VS-CONTROL WITH TIME-VARYING SLIDING SECTOR
— DESIGN AND APPLICATION TO PENDULUM —
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ABSTRACT

In general, a Variable Structure (VS) system is designed with a sliding mode. Recently a sliding sector, designed by an algebraic Riccati equation, has been proposed to replace the sliding mode for chattering-free VS controllers. In this paper we extend the design algorithm for the sliding sector to a time-varying sliding sector. The time-varying sliding sector is defined by functions dependent on both state and time, hence time-varying uncertainty can be considered. The VS controller is designed to stabilize an uncertain system, quadratically. The design procedure for real systems is introduced via an implementation to the control of “Furuta pendulum”. To enhance the stability it is necessary to compensate the time-varying nonlinear static friction of the actuator adequately, hence this problem is a good example to demonstrate the performance of the proposed VS control method. In the experiment, it will be shown that the VS control with the time-varying sliding sector is superior to an orthodox chattering-free VS control.

KeyWords: Variable structure control, sliding mode, sliding sector, quadratic stability, Furuta pendulum.

I. INTRODUCTION

The Variable Structure (VS) control system has been mainly considered for continuous time systems in the form of sliding mode [1]. The VS control law, where a sign function is often used with a switching function

\[ s(x) = Sx \]  (1)

is designed such that the state \( x \) reaches to the sliding mode \( s(x) = 0 \) in a finite time and stays there since then, where the reduced order system is stable. When the VS control with a sliding mode is implemented to practical systems or realized by digital controllers, not only the chattering around the sliding mode may occur because of the finite switching frequency, but also a stable sliding mode designed for continuous-time systems may become unstable after discretizing [2]. To overcome these problems some VS controllers with a boundary or a sliding sector have been proposed [3-6]. In the VS control with a boundary, a saturation function is introduced to replace the sign function and inside the boundary defined as

\[ |s(x)| \leq \Phi \]  (2)

a continuous feedback rule is used, where \( s(x) \) is the switching function defined by Eq. (1) and \( \Phi \) is a positive constant. With this kind of a VS control law, the chattering is removed but it is difficult to show the stability inside the boundary.

As an alternate design algorithm of the VS control, a Variable Structure Control with a Sliding Sector (VSC-SS) has been proposed to replace the sliding mode in the orthodox VS control for a chattering-free controller and for the implementation in discrete-time
control systems [4,7]. For a linear function \( s(x) \) and a positive scalar function \( \delta(x) \), the PR-sliding sector [7] defined as

\[
\mathcal{S} = \{ x \mid s(x) \leq \delta(x), x \in \mathbb{R}^n \},
\]

is a subset on the state space, and inside which the \( P \)-norm of the state defined as

\[
\| x \|_p = \sqrt{x^T P x}, \quad (P = P^T > 0),
\]
decreases with zero control. The corresponding VS controller is designed so that the state moves from the outside of the sector while the \( P \)-norm of the state keeps decreasing in the state space with specified negativity of its derivative as

\[
\frac{d}{dt} \| x \|_p^2 < -x^T R x, \quad (R = R^T > 0).
\]

When the sliding sector is designed for a nominal plant;

\[
\dot{x}(t) = A x(t) + B u(t),
\]

by using a positive definite symmetric solution \( P \) of the Riccati equation;

\[
A^T P + P A - P B B^T P = -Q, \quad (Q = Q^T > 0),
\]

the parameters of the sector defined by Eq. (3) are given as

\[
s(x) = S x(t),
\]

\[
S = B^T P, \quad \delta(x) = \sqrt{x^T (t) \Delta x(t)}
\]

\[
\Delta : = Q - R > 0.
\]

The linear function \( s(x) \) in the sliding sector of Eq. (3) is similar to the switching function of Eq. (1). Therefore it is possible to design a sliding sector based on the sliding mode [8]. In the beginning when the sliding sector was proposed in 1990 [4], it is dedicated for discrete-time systems but it has been found in [7] that the sliding sector can also be used to design a VS controller for continuous-time systems. And it has been shown in [7] that the VS controller with a sliding mode designed based on the sliding sector is feasible for mechanical system control. With the time-varying sliding sector by Eq. (8), a VS control law will be designed such that the state moves into the time-varying sliding sector in a finite time without chattering and some Lyapunov function keeps decreasing in the state space. Therefore the proposed VS control system is quadratically stable. Additionally because the time-varying sliding sector proposed in the paper is designed based on a sliding mode, it is possible to find an advantage of the controller with a sliding sector over the other VS controller with sliding mode by comparing them with a particular example. There are many algorithms for the design of the sliding mode. With the design algorithm given in this paper, all sliding mode algorithms can be extended to design a sliding sector.

On the other hand, a pendulum has been utilized as a popular mechanical system for evaluation of control theories and strategies, for instance, passivity based control [10], swing-up control [11-13], energy control [14], nonlinear observers [15], extension to an acrobot [16], and control of chaotic systems [17]. In this paper, a rotational type pendulum that is called “Furuta pendulum” is used to show the effectiveness of the proposed VSC-SS.

The organization of this paper is as follows. Section 2 defines the time-varying sliding sector and designs it based on the sliding mode; Section 3 proposes a VS control with the time-varying sliding sector; Section 4 shows an experimental result on the “Furuta pendulum” by the proposed VS control method and an orthodox VS control; Section 5 is conclusion.

**II. TIME-VARYING SLIDING SECTOR**

In this paper, a linear time-invariant continuous-time single input system with parameter uncertainties and external disturbances is taken into consideration.

\[
\dot{x}(t) = A x(t) + B u(t) + B d(x, t),
\]

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^1 \) are the state vector and the input respectively, \( A \) and \( B \) are constant matrices of appropriate dimensions, the pair \((A, B)\) is controllable, and \( d(x, t) \) represents parameter uncertainties and external disturbances. It is assumed that an absolute value of \( d(x, t) \) is bounded by a known positive function \( f(x, t) \), \( i.e., \)
Similar to a definition of the PR-sliding sector in [7], a time-varying PR-sliding sector for the plant given by Eq. (9) is defined at first.

**Definition 1.** The time-varying PR-sliding sector for a plant of Eq. (9) is defined on the state space $\mathbb{R}^n$ as

$$S = \{x| s(x) \leq \delta(x,t), x \in \mathbb{R}^n\}$$

inside which the P-norm $\|x\|_P = \sqrt{x^T(t)Px(t)}$ decreases by some control law, i.e., the derivative of a Lyapunov function candidate $L(t)$ satisfies that

$$\dot{L}(t) \leq -x^T(t)R\dot{x}(t), \quad \forall \dot{x}(t) \in S,$$

where

$$\dot{L}(t) := \|x\|_P^2 = x^T(t)Px(t).$$

In [7], the PR-sliding sector has been designed by using an algebraic Riccati equation. The time-varying sliding sector can also be designed in the same way but in this paper, we will give an other design algorithm for the time-varying PR-sliding sector, which is based on a sliding mode.

### 2.1 Sliding mode

A switching function defined as

$$s(x) = Sx, \quad S \in \mathbb{R}^{n \times n}$$

should be designed such that the reduced order system in the sliding mode:

$$\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) + d(x,t) \\
\dot{s}(t) = s_0 \leq \bar{s},
\end{cases}$$

is stable. Since it is assumed that the pair $(A,B)$ is controllable, there exists a nonsingular matrix $T_1 \in \mathbb{R}^{n \times n}$ that converts the plant of Eq. (9) to the following controlla-

$$\begin{bmatrix}
A \\
\bar{A} \\
0
\end{bmatrix} = T_1^{-1}AT_1 = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{21} & \bar{A}_{22}
\end{bmatrix},$$

where

$$\bar{A} = T_1^{-1}AT_1 = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\vdots & \ddots
\end{bmatrix},$$

$$\bar{B} = T_1^{-1}B = \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1
\end{bmatrix}.$$

Then the switching function given by Eq. (15) can be rewritten as

$$s(x) = Sx = \bar{S}_1 \bar{x}(t) = \bar{S}_1 \bar{\bar{x}}(t) + \bar{S}_2(t),$$

where

$$\bar{S} := ST_1 = [\bar{S}_1 \bar{S}_2]$$

and $\bar{S}_1 \in \mathbb{R}^{n \times (n-1)}$. An equivalent control input guaranteeing $\dot{s}(x) = 0$ is given by

$$u_{eq}(t) = -(SB)^{-1}SAx(t) = -SAx(t) \quad (SB = 1).$$

Taking a nonsingular transformation as

$$z(t) = T_2 \bar{\bar{x}}(t) = \begin{bmatrix}
\bar{x}_1(t) \\
\bar{x}_2(t)
\end{bmatrix},$$

where

$$T_2 = \begin{bmatrix}
I_{n-1} & 0_{(n-1) \times 1} \\
\bar{S}_1 & 1
\end{bmatrix},$$

the system of Eq. (18) with the equivalent control $u_{eq}(t)$ given by Eq. (20) is transformed into

$$\dot{z}(t) = \begin{bmatrix}
\bar{A}_1 & 0_{(n-1) \times 1} \\
0_{1 \times (n-1)} & 1
\end{bmatrix}z(t) + \begin{bmatrix}
0_{(n-1) \times 1} \\
1
\end{bmatrix}[v(t) + d(x,t)],$$

where

$$\bar{A}_1 := \bar{A}_{11} - \bar{A}_{12} \bar{S}_1,$$

and $v(t)$ is an alternative control input satisfying

$$u(t) = u_{eq}(t) + v(t).$$

In the sliding mode $s(x) = 0$, the reduced order system of Eq. (16) becomes

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}(u(t) + d(x,t)),\quad \text{with}\quad \bar{A} = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\vdots & \ddots
\end{bmatrix},$$

$$\bar{B} = \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1
\end{bmatrix}.$$
\[ \dot{x}_i(t) = (\bar{A}_1 - \bar{A}_2 \bar{S}_1) x_i(t). \]  

(23)

It is obvious that the pair \((\bar{A}_1, \bar{A}_2)\) is controllable. Therefore it is easy to design a feedback gain \(\bar{S}_1\) such that the above reduced order system is stable, for example, by using pole assignment algorithm, and LQR control algorithm \([18,19]\).

2.2 Design of time-varying sliding sector

With the switching function designed in the last subsection, the time-varying PR-sliding sector given by Eq. (11) can be easily designed by choosing the sliding mode \(s(x)\) of Eq. (15) as the linear function \(s(x)\) in Eq. (11). In this case, the problem is how to determine the matrices \(P\) and \(R\) and also the control law.

As the sliding mode is designed so that the reduced order system of Eq. (23) is stable, there exists a positive definite symmetric matrix \(\bar{P}\) such that the following Lyapunov equation holds for some positive definite symmetric matrix \(\bar{Q}\).

\[ \bar{Q} = \bar{A}_1^T \bar{P} + \bar{P} \bar{A}_1. \]  

(24)

Choosing positive definite symmetric matrices \(\bar{P}\) and \(\bar{R}\) as

\[ \bar{P} = \begin{bmatrix} \bar{P} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & \frac{h}{2} \end{bmatrix}, \]  

(25)

\[ \bar{R} = \begin{bmatrix} \bar{Q} & -\bar{P} \bar{A}_2 \\ -\bar{A}_2^T \bar{P} & \frac{h}{2} \end{bmatrix}, \]  

(26)

where \(h\) is a large enough positive constant such that the matrix \(\bar{R}\) is positive definite, then for the Lyapunov function

\[ L(t) = z^T(t) \bar{P} z(t), \]  

the following holds.

\[ \dot{L}(t) = -z^T(t) \bar{R} z(t) + h s(x) \cdot (s(x) + v(t) + d(x, t)). \]  

Therefore inside the time-varying PR-sliding sector (i.e., \(s^2(x) \leq \delta^2(x, t)\) holds), if the control law is given by

\[ v(t) = -k \delta(x, t) \cdot \text{sgn}(s(x)), \]  

(27)

where \(k\) is a large enough positive constant parameter satisfying

\[ k > 1 + \frac{1}{\eta}, \]  

(28)

then we have

\[ \dot{L}(t) \leq -z^T(t) \bar{R} z(t). \]  

(29)

The following theorem concludes the above discussion.

Theorem 1. The time-varying PR-sliding sector defined in Eq. (11) can be designed as

\[ S = \{ x \mid \| s(x) \| \leq \delta(x, t), x \in R^n \}, \]  

(30)

with the linear function \(s(x)\) is the switching function in Eq. (15) and the positive definite symmetric matrices \(P\) and \(R\) determined by

\[ P = T_1^{-T} T_2^T \bar{P} T_2 T_1^{-1}, \]  

(31)

\[ R = T_1^{-T} T_2^T \bar{R} T_2 T_1^{-1}, \]  

(32)

where \(\bar{P}\) and \(\bar{R}\) are given by Eqs. (25) and (26), respectively, and \(T_1\) and \(T_2\) are transformation matrices defined in the last subsection. Then inside the time-varying PR-sliding sector \(S\) with the control law given by Eq. (27), the \(P\)-norm decreases as

\[ \frac{d}{dt} L(t) = \frac{d}{dt} (x^T(t) P x(t)) \leq -x^T(t) R x(t). \]  

III. VS CONTROLLER WITH TIME-VARYING SLIDING SECTOR

With the time-varying PR-sliding sector \(S\) of Eq. (11), a VS control law should be designed such that the state moves into the sector in a finite time while the \(P\)-norm keeps decreasing. Together with the control law used inside the sector, the VS control law with the time-varying PR-sliding sector is given as follows.

\[ u(t) = \begin{cases} -S x(t) - k \delta(x, t) \cdot \text{sgn}(s(x)) & x(t) \in S \\ -S x(t) - k s(x) & x(t) \notin S \end{cases} \]  

(33)

Theorem 2. The VS control law given by Eq. (33) with the time-varying PR-sliding sector \(S\) described by Eq. (30) ensures that the state moves into the sector in a finite time and the resultant VS control system is quadratically stable.

Proof. It is assumed that the initial state is outside the time-varying PR-sliding sector, i.e., \(\| s(x) \| > \delta(x, t)\). In this case the VS control law is determined by

\[ u(t) = -S x(t) - k s(x), \]  

(34)

which will be active until the system’s state is moved into the inside of the sector. With the control by Eq. (34),
it can be shown by some calculation that
\[
\frac{d}{dt} s^2(x) = 2s(x)(d(x,t) - ks(x)) < -2s^2(x)
\]
\[
\frac{d}{dt} L(t) = -x^T(t)R x(t) + hs(x)(s(x) - ks(x) + d(x,t))
\]
\[
< -x^T(t)R x(t),
\]
\(i.e., s^2(x)\) and the Lyapunov function \(L(t)\) in Eq. (12) decrease. The decreasing of \(s^2(x)\) means that the state moves toward the inside of the time-varying PR-sliding sector. And also it can be shown that the state will move into the sector in a finite time with the decreasing of \(s^2(x)\) if the positive constant \(k\) is chosen large enough although it will cost infinite time for the state to converge to the sliding mode \(s(x) = 0\).

After the state being moved into the sector and until moving out of it, the control input is determined by
\[
u(t) = -\mathcal{A}_x x(t) - k\delta(x,t) \cdot \text{sgn}(s(x)). \tag{35}
\]
By the input of Eq. (35), it has been shown in Theorem 1 that the Lyapunov function decreases as
\[
\dot{L}(t) \leq -x^T(t)R x(t).
\]
If the state with the control law of Eq. (35) moves out of the time-varying PR-sliding sector, the control input by Eq. (34) will let it back to the inside of the sector in a finite time again while the Lyapunov function \(L(t)\) keeps decreasing.

In this way, the state will be moved from the outside into the inside of the time-varying PR-sliding sector in a finite time and the Lyapunov function in Eq. (12) keeps decreasing in the state space with the VS control law by Eq. (33), \(i.e., \]
\[
\dot{L}(t) \leq -x^T(t)R x(t), \forall x \in R^n,
\]
which means that the VS control law given by Eq. (33) stabilizes the plant (9) quadratically.

IV. APPLICATION OF VS CONTROL WITH TIME-VARYING SLIDING SECTOR

4.1 Furuta pendulum

Motors have always static friction, so it is difficult to control the position, speed and torque at the slow speed generally. Especially on stabilization control of an inverted pendulum, it is difficult to keep the pendulum strictly inverted at same motor position because the motor has to switch the rotation direction frequently to keep the balance. Hence it can be expected that the proposed VS control, which can take the time-varying friction into consideration, enhances the performance of the stabilization.

The apparatus is shown in Fig. 1. The pendulum-link is equipped on the output axis of a direct drive motor, and the link can rotate around the pivot freely. An encoder mounted on a rotated base detects the angle of a pendulum-link and the sensor’s resolution is 14400 [pulse/rev]. The motor generates torque according to commands from a computer and the angle sensor’s resolution is 409600 [pulse/rev]. The torque range is ±9.8 [N·m]. The controller devices are assembled from a 350MHz PC/AT computer and some peripheral I/O boards. Real time control program is built by using RT-MaTX [20]. The control interval is 1[msec], and a pseudo-differentiator of which continuous-time transfer function is given by \(s/(0.01s+1)\) is used to estimate the angular velocities.

4.2 Derivation of dynamic equation

The dynamic equation of the pendulum is obtained by using Lagrange method. To calculate the energies, the coordinate values of each Center Of Gravity (COG) are calculated. The coordinate systems attached according to DH-notation method are shown in Fig. 2. \(\theta_1, \theta_2\) and \(\tau\) are angles of the motor and the pendulum, and an input torque respectively. The DH-parameters are listed in Table 1 and the mechanical parameters and their meanings are described in Table 2.

Describing a homogeneous transformation matrix

\[\text{MaTX software has been developed by our colleague and is download free. For more information, refer the official site http://www.matx.org/ (the site is Japanese, but there are some English manuals).} \]
that translates $j$-coordinate system to $i$-coordinate by $"A^i_j"$, the following matrices are obtained.

\[ A^0_1 = \begin{bmatrix}
\cos \theta_1 & 0 & \sin \theta_1 & 0 \\
-\sin \theta_1 & 0 & -\cos \theta_1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

\[ A^1_2 = \begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\
\sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\
0 & 0 & 1 & l_1 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

Describing a COG of the link-2 on the coordinate-2 as $p_2 = [-b_2, 0, 0, 0]^T$, the coordinate value of this point on global coordinate is computed by

\[ p^0_2 = A^1_2 \cdot A^0_1 \cdot p^2_2. \]

Fig. 2. Modeling of 1-link Furuta pendulum.

Table 1. DH parameters.

<table>
<thead>
<tr>
<th>link</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\alpha_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$d_1$</td>
<td>$+\frac{\pi}{2}$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$l_2$</td>
<td>$l_1$</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
</tbody>
</table>

Table 2. Mechanical parameters of a Furuta pendulum.

<table>
<thead>
<tr>
<th>value meanings</th>
<th>$l_1$</th>
<th>$J_1$</th>
<th>$c_1$</th>
<th>$l_2$</th>
<th>$J_2$</th>
<th>$C_2$</th>
<th>$b_2$</th>
<th>$m_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>length of the link-1</td>
<td>0.245 [m]</td>
<td>$3.145 \times 10^{-3}$ [kg$\cdot$m$^2$]</td>
<td>0.334 [Nms/rad]</td>
<td>0.300 [m]</td>
<td>$3.885 \times 10^{-3}$ [kg$\cdot$m$^2$]</td>
<td>3.869 $\times 10^{-3}$ [Nms/rad]</td>
<td>0.183 [m]</td>
<td>0.080 [kg]</td>
</tr>
</tbody>
</table>

Therefore the kinematics energy $K$ is calculated as

\[ K = \frac{J_1}{2} \dot{\theta}_1^2 + \frac{J_2}{2} \dot{\theta}_2^2 + \frac{m_2}{2} \left( \frac{d}{dt} p^0_{2x} \right)^2 + \frac{m_2}{2} \left( \frac{d}{dt} p^0_{2y} \right)^2 + \frac{m_2}{2} \left( \frac{d}{dt} p^0_{2z} \right)^2, \]

(36)

where $p^0_{2x}$ means the $x$-component of vector $p^0_2$ and the others are similar. The potential energy $U$ is calculated as

\[ U = m_2 g \cdot p^0_{2z}. \]

(37)

Considering viscous of the joints, a disappeared term is

\[ R = \frac{1}{2} c_1 \dot{\theta}_1^2 + \frac{1}{2} c_2 \dot{\theta}_2^2, \]

(38)

Setting Lagrangian $L = K - U$, substituting Eqs. (36), (37) and (38) into the Lagrange equation:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = \tau_i, \]

(39)

replacing $\dot{\theta}_2$ as $\dot{\theta}_2 \rightarrow \dot{\theta}_2 + \pi/2$ to correspond $\theta_2 = 0$ to the upper equilibrium, and rewriting as $b_i = l_i - b_2$ for convenience yields the following dynamic equation.

\[ p_{11} \ddot{\theta}_1 + p_{12} \ddot{\theta}_2 + q_1 = \tau, \]

\[ p_{12} \ddot{\theta}_1 + p_{22} \ddot{\theta}_2 + q_2 = 0, \]

(40)

\[ q_1 = c_1 \dot{\theta}_1 + m_2 l_2 \cos \theta_2 \]

\[ q_2 = c_2 \dot{\theta}_2 - m_2 l_2 \left( \frac{1}{2} l_1 \sin \theta_2 \dot{\theta}_2^2 + g \sin \theta_2 \right). \]

Linearization of the nonlinear dynamic equation given by Eq. (40) around the unstable equilibrium ($\dot{\theta}_i = 0, \dot{\theta}_i = 0, i = 1, 2$) gives the following first order differential equation.

\[ E \dot{x} = F \cdot x + G \cdot u, \]

(41)


\( F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -c_1 & 0 \\ 0 & m_1 & 0 & -c_2 \end{bmatrix} \)

\( G = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \)

Substituting parameters written in Table 2 into Eq. (41) gives the following state-space model.

\[
\dot{x} = Ax + Bu ,
\]

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6.15 & -4.92 \times 10^1 & -2.59 \times 10^{-1} \\ 0 & 2.13 \times 10^1 & -2.27 \times 10^1 & -8.96 \times 10^{-1} \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 \\ 0 \\ 1.45 \times 10^2 \\ 6.68 \times 10^1 \end{bmatrix}^T.
\]

The eigen-values are \{3.95, 0, -4.66, -49.40\}.

### 4.3 Experiment

An equivalent input law in a VS controller is designed so that the reduced order system under VS control has same eigen-values as ones of closed loop system designed by a LQ optimal control. When weighting matrices are chosen as \( Q = \text{diag} \{300, 1, 1, 10\} \), \( R = [1000] \) for the quadratic performance index \( J = \int_0^\infty (x^T(t)Qx(t) + Ru^2(t))dt \), the feedback gain \( F = [-0.547, 10.65, -0.972, 2.255] \) is obtained and the resultant eigen-values of the closed-loop system are

\{−1.61, −4.11, −4.44, −50.0\}. \hspace{1cm} (43)

Now we chose the matrix \( S \) of a switching function in Eq. (15) so that the reduced order system described by Eq. (23) contains the same eigen-values as (43). After choosing the first three eigen-values \( p_1, p_2, p_3 = -1.61, -4.11, -4.44 \), a polynomial having \( p_1 \sim p_3 \) as roots is expanded as follows.

\[(s + 1.61)(s + 4.11)(s + 4.44) \Rightarrow s^3 + 10.2s^2 + 31.9s + 29.3, \]

\( S = [29.3, 31.9, 10.2, 1]^T \)

\[
= [-1.09, 19.6, -1.24, 4.18] \times 10^{-2} , \hspace{1cm} (44)
\]

where

\[
T = [B, AB, A^2B, A^3B].
\]

\[
|sI - A| = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0.
\]

Theorems 1 and 2 ensure that we can choose a time-varying function as a boundary function defining a sliding sector. For simplicity \( \delta(x, t) \) was chosen as

\[
\delta(x, t) = \sqrt{s^T(t) \cdot \text{diag} \{1, 1, 1, 1\} \times 10^{-2} \cdot x(t)}.
\]

The gain \( k \) was chosen as \( k = 15 \) by trial and error on the experiment. \(^2\)

The experimental result is shown in Fig. 3. The graph(a) is an angle of the pendulum-link and the 0 [degree] means the upright unstable equilibrium. Graph(b) is an angle of the base-link driven by the motor. The maximum slant angle of the pendulum-link is as small as 1.5 [degree] and the link is kept perfectly upright almost always. Amplitude of the input is small and there is less chattering. Figure 4 shows the changes of \( s(x) \) and \( \delta(x) \) that form the sliding sector. The solid line is \( s(x) \) and two dashed lines are \( \pm \delta(x) \) respectively. It can be verified that after disturbance is added the state returns quickly to the inside sector and stays in the inside. It will be found by comparison with other chattering-free variable structure control that this response is better than the other case. Figure 5 is the other result by a popular chattering-less VS-control: sliding-mode control with boundary layer. This controller is designed for the controllable system (4) by using a sliding-mode (1) with same matrix \( S \) described in Eq. (44). The control law is given by

\[
u(t) = -(SB)^{-1}SAx(t) - \tilde{k}(SB)^{-1} \cdot \text{sat}_\epsilon(s(x)),
\]

\[
\text{sat}_\epsilon(s) = \begin{cases} s, & |s| < \epsilon \\ \text{sgn}(s), & |s| \geq \epsilon \end{cases},
\]

where \( \text{sat}(*) \) is a saturation function and \( \epsilon \) is a threshold parameter. Considering the fact that \( \max |s(x)| \approx 6 \times 10^{-3} \) on the former experimental result shown in Fig. 3, we specified the threshold as \( \epsilon = 5 \times 10^{-3} \). Tuning the gain \( \tilde{k} \) so that the chattering of input becomes small and the

\(^2\) If an explicit form of \( f(x, t) \) in the inequality (10) is decided, we obtain \( k \) by using the inequality (28) with \( \delta(x, t) \) that is decided by Eq. (14) with a chosen parameter \( \eta \). If \( f(x, t) \) and \( \eta \) are introduced to prove the convergence of the proposed VS control theoretically, so it is enough in practice if large \( k \) is chosen.
overshoots of pendulum/motor get small, the best result under these conditions was obtained as shown in Fig. 5 (in this case, $k_1 = 1.2$). Maximums of the input and the overshoot became bigger than the case of a VS control with the time-varying sliding sector as $\max(|\dot{\theta}|) = 12 \rightarrow 18.5$ [degree] and $\max(|\theta|) = 1.5 \rightarrow 5.8$ [degree], and the response got worse. Expecting that quick response is recovered, we increased the gain a little as $k_1 = 1.2 \rightarrow 1.5$. The result is shown in Fig. 6. The response was improved as $\max(|\dot{\theta}|) = 18.5 \rightarrow 12.5$ [degree] and $\max(|\theta|) = 5.8 \rightarrow 3.7$ [degree], but the input chattered considerably. Bigger the gain $k$ is changed, more strongly the chattering occurs, and at last the stabilization of the inverted pendulum fails. With small gain, the motor moves more slowly at long distance, and the stabilization cannot be continued in this case also. From the above-mentioned results, it can be confirmed that the proposed VS control with a time-varying sliding sector could surmount an unfavorable effect of static friction and achieve robust/less-chattering stabilization.

V. CONCLUSIONS

In this paper, we extended the VS control algorithm by using a time-varying PR-sliding sector. The sector is designed based on a sliding mode. The resultant VS control system is quadratically stable even if there exist parameter uncertainties and external disturbances. We took the “Furuta pendulum” to evaluate the proposed VS control algorithm. The experiment results show that the VS controller with a time-varying PR-sliding sector is chattering-free and has satisfactory control perform-
The extension of the proposed time-varying sliding sector to multi-input systems is straightforward. And it is possible to implement the proposed VS controller with a time-varying sliding sector for continuous-time systems in a sampled-data system, which has been validated by the experimental results given in this paper.

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