ROBUST EIGENVALUE ASSIGNMENT IN DESCRIPTOR SYSTEMS
VIA OUTPUT FEEDBACK

Guang-Ren Duan, James Lam, and Guo-Ping Liu

ABSTRACT

Based on a recently proposed parametric approach for eigenstructure assignment in descriptor linear systems via output feedback, the robust eigenvalue assignment problem in descriptor linear systems via output feedback is solved. The problem aims to assign a set of finite closed-loop eigenvalues which have minimum sensitivities with respect to perturbations in the closed-loop coefficient matrices, while at the same time, guarantee the closed-loop regularity. The approach optimizes the design parameters existing in the closed-loop eigenvectors to achieve the minimum eigenvalue sensitivities, and use the extra degree of freedom existing in the solution of the gain matrix to further minimize the magnitude of the gain matrix and enhance the robustness of the closed-loop regularity. The approach allows the finite closed-loop eigenvalues to be optimized within desired regions, and is demonstrated to be simple and effective.

Keywords: Descriptor systems, output feedback, robust pole assignment, eigenvalue sensitivities, eigenstructure assignment.

I. INTRODUCTION

As is well known, solution to the problem of eigenvalue assignment in a multi-variable linear system is generally not unique. This fact was known in the early 70s but has only been fully revealed by recent parametric eigenstructure assignment approaches in [1-12]. The degrees of freedom provided by parametric eigenstructure assignment may be used to achieve some desired system specifications. Such an idea has resulted in many applications of parametric eigenstructure assignment ([1], and [13-16]).

By making use of these degrees of freedom in eigenvalue assignment in multivariable linear systems, the closed-loop eigenvalues may be made as insensitive as possible to perturbations in the components of the closed-loop system coefficient matrices. This problem is known, in the literature, as robust pole assignment, and has been extensively studied by many authors for the case of normal linear systems (e.g., [8,17-22]). However, for the case of descriptor systems, this problem has only been investigated by a few researchers ([23-27]). Kautsky and his coauthors ([23,24]) extended their earlier well known techniques in [17] developed for normal linear systems to the case of descriptor systems, and laid a special emphasis on the closed-loop regularity. Syrmos and Lewis ([25]) developed a robustness theory for the generalized spectrum of descriptor linear systems, and presented a compact theory for the robust eigenvalue assignment problem in descriptor linear systems using the concept of chordal metric. Different from the above, Duan and Patton ([26]) studied robust pole assignment in descriptor linear systems via proportional plus partial derivative state feedback. Due to the capacity of the derivative feedback, their work concentrates on the case that the closed-loop system possesses \( n \) (the system order) finite closed-loop eigenvalues. Very recently, a new approach is proposed in [28] for robust
pole assignment in descriptor linear systems via state feedback control. This work is based on the eigenstructure assignment result presented in [11], and like [26], it realizes robust pole assignment by minimizing the condition numbers associated with the closed-loop eigenvalues.

In this paper, robust pole assignment in multivariable descriptor linear systems via output feedback is investigated based on the eigenstructure assignment approach recently proposed in [12], where parametric expressions for both the left and right closed-loop eigenvectors in terms of the closed-loop eigenvalues and two groups of parameter vectors are proposed. By using these parametric expressions of the closed-loop eigenvectors and the perturbation theory of generalized eigenvalue problems proposed by [29], closed-loop eigenvalue sensitivities in terms of these design parameters are obtained. These parameters existing in the closed-loop eigenvectors are then optimized to minimize the closed-loop eigenvalue sensitivities. When the closed-loop system resulted in by these parameters is sub-normal ([12]), there exists a unique corresponding closed-loop system.

When the following output feedback control law

\[ E \delta x = Ax + Bu \]

\[ y = Cx \] (1)

where \( \delta \) denotes the differential operator d/dt for continuous-time systems, or the one-step forward operator \( q \) (defined by \( qx(k) = x(k+1) \)) for discrete-time systems; \( x \in \mathbb{R}^n, u \in \mathbb{R}^r, y \in \mathbb{R}^m \) are, respectively, the descriptor-variable vector, the input vector and the output vector; \( A, E \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^m \) are known matrices with, \( \text{rank}(E) = n_0 \leq n, \text{rank}(B) = r, \text{rank}(C) = m \), and they satisfy the following controllability and observability assumption

**Assumption A1.** \( \text{rank}([sE - A, B]) = \text{rank} \{(sE^T - A^T, C^T)\} = n \) for \( s \in C \)

When the following output feedback control law

\[ u = Ky, \quad K \in \mathbb{R}^{r \times m} \] (2)

is applied to the system (1), the closed-loop system is obtained in the following form

\[ E \delta x = Ax, \quad A_1 = A + BKC \] (3)

Refer to the fact that non-defective matrix pair \([E, A_1]\) possesses relative eigenvalues which are less sensitive to the matrix parameter perturbations ([24]), we here restrict the closed-loop finite eigenvalues to be a set of \( n_0 \) distinct, but self-conjugate complex numbers. The robust pole assignment problem for system (1) via the output feedback control law (2) can then be stated as follows.

**Problem RPA.** Given system (1) satisfying Assumption A1, and a series of regions \( \Omega_i, i = 1, 2, \ldots, n_0 \), on the complex plane, seek an output feedback controller in the form of (2), such that the following requirements are met:

1. The closed-loop system (3) is regular, and has \( n_0 \) number distinct finite relative eigenvalues.
2. The finite closed-loop eigenvalues \( s_i, i = 1, 2, \ldots, n_0 \), satisfy the location conditions \( s_i \in \Omega_i, i = 1, 2, \ldots, n_0 \), and are as insensitive as possible to parameter perturbations in the closed-loop system matrices \( E \) and \( A_1 \).

**Remark 2.1.** The conditions \( s_i \in \Omega_i, i = 1, 2, \ldots, n_0 \), in the above problem represent the requirement on the closed-loop stability and performance property. For a real closed-loop eigenvalue \( s_i \), the region \( \Omega_i \) may be chosen to be an interval \([a_i, b_i]\). For a pair of complex eigenvalues \( s_i \) and \( s_j \), the regions \( \Omega_i \) and \( \Omega_j \) may often be chosen as
\[ \Omega_{ij} = \left\{ s = -\sigma_j \pm \sigma_j, \sigma_j \in [a_i, b_i], \sigma_j \in [a_j, b_j] \right\} \] (4a)

or

\[ \Omega_{ij} = \left\{ s = -\sigma_j \pm \sigma_j, (\sigma_j - \sigma_j^0)^2 + (\sigma_j - \sigma_j^0)^2 \leq \rho_j^2 \right\} \] (4b)

where \( a_i, b_i, \sigma_j^0 \) and \( \rho_j \) are some properly chosen scalars.

### III. PRELIMINARIES

#### 3.1 An eigenstructure assignment result

Under Assumption A1, there exist a pair of right coprime polynomial matrices \( N(s) \in \mathbb{R}^{m_0 \times n} \) and \( D(s) \in \mathbb{R}^{r \times r} \), and a pair of right coprime polynomial matrices \( H(s) \in \mathbb{R}^{m_0 \times n} \) and \( L(s) \in \mathbb{R}^{r \times r} \), satisfying

\[ (A - sE)N(s) + BD(s) = 0 \] (5)

and

\[ (A - sE)^T H(s) + C^T L(s) = 0 \] (6)

Let the infinite eigenvalue of the closed-loop system be denoted by \( s_\infty \). Then \( s_\infty \) is a multiple eigenvalue with both geometric and algebraic multiplicities being equal to \( n - n_0 \). Therefore, there are \( n - n_0 \) left and right eigenvectors associated with \( s_\infty \). Denote the left and the right eigenvector matrices of the closed-loop system (3) by \( T_\infty \in \mathbb{C}^{n \times n_0} \) and \( V_\infty \in \mathbb{C}^{m \times n_0} \), then by definition,

\[ EV_\infty = 0, \quad \text{rank}(V_\infty) = n - n_0 \] (7)

and

\[ T_\infty^T E = 0, \quad \text{rank}(T_\infty) = n - n_0 \] (8)

Following the main result in [12], we can obtain the following general form for all the output feedback controllers which meet the first condition in Problem RPA.

**Lemma 3.1.** Let Assumption A1 be satisfied, and \( T_\infty \) and \( V_\infty \) be the infinite left and right closed-loop eigenvector matrices given in (7) and (8), respectively. Then

(1) All the output feedback controllers in the form of (2) for the descriptor linear system (1), which satisfy the first condition in Problem RPA can be parameterized by

\[ K = (W V^T + W V^T) C^T [C (V V^T + V V^T) C^T]^{-1} \] (9)

or

\[ K = [B^T (T T^T + T T^T) B]^{-1} B^T (T Z^T + T Z^T) \] (10)

with the matrices \( T, V, W \) and \( Z \) given by

\[ V = \begin{bmatrix} N(s_1) & N(s_2) & \cdots & N(s_{n_0}) \end{bmatrix} \] (11a)

\[ W = \begin{bmatrix} D(s_1) & D(s_2) & \cdots & D(s_{n_0}) \end{bmatrix} \] (11b)

\[ T = \begin{bmatrix} H(s_1) g_1 & H(s_2) g_2 & \cdots & H(s_{n_0}) g_{n_0} \end{bmatrix} \] (12a)

\[ Z = \begin{bmatrix} L(s_1) g_1 & L(s_2) g_2 & \cdots & L(s_{n_0}) g_{n_0} \end{bmatrix} \] (12b)

where \( W_\infty, Z_\infty, f_i \) and \( g_i, i = 1, 2, \ldots, n_0 \), are the design parameters satisfying the following constraints:

**C1:** \( f_i = \bar{f}_i \), \( g_i = \bar{g}_i \) if \( s_i = \bar{s}_i \)

**C2:** \( g_j^T H^T(s_j) EN(s_i) f_i = \delta_{ij}, i, j = 1, 2, \ldots, n_0 \)

**C3:** \( (t_j^T)^N B(s_j) f_i = (z_j^T)^N C N(s_i) f_i \) \( i = 1, 2, \ldots, n_0 \), \( j = 1, 2, \ldots, n - n_0 \)

**C4:** \( (t_j^T)^N B W_\infty = (z_j^T)^N C V_\infty \), \( j = 1, 2, \ldots, n - n_0 \)

**C5:** \( \det(T_\infty^T A V_\infty + T_\infty^T B W_\infty) \neq 0 \) or \( \det(T_\infty^T A V_\infty + Z_\infty^T C V_\infty) \neq 0 \)

where \( \delta_{ij} \) represents the Kronecker function, and \( t_j^T, v_j^T, w_j^T \) and \( z_j^T \) are the columns of matrices \( T_\infty, V_\infty, W_\infty \), and \( Z_\infty \), respectively.

(2) The matrices \( T \) and \( V \) given above are a pair of normalized left and right finite eigenvector matrices for the closed-loop system, that is, they satisfy

\[ T^T A_s = \Lambda^T T E, \quad A V = E V \Lambda \] (13)

and

\[ T^T A V = \Lambda, \quad T^T E V = I_{n_0} \] (14)

with

\[ A_s = A + B K C, \Lambda = \text{diag}[s_1, s_2, \cdots, s_{n_0}] \] (15)

#### 3.2. Closed-loop eigenvalue sensitivities

To derive the closed-loop eigenvalue sensitivity measures, we need the following perturbation result of generalized eigenvalue problems of matrix pairs.

**Definition 3.1** [29]. Let \( M, N \in \mathbb{R}^{m \times n} \), \( \lambda \) be a simple finite relative eigenvalue of the matrix pair \( [M \ N] \). A pair of right and left eigenvectors \( x \) and \( y \) of the matrix pair \( [M \ N] \) associated with eigenvalue \( \lambda \) are said to be a normalized pair if

\[ A \]
Proposition 3.1 [29]. Let \( M, N \in \mathbb{R}^{n \times n} \), \( \lambda \) be a simple finite relative eigenvalue of the matrix pair \( [M \ N] \). Then perturbations of order \( O(\epsilon) \) in the components of matrices \( M \) and \( N \) cause perturbations of order \( O(\epsilon c(\lambda)) \) in the eigenvalue \( \lambda \), here \( c(\lambda) \) is the condition number corresponding to the eigenvalue \( \lambda \), defined by

\[
c(\lambda) = \frac{\| y \|_2 \| x \|_2}{\| (1 + |\lambda|^2)^{1/2} \|_2} \quad (17)
\]

with \( x \) and \( y \) being a pair of normalized right and left eigenvectors of the matrix pair \( [M \ N] \) associated with eigenvalue \( \lambda \).

Based on Lemma 3.1 and the above proposition, we can easily prove the following result.

Lemma 3.2. Let Assumption A1 be satisfied, and the matrices \( K \), \( T \) and \( V \) be given by Lemma 3.1. Then the closed-loop system (3) takes \( s_i \), \( i = 1, 2, \ldots, n_0 \), as the set of closed-loop finite eigenvalues, and the condition number corresponding to the eigenvalue \( s_i \) is given by

\[
c_i = \frac{\| s_i \|_2 \| v_i \|_2}{\| (1 + |s_i|^2)^{1/2} \|_2} \frac{\| H(s_i)g_i \|_2 \| N(s_i)f_i \|_2}{\| (1 + |s_i|^2)^{1/2} \|_2} \quad (18)
\]

\( i = 1, 2, \ldots, n_0 \)

Proof. It follows from Lemma 3.1 that the matrices \( T \) and \( V \) given by (12a) and (11a) are a pair of normalized left and right finite eigenvectors for the closed-loop system. Therefore, \( t_i = H(s_i)g_i \) and \( v_i = N(s_i)f_i \) form a pair of normalized left and right eigenvectors for the closed-loop system associated with the eigenvalue \( s_i \). It thus follows from Proposition 3.1 that the sensitivity measures of the closed-loop finite eigenvalues are given by (18).

Remark 3.1. The general parametric expressions for the closed-loop eigenvectors can be immediately written out as soon as the two pairs of right coprime polynomial matrices \( N(s) \) and \( D(s) \), \( H(s) \) and \( L(s) \), satisfying (5) and (6), are derived. For solutions of such right coprime polynomial matrices \( N(s) \) and \( D(s) \) satisfying (5) (or \( H(s) \) and \( L(s) \) satisfying (6)), several ways have been given in [30] under the controllability condition of the open-loop system (1). General computational methods for such right coprime polynomial matrices can also be found in [21,31,32,33] and [34].

IV. ROBUST POLE ASSIGNMENT

It follows from Lemma 3.1 that the design freedom existing in the closed-loop eigenstructure assignment consists of the following three parts:

1. The closed-loop eigenvalues \( s_i \), \( i = 1, 2, \ldots, n_0 \).
2. The group of parameter vectors \( f_i, g_i \), \( i = 1, 2, \ldots, n_0 \).
3. The parameter matrix \( W_\infty \) or \( Z_\infty \).

These parameters are required to satisfy Constraints C1–C5. In many cases, these parameters exist and are generally not unique. The Proposition 1 in [12] has stated a sufficient condition for existence of these parameters based on the pole assignment result proposed in [35]. To solve Problem RPA, in the following we need to seek proper choices of these three parts of design parameters to meet the two requirements in Problem RPA stated in Section 2.

4.1. Optimizing parameters \( f_i, g_i \) and \( s_i \), \( i = 1, 2, \ldots, n_0 \)

It follows from Lemma 3.1 that the feedback gain matrix given by (9) or (10), with the matrices \( V, W \) and \( T, Z \) given by (11) and (12), and the design parameters \( f_i \) and \( g_i \), \( W_\infty \) and \( Z_\infty \), satisfying constraints C1–C5, meet the first requirement in Problem RPA. Note that the eigenvalue sensitivity measures given in Lemma 3.2 have relations only with the parameters \( f_i, g_i \), and \( s_i \), \( i = 1, 2, \ldots, n_0 \), to further meet the second requirement in Problem RPA, a natural idea is to minimize the closed-loop eigenvalue sensitivity measures \( c_i \), \( i = 1, 2, \ldots, n_0 \), defined in Lemma 3.2 by optimizing the parameters \( f_i, g_i \) and \( s_i \), \( i = 1, 2, \ldots, n_0 \).

Define the objective

\[
J = J(s_i, f_i, g_i, i = 1, 2, \ldots, n_0) = \sum_{i=1}^{n_0} \tau_i c_i^2 \quad (19)
\]

with \( c_i \), \( i = 1, 2, \ldots, n_0 \), being the closed-loop eigenvalue measures defined by (18), and \( \tau_i \), \( i = 1, 2, \ldots, n \), being a group of positive scalars representing the weighting factors. Then the parameters \( f_i, g_i \) and \( s_i \), \( i = 1, 2, \ldots, n_0 \), can be sought by the following optimization problem:

\[
\text{minimize} \quad J(s_i, f_i, g_i, i = 1, 2, \ldots, n_0) \quad \text{s.t.} \quad s_i \in \Omega_s, \quad i = 1, 2, \ldots, n_0 \quad (20)
\]

Constraints C1 and C2

4.2. Optimizing parameter \( W_\infty \) or \( Z_\infty \)

The third part of parameter \( W_\infty \) or \( Z_\infty \) is associated with the solution of the feedback gain \( K \). It follows from the Theorem 4 in [12] that, for arbitrary \( f_i, g_i \) and \( s_i \), \( i = 1, 2, \ldots, n_0 \), satisfying Constraints C1 and C2, there always exist \( W_\infty \) or \( Z_\infty \) satisfying Constraints C3 and C4. In this subsection we assume that, with the parameters \( f_i, g_i \) and \( s_i \), \( i = 1, 2, \ldots, n_0 \) obtained by solving the optimization problem...
problem (20), the matrix \( W_{\infty} \) or \( Z_\infty \) satisfying Constraints C3 and C4 is not unique. In this case, we should make a good use of this extra degree of freedom.

There are two criteria to select the parameter \( W_{\infty} \) or \( Z_\infty \) satisfying Constraints C3–C5. One is to make the magnitude of the feedback gain matrix \( K \) given by (9) or (10) be small. The other is to make the solution to the inverse of the matrix

\[
\Sigma_\infty = T_{\infty}^T A V_{\infty} + T_{\infty}^T B W_{\infty}
\]
or

\[
\Sigma_\infty = T_{\infty}^T A V_{\infty} + Z_{\infty}^T C V_{\infty}
\]

be well-conditioned. This is because the non-singularity of this matrix determines the closed-loop regularity ([12]) and making the inverse of this matrix well-conditioned enhances the robustness of the closed-loop regularity against system parameter perturbations. Combining these two aspects yields the following objective

\[
J_\infty = J_\infty(W_{\infty} \text{ or } Z_\infty) = \beta \| K \|_\infty + \gamma \| \Sigma_\infty \|_\infty + \| \Sigma_\infty^{-1} \|_\infty
\]

where \( K \) is given by (9) or (10), \( \beta \) and \( \gamma \) are two positive scalars which represent the weighting factors. Then the parameter \( W_{\infty} \) or \( Z_\infty \) can be found by solving the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad J_\infty(W_{\infty} \text{ or } Z_\infty) \\
\text{s.t.} & \quad \text{Constraints C3 ~ C5}
\end{align*}
\]

### 4.3. Closed-loop regularity and the algorithm

The closed-loop regularity is actually guaranteed by Constraint C5 ([12]).

When the parametric matrix \( W_{\infty} \) or \( Z_\infty \) satisfying Constraints C3 and C4 is not unique, solving the optimization problem (22) produces a parameter matrix \( W_{\infty} \) or \( Z_\infty \) satisfying Constraint C5. Note that Constraint C5 is actually the closed-loop regularity condition, the final gain matrix determined by the parameters \( f_\infty, g_\infty \) and \( s_\infty, i = 1, 2, \ldots, n_0 \) and \( W_{\infty} \) or \( Z_\infty \) naturally meets the closed-loop regularity requirement. However, when the parameter matrix \( W_{\infty} \) or \( Z_\infty \) satisfying Constraints C3 and C4 is unique, there is obviously no need to optimize this parameter matrix. Such a case is defined in [12] as the sub-normal case because, as in the case of normal linear systems, the feedback gain is now totally determined by the closed-loop eigenvectors (or the design parameters \( f_\infty, g_\infty \) and \( s_\infty, i = 1, 2, \ldots, n_0 \)). Thus in this case, the closed-loop regularity is also completely determined by the parameters \( f_\infty, g_\infty \) and \( s_\infty, i = 1, 2, \ldots, n_0 \).

It follows from the Theorem 7 in [12] that the sub-normal case occurs if and only if

\[
(CV)(CV)^{-1} = I \quad \text{or} \quad (T^T B)^{-1}(T^T B) = I
\]

where \( M^{-1} \) represents the inner inverse of the matrix \( M \), which is defined to be any matrix \( M^{-1} \) satisfying \( MM^{-1} = M \) (see [4]). When (23) is met, the gain matrix is given by

\[
K = W(CV)^{-1} \quad \text{or} \quad K = (T^T B)^{-1} Z^T
\]

and Constraint C5 is given by

\[
C_{5}^*:
\begin{cases}
\det\left\{T_{\infty}^T \left(A + BW(CV)^{-1} C\right)V_{\infty}\right\} \neq 0 \\
\det\left\{T_{\infty}^T \left(A + B(T^T B)^{-1} Z^T C\right)V_{\infty}\right\} \neq 0
\end{cases}
\]

A special case of the above condition (23) is

\[
\operatorname{rank}(CV) = m \leq n_0 \quad \text{or} \quad \operatorname{rank}(T^T B) = r \leq n_0
\]

When this condition is met, the gain matrix is given by

\[
K = W(CV)^{-1} [(CV)(CV)^{-1}]^{-1} \quad \text{or} \quad K = (T^T B)^{-1} (T^T B)^{-1} Z^T
\]

and Constraint C5 becomes

\[
C_{5}^*:
\begin{cases}
\det\left\{T_{\infty}^T \left(A + BW(CV)^{-1} C\right)V_{\infty}\right\} \neq 0 \\
\det\left\{T_{\infty}^T \left(A + B(T^T B)^{-1} Z^T C\right)V_{\infty}\right\} \neq 0
\end{cases}
\]

When equalities hold in (25), the solution in (26) reduces to

\[
K = W(CV)^{-1} \quad \text{or} \quad K = (T^T B)^{-1} Z^T
\]

while Constraint C5 is changed to

\[
C_{5}^*:
\begin{cases}
\det\left\{T_{\infty}^T \left(A + BW(CV)^{-1} C\right)V_{\infty}\right\} \neq 0 \\
\det\left\{T_{\infty}^T \left(A + B(T^T B)^{-1} Z^T C\right)V_{\infty}\right\} \neq 0
\end{cases}
\]

It follows from the above that, after solving the optimization problem (20), we need to check the sub-normal condition (23) (or its special one (25)). If this condition is not met, solve the optimization problem (22). If the sub-normal condition (23) (or its special one (25)) is met, check Constraint C5 (or its special ones C5 or C5*). If this Constraint is met, calculate the gain matrix
Algorithm RPA:

1. Solve the two pairs of matrix polynomials \(N(s)\) and \(D(s)\), \(H(s)\) and \(L(s)\) satisfying the two equations (5) and (6);
2. Solve the left and right eigenvector matrices \(T_\infty\) and \(V_\infty\) associated with the infinite eigenvalues of the system defined by the equations in (7) and (8);
3. Find parameters \(f_i, g_i\) and \(s_i\), \(i = 1, 2, \ldots, n_0\), by solving the minimization problem (20). If such parameters do not exist, the robust pole assignment problem for this system does not have a solution;
4. Compute matrices \(T\) and \(V\), \(s_i\), or \(W\) and \(Z\), according to the formulae (11) or (12) based on the parameter vectors \([f_i]\) or \([g_i]\) obtained;
5. Check condition (23) or (25), if satisfied, carry on with the next step. Otherwise, go to Step 7.
6. Verify Constraint C5 (or C5). If Constraint C5 (or C5) is met, compute the feedback gain matrix \(K\) according to (24) or (26) and then terminate the program. Otherwise, solve the optimization problem (28) and then go back to Step 4.
7. Find parameters \(W_\infty\) and \(Z_\infty\) by solving the minimization problem (22).
8. Compute the feedback gain matrix \(K\) by formula (9) or (10) based on the matrices \(T\) and \(V\), or \(W\) and \(Z\), obtained in Step 4 and the parameter matrix \(W_\infty\) or \(Z_\infty\) obtained in Step 7.

Remark 4.1. Since the closed-loop finite eigenvalues are also included in the design parameters and are optimized within desired regions in the complex plane, this algorithm can usually give a closed-loop system with better robustness. Moreover, because the completeness of the eigenstructure assignment approach used, the optimality of the solution to the robust pole assignment problem obtained through Algorithm RPA is totally dependent on the solution to the optimization problem (20) (or (28) and (29)). However, since the optimization (20) or (28) is generally a non-convex nonlinear programming, it is very difficult to produce an algorithm for solving this optimization with theoretical guarantee of global optimality. It is recommended that proper functions in the Matlab Optimization Toolbox are used to handle such optimization problems.

Remark 4.2. It is obvious that an alternative way of solving Problem RPA is to optimize parameters \(f_i, g_i\) and \(s_i\), \(i = 1, 2, \ldots, n_0\) and \(W_\infty\) or \(Z_\infty\) all at once by solving

\[
\text{minimize } \{J(s, f_i, g_i, s_i, i = 1, 2, \ldots, n_0) + J_\infty(W_\infty \text{ or } Z_\infty)\}
\]

\[\text{s.t. } \begin{align*}
& s_i \in \Omega_i, \ i = 1, 2, \ldots, n_0 \\
& \text{Constraints C1 \sim C5}
\end{align*}\]

However, this minimization requires much more computational load. Furthermore, without guarantee of global optimality, the results obtained practically via this simultaneous optimization may not be better than those obtained using Algorithm RPA.

Remark 4.3. The result proposed in this section generalizes a few previously reported ones. Particularly, when the observation matrix \(C\) reduces to the identity matrix, the result becomes the one for robust pole assignment in descriptor linear systems via state feedback proposed in [28]; and when the open-loop system (1) becomes a normal linear system, the proposed result becomes the one in [21]. Other very closely related works are [26], which treats robust pole assignment in descriptor linear systems via proportional plus derivative state feedback, and [8], which treats robust pole assignment in normal linear systems via dynamical output feedback.

V. AN ILLUSTRATIVE EXAMPLE

Consider a system in the form of (1) with the following coefficient matrices ([4,10] and [12]):

\[
E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
1 & -1 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

It is easy to verify that with this example system Assumption A1 holds. Moreover, it can be obtained that the left and right infinite eigenvector matrices \(T_\infty\) and \(V_\infty\) defined by (7) and (8) are

\[
V_\infty = T_\infty = \begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}^T
\]

5.1. Eigenstructure assignment

Restrict the closed-loop eigenvalues \(s_i\), \(i = 1, 2, 3\) to
be distinct and real, and denote

\[ f_i = f_i^1 = \begin{bmatrix} \alpha_i \\ \alpha_i \\ \alpha_i \end{bmatrix}, \quad g_i = g_i^1 = \begin{bmatrix} \beta_i \\ \beta_i \\ \beta_i \end{bmatrix} \quad i = 1,2,3 \]

and

\[ W_\infty = w_\infty^\prime = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad Z_\infty = z_\infty^\prime = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \]

then the matrices \( T, V, W \) and \( Z \) are given as follows

\[ T = \begin{bmatrix} \beta_{i1} & \beta_{i2} & \beta_{i3} \\ s_i\beta_{i1} & s_i\beta_{i2} & s_i\beta_{i3} \\ \beta_{i2} & \beta_{i3} & \beta_{i1} - \beta_{i3} \\ s_i\beta_{i2} & s_i\beta_{i3} - \beta_{i1} & s_i\beta_{i3} - \beta_{i1} \end{bmatrix} \quad i = 1,2,3 \]

\[ Z = \begin{bmatrix} \beta_{i1} & -\beta_{i2} & -\beta_{i3} \\ \beta_{i2} & -\beta_{i3} & -\beta_{i1} \end{bmatrix} \]

and

\[ V = \begin{bmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \alpha_{i2} & \alpha_{i3} & \alpha_{i1} \\ \alpha_{i3} & \alpha_{i1} & \alpha_{i2} \\ \psi_1 & \psi_2 & \psi_3 \end{bmatrix} \]

\[ W = \begin{bmatrix} s_i\alpha_{i1} & s_i\alpha_{i2} & s_i\alpha_{i3} & s_i\alpha_{i1} & s_i\alpha_{i2} & s_i\alpha_{i3} \\ \psi_1 & \psi_2 & \psi_3 & -\alpha_{i3} & -\alpha_{i3} & -\alpha_{i3} \end{bmatrix} \]

with

\[ \varphi_i = (-\alpha_{i1} + \alpha_{i2} + \alpha_{i3})s_i - \alpha_{i1} - \alpha_{i2} + 3\alpha_{i3} \]

\[ \psi_i = (s_i + 1)\alpha_{i1} - s_i\alpha_{i2} - 3\alpha_{i3} \]

Since all the finite closed-loop eigenvalues are real, we can also restrict the parameters \( \alpha_i \)'s and \( \beta_i \)'s to be real. Therefore, Constraint C1 holds automatically, while Constraints C2~C5 give the following set of equations:

\[ \beta_{ij}(s_j + 1)\xi_{ij} + (\beta_{ij} - s_i\beta_{ij})\xi_{ij} = \rho_j, \quad i = 1,2,3 \]

where

\[ \rho_j = -(2s_j - 1)\beta_{ij} + s_i\beta_{ij}\eta_j - \beta_{ij}, \quad i = 1,2,3 \]

For a thorough treatment of this set of constraints, refer to the subsections 4.1 and 4.2 in [12].

5.2. Robust pole assignment

This subsection considers robust pole assignment in this example system based on the above general result of eigenstructure assignment. The main task in solving the robust pole assignment problem is to solve the optimization problem (20).

Obviously, the condition numbers defined by (18) are

\[ c_i = \frac{\left\{ (1 + s_i^2)\beta_{i1}^2 + \beta_{i2}^2 + (s_i\beta_{i2} - \beta_{i1})^2 \right\}^{1/2}}{\left\{ s_i^2 + \beta_{i2}^2 + (s_i\beta_{i2} - \beta_{i1})^2 \right\}^{1/2}} \]

\[ i = 1,2,3 \]

(43)

where \( \varphi_i, \varphi_i, i = 1,2,3, \) are given by (35). It follows from the Fact 7.1 in [12] that Constraint C2, that is, condition (40), is satisfied if and only if

\[ \Delta_i = \det \begin{bmatrix} \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \\ \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \\ \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \end{bmatrix} \neq 0 \]

(44)

and in this case, there holds

\[ \begin{bmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \alpha_{i2} & \alpha_{i3} & \alpha_{i1} \\ \alpha_{i3} & \alpha_{i1} & \alpha_{i2} \end{bmatrix} = \begin{bmatrix} \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \\ \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \\ \beta_{i1} & s_i\beta_{i1} & \beta_{i2} \end{bmatrix}^{-1} \]

(45)

Through substituting (45) into (43), these condition numbers can be finally arranged into the form represented only by the part of parameters \( \beta_{ij} \)'s. In this case, Constraint C2 in the optimization problem (22) becomes condition (44). Further note that the closed-loop eigenvalues are restricted to be real, Constraint C1 holds automatically. Therefore, the optimization problem (22) is finally turned into the following
minimise $\sum_{i=1}^{3} \tau c_{i}^{2}$

s.t. $s_{i} \in \Omega_{i}$, $i = 1, 2, 3$

Condition (44)

where the condition numbers $c_{i}$, $i = 1, 2, 3$ are given by (43) and (45).

In the following, we will consider two different cases.

5.2.1. Case I: $\Omega_{i} = \{ -i \}$, $i = 1, 2, 3$,

In this case the closed-loop eigenvalues are pre-assigned to $-1$, $-2$ and $-3$, and are not optimized within any fields on the complex plane by minimizing the condition numbers. Therefore, the above optimization problem has only 6 parameters $\beta_{ij}$, $i = 1, 2, 3$; $j = 1, 2$. By using the MATLAB command `constr`, we obtain the following solution, with 7 decimal places, to the optimization problem (46):

$$
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22} \\
\beta_{31} & \beta_{32}
\end{bmatrix}
= \begin{bmatrix}
-0.2365024 & 0.0113553 \\
-0.0547700 & 0.1490114 \\
0.6877025 & 0.0849793
\end{bmatrix}
$$

With this group of parameters, we have

$$
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22} \\
\beta_{31} & \beta_{32}
\end{bmatrix}
= \begin{bmatrix}
0.2365024 & -0.0547700 & 0.1490114 & 0.0849793
\end{bmatrix}
$$

Since $(T^T B)$ is nonsingular, and note

$$
Z = \begin{bmatrix}
-0.2478577 & -0.3680914 & 6.1043432 \\
-0.0113553 & -0.1490114 & -0.0849793
\end{bmatrix}
$$

we have the following robust pole assignment solution (with 7 decimal places) for this system:

Solution 1:

$$
K = (T^T B)^{-1} Z^T
$$

$$
= \begin{bmatrix}
4.5998628 & -0.9960638 \\
-3.9379684 & 2.9943938 \\
-1.6249757 & -0.9819272
\end{bmatrix}
$$

With this solution, the closed-loop finite eigenvalues are given by

$$
s_{1} = -0.9999998, \ s_{2} = -2.0000000, \ s_{3} = -3.0000005
$$

Further, note $T^T (A + BKC) V_{\infty} = -0.9819272 \neq 0$, the closed-loop system is regular.

5.2.2. Case II: $\Omega_{1} = \{ -1 \}$, $\Omega_{2} = \{ -3 \}$, $\Omega_{3} = \{ -5 \}$

In this case the closed-loop eigenvalues are included as the optimizing parameters, and now the optimization problem (46) has 9 parameters to be optimized. Again by using the MATLAB command `constr`, we obtain the following solution, with 7 decimal places, to the optimization problem (46):

$$
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22} \\
\beta_{31} & \beta_{32}
\end{bmatrix}
= \begin{bmatrix}
3.2642800 & 1.7657230 \\
-3.1715436 & 4.0664588 \\
-2.9145538 & -1.7866234
\end{bmatrix}
$$

With this solution, the closed-loop finite eigenvalues are calculated exactly the same as given in the sought parameters above when 7 decimal places are used. Further, note $T^T (A + BKC) V_{\infty} = -0.9819272 \neq 0$, the closed-loop system is regular.
5.2.3 Comparison of solutions

By simply choosing $s_i = -i, i = 1, 2, 3,$ and

$$
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22} \\
\beta_{31} & \beta_{32}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}
$$

we obtain the following solution:

**Solution 0:**

$$K = \begin{bmatrix} -6 & -1 \\ 6 & 0.5 \\ 3 & 0.5 \end{bmatrix}$$

The condition numbers and the norms of the above Solutions 0, 1, and 2, are listed in Table 1, from which we can see that

1. The robust solutions, Solutions 1 and 2, have much smaller eigenvalue sensitivities than the non-robust one, Solution 0.
2. Including the closed-loop eigenvalues as the optimizing parameters can further reduce the closed-loop eigenvalue sensitivities.

<table>
<thead>
<tr>
<th>Solution</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$|K|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.7211110</td>
<td>7.3484692</td>
<td>2.5690465</td>
<td>9.0754289</td>
</tr>
<tr>
<td>1</td>
<td>2.1126301</td>
<td>1.2306327</td>
<td>1.3207895</td>
<td>6.9291089</td>
</tr>
<tr>
<td>2</td>
<td>1.5783405</td>
<td>0.8816220</td>
<td>0.4842132</td>
<td>6.3098942</td>
</tr>
</tbody>
</table>

VI. CONCLUDING REMARKS

This paper proposes, based on a recently proposed eigenvalue assignment approach, a simple and effective algorithm for eigenstructure assignment with minimum sensitivities in descriptor linear systems via output feedback. The degrees of the design freedom existing in the proposed approach is composed of three parts: (i) the closed-loop finite eigenvalues $s_i, i = 1, 2, \ldots, m$, (ii) two groups of parameter vectors $f_i$ and $g_i$ existing in the expressions of the closed-loop eigenvectors, and (iii) a parameter matrix $W_c$ or $Z_c$ existing in the general expression for the output feedback gain. The eigenvalue condition numbers for matrix pairs are used to measure the eigenvalue sensitivities with respect to the closed-loop parameter perturbations. Explicit parametric forms for these eigenvalue sensitivities are established in terms of the design parameters. The algorithm first minimizes the closed-loop eigenvalue sensitivities by optimizing the first two parts of the design parameters, and then use the third part of design parameters (if exists) to minimize the magnitude of the gain matrix and enhance the robustness of the closed-loop regularity. An example is presented to illustrate the effect of the proposed approach.

ACKNOWLEDGEMENTS

This work was supported in part by the Outstanding Youth Foundation of the Chinese Ministry of Education, the Chinese Outstanding Youth Science Foundation and the Chinese National Natural Science Foundation under Grant No. 69925308.

The authors are grateful to the referees for their helpful comments and suggestions.

REFERENCES
