4-6] to the case of static output feedback, extension to dynamical output feedback is not easy.

Motivated by the above observations, we in this paper extend the results of [4,6] to the dynamical output feedback case. The system we consider is described by equations of the form

$$\begin{cases}
\dot{x} = Ax + B_1w + B_2u \\
z = C_1x + D_{11}w + D_{12}u \\
y = C_2x
\end{cases}$$

(1)

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the disturbance input, $y \in \mathbb{R}^p$ is the measurement output, $z \in \mathbb{R}^r$ is the controlled output, and $A$, $B_1$, $B_2$, $C_1$, $C_2$, $D_{11}$, and $D_{12}$ are constant matrices of appropriate dimension. Throughout this paper, we assume that (i) $A$ is not stable; (ii) the triple $(A, B_1, C_1)$ is stabilizable and detectable; (iii) the dynamical output feedback controller

$$\begin{cases}
\dot{x}_c = A_c x_c + B_c y \\
u = C_c x_c + D_c y
\end{cases}$$

(2)

has been designed so that the closed-loop system composed of (1) and (2) has the desired property (exponential stability with a certain decay rate or a certain $H_\infty$ disturbance attenuation level), where $x_c \in \mathbb{R}^n_c$ is the controller’s state, $n_c$ is the controller’s order, and $A_c$, $B_c$, $C_c$, and $D_c$ are constant matrices. However, for physical or purposeful reasons, the designed controller sometimes fails over a (not necessarily constant) time interval until we recover it. In this setting, we derive the condition of the controller failure time, under which the system’s exponential stability or its $H_\infty$ disturbance attenuation property is preserved to a desired level. As in [4-6], we use the word “controller failure” in this paper to mean complete breakdown of the controller ($u = 0$) for a certain time interval, not to mean that part of the controller fails [1], nor to mean that the controller decays slowly at a given rate [2].

To proceed, we will first introduce some notations. For any given $t > 0$, we denote by $T_d(t)$ the total time interval of controller failures during $[0, t)$, and call the ratio $\frac{T_d(t)}{t}$ the “unavailability rate” of the controller in the system. We denote by $N_t$ the number of instances of controller failure during $[0, t)$. If for some constant $T_f > 0$, the inequality $N_t \leq \frac{t}{T_f}$ holds for any $t > 0$, then $T_f$ is called as a lower bound of the “average time between controller failures” (ATBCF). The idea is that the average time interval between subsequent controller failures is not less than $T_f$ according to the equivalent inequality $\frac{t}{N_t} \geq T_f$. For stability analysis, we will prove that if the unavailability rate of the controller is smaller than a specified constant and the ATBCF is large enough, then the global exponential stability of the system is guaranteed. For $H_\infty$ disturbance attenuation analysis, we will show that if the unavailability rate of the controller is smaller than a specified constant, then a system with an ATBCF achieves a reasonable weighted $H_\infty$ disturbance attenuation level, and the weighted $H_\infty$ disturbance attenuation approaches normal $H_\infty$ disturbance attenuation if the ATBCF is sufficiently large.

To analyze stability and the $H_\infty$ disturbance attenuation property of a system with controller failures, we utilize a piecewise Lyapunov function approach (e.g., [7,8]). It is well known that Lyapunov function theory is the most general and useful approach to studying the qualitative properties of various control systems. However, for the system on hand, traditional Lyapunov functions do not exist since the system is unstable when the controller fails. Instead of traditional single Lyapunov functions, we construct a piecewise Lyapunov function along with the situation of the controller. It should be noted here that the idea of using a piecewise Lyapunov function for $H_\infty$ control systems with controller failures originated in recent papers [9-11], where the stability and $L_2$ gain of switched systems composed of both stable and unstable subsystems were analyzed using the concept of the average dwell time. In this paper, we will use the more practical concept “ATBCF,” which is quantitatively different from the average dwell time in [9-11], and we also define a new quantity, the “unavailability rate,” so as to deal with controller failures. We will show that these modifications help us to analyze the controller failure time of the control systems under consideration.

\section*{II. STABILITY ANALYSIS}

In this section, we set $w \equiv 0$ in system (1) to analyze the stability of the system with controller failures. More precisely, we assume that controller (2) has been designed so that the closed-loop system
\[
\dot{x} = A\dot{x}, \quad A = \begin{bmatrix} A + B_1D_1C_2 & B_1C_2 \\ B_2C_2 & A_2 \end{bmatrix}
\]  

(3)

is exponentially stable, where \( \dot{x} = [x^T \quad \dot{x}^T]^T \) is the state of the closed-loop system.

We will first give a quantitative definition of the exponential stability of an autonomous system. Throughout this paper, we denote by \( \| \cdot \| \) the Euclidean norm of a vector.

**Definition 1.** The system \( \dot{x} = f(x) \) with \( f(0) = 0 \) is said to be **globally exponentially stable** with decay rate \( \lambda > 0 \) if \( \|x(t)\| \leq ce^{-\lambda t} \|x_0\| \) holds for any \( x_0 \), any \( t \geq 0 \) and a constant \( c > 0 \).

We suppose that the designed controller (2) sometimes fails, and that we need a (not necessarily constant) time interval to recover it. Obviously, when the controller fails, the closed-loop system assumes the form of

\[
\dot{x} = A_\varepsilon \dot{x}, \quad A_\varepsilon = \begin{bmatrix} A & 0 \\ B_2C_2 & A_2 \end{bmatrix},
\]

(4)

which is obtained by substituting \( u = 0 \) in (1). Hence, the performance of the entire system is dominated by the following piecewise differential equation:

\[
\dot{x} = \begin{cases} A\dot{x} & \text{when the controller works,} \\ A_\varepsilon \dot{x} & \text{when the controller fails.} \end{cases}
\]

(5)

Since \( A_\varepsilon \) is stable and \( A_\varepsilon \) is unstable, we can always find two positive scalars \( \lambda_\varepsilon \) and \( \lambda_\varepsilon \) such that \( A_\varepsilon + \lambda_\varepsilon J \) remains stable and \( A_\varepsilon - \lambda_\varepsilon J \) becomes stable. As will be shown later, large \( \lambda_\varepsilon \) and small \( \lambda_\varepsilon \) are desirable. Then, there are two matrices, \( P_\varepsilon > 0 \) and \( P_\varepsilon > 0 \) such that

\[
A_\varepsilon + \lambda_\varepsilon J \preceq P_\varepsilon + P_\varepsilon \begin{bmatrix} A_\varepsilon + \lambda_\varepsilon J \\ A_\varepsilon - \lambda_\varepsilon J \end{bmatrix} < 0,
\]

(6)

\[
A_\varepsilon - \lambda_\varepsilon J \preceq P_\varepsilon + P_\varepsilon \begin{bmatrix} A_\varepsilon + \lambda_\varepsilon J \\ A_\varepsilon - \lambda_\varepsilon J \end{bmatrix} < 0.
\]

Note that the above inequalities are LMIs [12] with respect to \( P_\varepsilon \) and \( \varepsilon \), and thus are easily solved using any one of several existing software programs, such as the LMI Control Toolbox [13].

Using the solutions \( P_\varepsilon \) and \( P_\varepsilon \) of (6), we define the **piecewise Lyapunov function** candidate

\[
V(t) = V_{\sigma(t)}(\dot{x}) = \dot{x}^T P_{\sigma(t)} \dot{x}
\]

(7)

for the system. Here, \( P_{\sigma(t)} \) is a two-valued piecewise constant matrix function as

\[
P_{\sigma(t)} = \begin{cases} P_\varepsilon & \text{when the controller works,} \\ P_\varepsilon & \text{when the controller fails,} \end{cases}
\]

and \( V_{\sigma(t)}(\dot{x}) \) is defined correspondingly. Then, the following properties of (7) are obtained:

(i) Both \( V_{\sigma}(\dot{x}) = \dot{x}^T P_{\sigma} \dot{x} \) and \( V_{\sigma}(\dot{x}) = \dot{x}^T P_{\sigma} \dot{x} \) are continuous, and their derivatives along the solutions of the corresponding system satisfy

\[
\dot{V}_\sigma \leq -2\lambda_\sigma V_\sigma, \quad \dot{V}_\sigma \leq 2\lambda_\sigma V_\sigma.
\]

(9)

(ii) There exist constant scalars \( \alpha_2 \geq \alpha_1 > 0 \) such that

\[
\alpha_1 \|x(t)\|^2 \leq \{V_{\sigma}(\dot{x}),V_{\sigma}(\dot{x}) \} \leq \alpha_2 \|x(t)\|^2, \forall \dot{x} \in \mathbb{R}^{n_x}.\]

(10)

(iii) There exists a constant scalar \( \mu \geq 1 \) such that

\[
V_{\sigma}(\dot{x}) \leq \mu V_{\sigma}(\dot{x}), \quad \mu V_{\sigma}(\dot{x}) \leq \mu V_{\sigma}(\dot{x}), \forall \dot{x} \in \mathbb{R}^{n_x}.
\]

(11)

The first property is a straightforward consequence of (6), while the second and third properties hold, for example, with \( \alpha_2 = \min\{\lambda_\sigma(P_\varepsilon), \lambda_\sigma(P_\varepsilon)\} \), \( \sigma_2 = \{\lambda_\sigma(P_\varepsilon), \lambda_\sigma(P_\varepsilon)\} \), and \( \mu = \sigma_2 \alpha_2 \), respectively. Here, \( \lambda_\sigma() \) denotes the largest (smallest) eigenvalue of a symmetric matrix.

Now, without loss of generality, we assume that the designed controller works during \([t_{2j}, t_{2j+1}]\), and that the controller fails during \([t_{2j+1}, t_{2j+2}], j = 0, 1, \ldots, \), where \( t_0 = 0 \). Then, using the differential inequality theory (for example, [15]) for (9), we get for any \( t > 0 \) that

\[
V(t) \leq \begin{cases} \mu^t e^{2\lambda_\sigma t_{2j}} V(t_{2j}) & \text{if } t_{2j} \leq t < t_{2j+1}, \\ e^{2\lambda_\sigma (t-t_{2j+1})} V(t_{2j+1}) & \text{if } t_{2j+1} \leq t < t_{2j+2}. \end{cases}
\]

(12)

Noting that \( V(t_0) \leq \mu V(t_0) \) holds for any \( t > 0 \) according to (11), where \( t_j = \lim_{t \to t_j} t \), we obtain from (12) that for \( t_{2j} \leq t < t_{2j+1} \),

\[
V(t) \leq e^{-2\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu e^{-2\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu_2 e^{-2\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu_2 e^{-2\lambda_\sigma (t-t_{2j})} V(t_{2j})
\]

(13)

and then by induction that

\[
V(t) \leq \mu_2^{2j} e^{\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu_2^{2j} e^{\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu_2^{2j} e^{\lambda_\sigma (t-t_{2j})} V(t_{2j}) \leq \mu_2^{2j} e^{\lambda_\sigma (t-t_{2j})} V(t_{2j})
\]

(14)

It is easy to confirm that the above inequality is also true for \( t_{2j+1} \leq t < t_{2j+2} \). Since \( V(t) \geq \alpha_1 \|x(t)\|^2 \) and \( V(0) \leq \alpha_2 \|x(0)\|^2 \), we obtain from the above inequality that

\[
\|x(t)\| \leq \left[ \frac{\alpha_2}{\alpha_1} \right]^{1/2} \left[ e^{\lambda_\sigma (t-t_{2j})} + \lambda_\sigma (t-t_{2j}) \right] \|x(0)\|.
\]

(15)

From now on, we will consider the convergence property of \( x(t) \) in (15) in two different cases.
First, when \( \mu = 1 \), we get from (15) that
\[
\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\lambda^* t} \|x_0\|. \tag{16}
\]
If there exists a positive scalar \( \lambda < \lambda_* \) such that
\[
T_f(t) \leq \frac{\lambda_* - \lambda}{\lambda_* + \lambda_*}, \tag{17}
\]
which is a condition for the unavailability rate of the controller, then we can easily obtain
\[
\lambda_* T_f(t) - \lambda_* (t - T_f(t)) \leq -\lambda t \tag{18}
\]
and, thus,
\[
\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\lambda^* t} \|x_0\|. \tag{19}
\]
This implies that the entire system is globally exponentially stable with decay rate \( \lambda^* \).

Secondly, when \( \mu > 1 \), in addition to (17), if there exists a scalar \( \lambda^* \in (0, \lambda_*) \) such that
\[
N_f \leq \frac{t}{T_f}, \quad T_f = \frac{\ln \mu}{\lambda - \lambda^*}, \tag{20}
\]
holds for all \( t > 0 \), then we can easily get
\[
-\lambda t + N_f \ln \mu \leq -\lambda^* t, \tag{21}
\]
and, thus,
\[
\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\lambda^* t} \|x_0\|. \tag{22}
\]
This means that the entire system is globally exponentially stable with decay rate \( \lambda^* \).

We observe that condition (20) is the requirement for the ATBCF. More precisely, if the ATBCF in system (1) is larger than or equal to \( T_f \) given in (20), then (21) and (22) still hold and the system’s exponential stability is guaranteed.

The above discussion indicates that (7) with (6) constitutes a piecewise Lyapunov function for system (1) with controller failures satisfying (17) and (20). We state this result in the following theorem.

**Theorem 1.** If the unavailability rate of the controller in system (1) is small in the sense of satisfying (17) for some \( \lambda > 0 \), then for any positive scalar \( \lambda^* \) smaller than \( \lambda \), there exists a finite constant \( T_f \) such that system (1) is globally exponentially stable with decay rate \( \lambda^* \), for any ATBCF larger than or equal to \( T_f \).

The next two remarks give more precise explanations of conditions (17) and (20).

**Remark 1.** Condition (17) implies that if we expect the entire system to potentially have a decay rate close to \( \lambda_* \) (i.e., \( \lambda \rightarrow \lambda_* \)), then we should restrict the total controller failure time small to be enough (i.e., \( T_f(t) \rightarrow 0 \)). This is reasonable when we consider that the control system with the designed controller breaks down only occasionally, and that we can recover it very quickly. In this case, we definitely expect that the system stability will not degenerate greatly.

Concerning the other two stability indices, \( \lambda_* \) and \( \lambda_* \), we observe that according to the unavailability rate condition (17), a comparatively long controller failure time \( T_f(t) \) is tolerable for large \( \lambda_* \) and small \( \lambda_* \). This is reasonable since the closed-loop system has a large decay rate (thus, a good stability property) when the controller works with large \( \lambda_* \) and the open-loop system does not diverge greatly when the controller fails with small \( \lambda_* \). Therefore, if we concentrate on the stability property of the system, we should design the original output feedback controller so that a large \( \lambda_* \) can be obtained.

**Remark 2.** While the unavailability rate condition (17) of the controller is easy to understand, the ATBCF condition (20) is not so straightforward. The key point is that if the open-loop system (when the controller fails) has very poor stability and controller failures occur very frequently, then the entire system will not perform well even when the total controller failure time interval is not long. If we expect the entire system to have a decay rate close to \( \lambda_* \), then we should require that \( T_f \) be large enough and, thus, that \( N_f \) be small enough, which means that the controller will not fail very frequently. Therefore, condition (20) is a balanced requirement for the decay rate and the number of controller failures.

The next remark discusses the case in which complete controller breakdown does not occur.

**Remark 3.** Although we concentrate on the case of complete controller breakdown \( (u = 0) \) in this paper, it is an easy matter to extend our discussion to the case where for various reasons the output feedback controller (2) (written as \( u = K(s)y \) abbreviatedly) decays in the sense of \( u \rightarrow aw \) with \( a \) being a fixed constant satisfying \( 0 \leq a < 1 \). In this case, if the closed-loop system composed of (1) and \( u = \alpha K(s)y \) is unstable, then our discussion up to now applies if we modify some notations. If this is not the case, then the entire system can be considered as a switched system composed of two stable subsystems; thus, it is globally exponential stable if the ATBCF is large enough (see detailed discussions in [9, 10]).

In the remainder of this section, we will give a numerical example to demonstrate Theorem 1.
**Example.** Consider the following mechanical system model, which was used in the benchmark problem proposed in [14] and discussed in [13]:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
y
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k & k & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u
\]

(23)

where \( k \) is the stiffness parameter changing in [0.5, 2.0].

For the above system with uncertain \( k \), the following dynamical output feedback controller was proposed in [14]:

\[
\begin{bmatrix}
\hat{x} \\
\hat{u}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} w
\]

is stable and the \( H_\infty \) norm of the transfer function from \( w \) to \( z \) in (25) is smaller than a prespecified constant \( \gamma \). Since our interest is in analyzing the \( H_\infty \) disturbance attenuation property of the system, we assume that \( \hat{x}(0) = 0 \) in (25).

Also, we assume that the designed controller (2) sometimes fails, and that we need a (not necessarily constant) time interval to recover it. When the controller fails, the closed-loop system assumes the form

\[
\begin{bmatrix}
\hat{x} \\
\hat{u}
\end{bmatrix} =
\begin{bmatrix}
0 & -0.7195 & 1 & 0 \\
0 & -2.9732 & 0 & 1 \\
-2.5133 & 4.8548 & -1.7287 & -0.9616 \\
1.0063 & -5.4097 & -0.0081 & 0.0304
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} w
\]

(26)

Now, we set \( k = 1.5 \) in system (23) and consider the case where the above designed controller fails from time to time. We can easily obtain \( \lambda_s = 0.18 \) and \( \lambda_u = 0.02 \), and solve the LMIs (6) to obtain the parameter \( \mu = 2.5 \).

If for \( \lambda = 0.16 \) and \( \lambda^* = 0.01 \), the unavailability rate of the controller satisfies \( T_u \leq \frac{\lambda_s - \lambda}{\lambda_s + \lambda_u} = 0.1 \), and the ATBCF is larger than \( T_f = \frac{\ln \mu}{\lambda - \lambda^*} \leq 2.80 \), then the entire system should be exponentially stable. In fact, Fig. 1 shows us that when the designed output feedback controller works with a time interval of 3.00 and then fails with a time interval of 0.10, which satisfies the unavailability rate and the ATBCF conditions, the norm of the solution quickly decreases to zero, where the initial condition is \( x_0 = [200 \quad 150 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \).

**III. \( H_\infty \) DISTURBANCE ATTENUATION ANALYSIS**

In this section, we assume that the dynamical output feedback controller (2) has been designed so that the closed-loop system

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
A_x & \bar{B}_w \\
C_x & D_{xz}
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
\]

(25)

is stable and the \( H_\infty \) norm of the transfer function from \( w \) to \( z \) in (25) is smaller than a prespecified constant \( \gamma \). Since our interest is in analyzing the \( H_\infty \) disturbance attenuation property of the system, we assume that \( \hat{x}(0) = 0 \) in (25).

Also, we assume that the designed controller (2) sometimes fails, and that we need a (not necessarily constant) time interval to recover it. When the controller fails, the closed-loop system assumes the form

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
A_x & \bar{B}_w \\
C_x & D_{xz}
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
\]

(26)

Then, the behavior of the entire system is dominated by a piecewise LTI system: system (25) when the controller works and system (26) when the controller fails.

Since \( A_x \) is stable and \( \left\| \bar{C}_s (sI - A_x)^{-1} \bar{B}_w + D_{xz} \right\| < \gamma \) according to the Bounded Real Lemma [16], we know immediately that there exist a scalar \( \lambda_s > 0 \) and a matrix \( P_s > 0 \) such that

\[
\begin{bmatrix}
P_{s} A_x + \lambda_s P_s + \frac{1}{\gamma} C^T_s C_x & \bar{B}_w \bar{B}_w^T
\end{bmatrix} < 0,
\]

(27)

or, equivalently,

\[
\begin{bmatrix}
P_{s} A_x + \lambda_s P_s + \frac{1}{\gamma} C^T_s C_x & \bar{B}_w \bar{B}_w^T
\end{bmatrix} < 0.
\]

(28)

Though \( A_u \) is unstable, we can always find a posi-
tive scalar $\lambda_u$ and a matrix $P_u > 0$ such that $A_u - \frac{\lambda_u}{2} I$ is stable, and
\[
\begin{bmatrix}
A_u^T P_u + P_u A_u - \lambda_u P_u & P_u \hat{B}_1 & C_u^T \\
\hat{B}_1^T P_u & -\gamma I & D_{11}^T \\
C_u & D_{11} & -\gamma I
\end{bmatrix} < 0,
\tag{29}
\]
or, equivalently,
\[
\begin{bmatrix}
A_u^T P_u + P_u A_u - \lambda_u P_u + \frac{1}{\gamma} C_u^T C_u & P_u \hat{B}_1 + \frac{1}{\gamma} C_u^T D_{11} \\
\hat{B}_1^T P_u + \frac{1}{\gamma} D_{11}^T C_u & -\gamma I + \frac{1}{\gamma} D_{11}^T D_{11}
\end{bmatrix} < 0.
\tag{30}
\]
This is always possible because $-\gamma I + \frac{1}{\gamma} D_{11}^T D_{11} < 0$ is guaranteed by (28); thus, we can choose the positive scalar $\lambda_u$ to be large enough so that the (1, 1)-th element of (29) or (30) is negative definite “enough.” Note that the above inequalities are LMIs [12] with respect to $P_u$ and $P_u$, and thus are easily solved using any one of several existing softwares.

Using solutions $P_u$ and $P_u$, we define the same piecewise Lyapunov function candidate (7) for system (1). Then, properties (ii) and (iii) stated in the above section are satisfied with $V_i(x) = \tilde{x}^T P_u \tilde{x}$ and $V_j(x) = \tilde{x}^T P_u \tilde{x}$, while the first property assumes the form (i') $V_i(x) = \tilde{x}^T P_u \tilde{x}$ and $V_j(x) = \tilde{x}^T P_u \tilde{x}$, which are continuous and have derivatives along solutions of the corresponding system that satisfy
\[
\dot{V}_i \leq -\lambda_i V_i - \frac{1}{\gamma} (\tilde{z}^T z - \gamma^2 w^T w),
\tag{31}
\]
\[
\dot{V}_u \leq -\lambda_u V_u - \frac{1}{\gamma} (\tilde{z}^T z - \gamma^2 w^T w).
\tag{32}
\]
The above two inequalities can be obtained easily. For example, the computation of $\dot{V}_i$ is as follows.
\[
\dot{V}_i = (A, \tilde{x} + \hat{B}_1 w)^T P_u \tilde{x} + \tilde{x}^T P_u (A, \tilde{x} + \hat{B}_1 w)
= \begin{bmatrix}
\tilde{x}^T & w^T
\end{bmatrix}
\begin{bmatrix}
A_u^T P_u + P_u A_u - \lambda_u P_u & P_u \hat{B}_1 \\
\hat{B}_1^T P_u & -\gamma I
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
w
\end{bmatrix}
\leq -\begin{bmatrix}
\tilde{x}^T & w^T
\end{bmatrix}
\begin{bmatrix}
\lambda_u P_u + \frac{1}{\gamma} C_u^T C_u & \frac{1}{\gamma} C_u^T D_{11} \\
\frac{1}{\gamma} D_{11}^T C_u & -\gamma I + \frac{1}{\gamma} D_{11}^T D_{11}
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
w
\end{bmatrix}
\leq -\lambda_i V_i - \frac{1}{\gamma} (\tilde{z}^T z - \gamma^2 w^T w)
\]
The second inequality in (31) can be obtained in a similar manner.

Now, without loss of generality, we assume that the designed controller works during $[t_{2i}, t_{2i+1})$, and that the controller fails during $[t_{2i+1}, t_{2i+2})$, $j = 0, 1, ...$, where $t_0 = 0$. Then, using the differential inequality theory (e.g., [15]) in (31), we get for any $t > 0$ that
\[
V(t) \leq \begin{cases}
\int_{t_{2i}}^{t_{2i+1}} \exp(-\lambda_i (t - \tau)) V(t) d\tau 
& \text{if } t_{2i} \leq t < t_{2i+1}, \\
\int_{t_{2i+1}}^{t} \exp(-\lambda_i (t - \tau)) V(t) d\tau 
& \text{if } t_{2i+1} \leq t < t_{2i+2},
\end{cases}
\tag{33}
\]
where $\Gamma(t) \triangleq \frac{1}{\gamma} (z^T (z - \gamma^2 w^T w)$. Therefore, for a given time instant $t$ located at $[t_{2i}, t_{2i+1})$ $(i \geq 0)$, we get from (31) by induction that
\[
V(t) \leq \mu^{2N_i} \exp(\lambda_i (t - t_{2i})) V(0) - \sum_{j=0}^{\lfloor \frac{t - t_{2i}}{1 + 2} \rfloor} \mu^{2N_j} \exp(\lambda_i (t - t_{2i+1})) \exp(-\lambda_i (t_{2i+1} - t_{2i+2})) \Gamma(t) d\tau.
\tag{34}
\]
Similarly, for a time instant $t$ located at $[t_{2i+1}, t_{2i+2})$ $(i \geq 0)$, we get
\[
V(t) \leq \mu^{2N_i} \exp(\lambda_i (t - t_{2i})) V(0) - \sum_{j=0}^{\lfloor \frac{t - t_{2i+1}}{1 + 2} \rfloor} \mu^{2N_j} \exp(\lambda_i (t - t_{2i+2})) \exp(-\lambda_i (t_{2i+2} - t_{2i+3})) \Gamma(t) d\tau.
\tag{35}
\]
When $\mu = 1$, we get from both (34) and (35) with $x(0) = 0$ and $V(t) \geq 0$ that
\[
\int_0^T \exp(\lambda_i (t - \tau)) \exp(-\lambda_i (t - \tau)) \Gamma(t) d\tau \leq 0. \tag{36}
\]
Note that the integral term $\exp(\lambda_i (t - \tau)) \exp(-\lambda_i (t - \tau)) \Gamma(t)$ in the above inequality is the transition matrix from time instant $t$ to $T$. Then, according to the stability analysis results given in the previous section, the inequality
\[
\left\| \exp(\lambda_i (t - \tau)) \exp(-\lambda_i (t - \tau)) \right\| \leq c e^{-\lambda (t - \tau)} \tag{37}
\]
holds with $c = \frac{\sqrt{e \alpha_i}}{\alpha_i}$, under the assumption that there
exists a positive scalar \( \lambda < \lambda_s \) such that

\[
\frac{T_f}{t} \leq \frac{\lambda - \lambda_s}{\lambda + \lambda_s},
\]

(38)

which is a condition imposed on the unavailability rate of the controller. Combining (36) and (37), we obtain

\[
\int_0^t e^{-\lambda(t-\tau)} z^T(\tau) d\tau \leq c\gamma^2 \int_0^t \frac{e^{-\lambda(t-\tau)}}{\lambda} w^T(\tau) w(\tau) d\tau.
\]

(39)

We integrate both sides of the above inequality from \( t = 0 \) to \( t = \infty \) to obtain (by rearranging the double-integral area)

\[
\frac{1}{\lambda_s} \int_0^t z^T(\tau) z(\tau) d\tau \leq c\gamma^2 \int_0^t w^T(\tau) w(\tau) d\tau,
\]

(40)

which means that the \( H_\infty \) disturbance attenuation level

\[
\sqrt{\frac{c\mu \lambda}{\lambda}} \gamma
\]

is achieved under the unavailability rate condition (38).

Next, when \( \mu > 1 \), we get from (34) and (35) that

\[
\int_0^t \mu^{t-N_s(t)} e^{\mu(t-t')-\lambda_s(t-t')} \lambda_s(t-t') z^T(\tau) z(\tau) d\tau
\]

\[
\leq \gamma^2 \mu^{-N_s(t)} \int_0^t \lambda_s(t-t') z^T(\tau) z(\tau) d\tau
\]

\[
\cdot \int_0^t w^T(\tau) w(\tau) d\tau.
\]

(41)

In this case, if in addition to (38), there exists a positive scalar \( \lambda^* \) such that

\[
N_s \leq \frac{t}{T_f}, \quad T_f = \frac{\ln \mu}{\lambda}
\]

(42)

hold for all \( t > 0 \), then we know that \( \mu^{N_s} \leq e^{\lambda^* t} \) holds for any \( t > 0 \). Using this inequality, we multiply both sides of (41) by \( \mu^{-N_s} \) to obtain

\[
\int_0^t e^{-\lambda^* (t-\tau)} z^T(\tau) z(\tau) d\tau \leq c\mu \gamma^2 \int_0^t e^{-\lambda(t-\tau)} w^T(\tau) w(\tau) d\tau.
\]

(43)

Integrating both sides of the above inequality from \( t = 0 \) to \( t = \infty \) yields

\[
\frac{1}{\lambda_s} \int_0^t e^{-\lambda(t-\tau)} z^T(\tau) z(\tau) d\tau \leq c\mu \gamma^2 \int_0^t w^T(\tau) w(\tau) d\tau,
\]

(44)

which means that a weighted \( H_\infty \) disturbance attenuation level

\[
\sqrt{\frac{c\mu \lambda}{\lambda}} \gamma
\]

is achieved.

We observe that condition (42) is the requirement imposed on the ATBCF. More precisely, if the ATBCF in system (1) is larger than or equal to \( T_f \) given in (42), then (44) still holds, and the system achieves the same weighted \( H_\infty \) disturbance attenuation level. We summarize the above discussion in the following theorem.

**Theorem 2.** If the unavailability rate of the controller in system (1) is small in the sense of satisfying (38) for some \( \lambda > 0 \), then there exists a finite constant \( T_f \) such that system (1) achieves a weighted \( H_\infty \) disturbance attenuation level

\[
\sqrt{\frac{c\mu \lambda}{\lambda}} \gamma
\]

in the sense of (44), for any ATBCF larger than or equal to \( T_f \).

**Remark 4.** The inequality (44) describes a weighted \( H_\infty \) disturbance attenuation level from \( w(t) \) to \( z(t) \) due to the existence of \( e^{-\lambda^* t} \). When \( \lambda^* \) is small enough (close enough to zero), which means that the ATBCF is sufficiently large according to (42), obviously, inequality (44) approaches normal \( H_\infty \) disturbance attenuation.

**Remark 5.** We can easily extend the discussion presented here to the case where the output feedback controller (2) (written as \( u = K(s)y \) abbreviatedly) decays in the sense of \( u \rightarrow \alpha u \) with \( \alpha \) being a fixed constant satisfying \( 0 \leq \alpha < 1 \). In this case, if the closed-loop system composed of (1) and \( u = \alpha K(s)y \) is unstable, the discussions up to now are the same after making changes to some notations. If this is not the case, then the entire system can be viewed as a switched system composed of two stable subsystems; thus, a weighted \( H_\infty \) disturbance attenuation level is achieved under an ATBCF scheme (without considering the unavailability rate of the controller), and the achieved weighted \( H_\infty \) disturbance attenuation level approaches normal \( H_\infty \) disturbance attenuation level if the ATBCF is large enough. The reader may refer to the detailed discussions in [11].

**IV. CONCLUSION**

We have studied stability and \( H_\infty \) disturbance attenuation properties for LTI control systems controlled by a pre-designed dynamical output feedback controller which fails from time to time due to physical or purposeful reasons. For stability, by using a piecewise Lyapunov function, we have shown that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures (ATBCF) is large enough, then the global exponential stability of the system is guaranteed. For \( H_\infty \) disturbance attenuation, also by using a piecewise Lyapunov function, we have shown that if the unavailability rate of the controller is smaller than a specified constant, then the system with an ATBCF achieves a reasonable weighted \( H_\infty \) disturbance attenuation level, and the weighted \( H_\infty \) disturbance attenuation approaches normal \( H_\infty \) disturbance attenuation if the ATBCF is suf-
ficiently large. We suggest that the methodology used here can also be applied to other types of performance specification analysis of control systems with controller failures.

REFERENCES


