A CLOSED-LOOP SUBSPACE IDENTIFICATION METHOD FOR CONTINUOUS-TIME SYSTEMS BASED ON $\delta$-OPERATOR MODEL

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ABSTRACT

This paper is concerned with a subspace identification of a continuous-time plant operating in closed-loop in the framework of the joint input-output approach. The main procedure consists of two steps. Firstly, the dual-Youla parametrization of the plant is used for obtaining an equivalent open-loop problem to the original closed-loop identification problem. Then, a $\delta$-operator based IV-MOESP type subspace identification algorithm is developed to estimate the state space model for the joint input-output process, whereby a higher-order state space model of the plant is obtained by an algebraic operation. Subsequently, a model reduction procedure is employed to derive a lower-order plant model removing irrelevant modes from the higher order model. Simulation results by using numerical and chemical plant models demonstrate the feasibility of the proposed method.

KeyWords: Closed-loop systems, subspace identification, $\delta$-operator model.

I. INTRODUCTION

Identification of dynamic systems operating in closed-loop has been studied since the sixties because many industrial processes are operating under feedback control [1,2]. Moreover, with increasing interest in joint identification and control, the closed-loop identification problems have received much attention in recent years; see [3,4] for overview.

The closed-loop identification methods based on the coprime representation, or the dual-Youla parametrization, of plants have been proposed by Hansen et al. [5,6], and further examined by Schrama [7] for the control-oriented approximate identification. The main features of the approach based on the dual-Youla parametrization are that the closed-loop identification problem can be transformed into an equivalent open-loop one and the identified plant models are stabilizable by the present controller. In [8], based on the MOESP method, a technique of identifying state space model of the plant operating in closed-loop has been developed by identifying an overall open-loop state space model, followed by model reduction. We have also presented a note on a closed-loop state space identification method by using the dual-Youla parametrization combined with subspace methods [9]. Moreover, modifying the N4SID method, a closed-loop subspace identification method has been derived under the assumption that a finite number of Markov parameters of the controller are given [10].

It is well known [11] that the identification of continuous-time system based on sampled data is desirable from the practical viewpoint since in practice many processes are continuous in time. For example, a benefit comes from the harmony between the parameters of continuous-time systems and the physical ones. In the identification of continuous-time systems, usually a discrete-time model is identified using sampled data, and then a continuous-time system is recovered by, say, the inverse bilinear transform. However, serious problem
such as numerical ill-conditioning arises when a small sampling period is selected for the least loss of information. To overcome this problem, a number of techniques have been developed in the area of continuous-time identification [12,11,13]. Recently, the continuous-time subspace identification methods are proposed, see e.g. Haverkamp et al. [14], Johansson et al. [15], Yang et al. [16], where \( \lambda \)- and \( \alpha \)-operator are employed to transform the continuous-time system into an equivalent discrete-time system. In [17], we have also developed subspace state space identification methods by using the \( \delta \)-operator model, where it has been shown by numerical simulations that the \( \delta \)-operator based methods yield better performance compared with other operator based methods.

In this paper, in the framework of the joint input-output approach, we derive a method of identifying a continuous-time closed-loop system based on subspace methods [18-20]; the advantage being that the difficulty associated with the classical prediction error identification methods is avoided by using the singular value decomposition (SVD) and QR decomposition. We first transform the closed-loop identification problem into an equivalent open-loop identification problem based on the dual-Youla parametrization of the plants and then develop a \( \delta \)-operator subspace identification method for the equivalent open-loop one in order to obtain the estimate of open-loop plant model. An additional step of model reduction is included to derive a plant model from the estimated higher order model by deleting irrelevant modes.

This paper is organized as follows. In Section 2, the problem of continuous-time closed-loop identification is stated. A transformation of the closed-loop identification problem into an equivalent open-loop one is described in Section 3 based on the dual-Youla parametrization. Section 4 presents a \( \delta \)-operator based IV-MOESP type subspace identification method, coupled with the prefiltering technique. A procedure of model reduction is briefly summarized in Section 5. The effectiveness of the proposed method is verified by means of numerical examples in Section 6. Conclusions are given in Section 7. The appendices contain some proofs of lemmas.

In the sequel, \( \mathcal{R}^n \) denotes the set of \( n \)-dimensional real vectors, and \( \mathcal{RH}_{\infty}^{p \times m} \) denotes the set of \( p \times m \) real rational proper stable matrices.

**II. PROBLEM DESCRIPTION**

Consider a continuous-time closed-loop linear time-invariant (LTI) system shown in Fig. 1, where \( u(t) \in \mathcal{R}^n \) is the control input, \( y(t) \in \mathcal{R}^p \) is the system output, \( r_1(t) \in \mathcal{R}^n \) and \( r_2(t) \in \mathcal{R}^p \) are reference signals and \( v(t) \in \mathcal{R}^p \) is the disturbance. The \( G(s) \in \mathcal{RH}_\infty^{p \times n} \) and \( C(s) \in \mathcal{RH}_\infty^{m \times p} \) denote transfer matrices of the plant and controller, respectively. It is assumed that the output disturbance is modeled by \( v(t) = H(s)e(t) \), where \( H(s) \in \mathcal{RH}_\infty^{p \times p} \) is a minimal phase noise filter, and \( e(t) \in \mathcal{R}^p \) is a white noise.

We see from Fig. 1 that the closed-loop system equations are written as
\[
\begin{align*}
y(t) &= G(s)u(t) + H(s)e(t) \\
u(t) - C(s)y(t)
\end{align*}
\]
where \( r(t) = r_1(t) + C(s)r_2(t) \in \mathcal{R}^n \) is a combination of the reference inputs.

In this paper, we give the following assumptions.

**Assumption 1.** The exogenous signal \( r(t) \) is uncorrelated with the disturbance \( v(t) \), and satisfies a sufficiently high-order persistently exciting (PE) condition.

**Assumption 2.** The controller \( C(s) \) is known and the orders \( n_p, n_c \) of the plant and the controller are given.

**Assumption 3.** The closed-loop system is asymptotically stable while the plant \( G(s) \) and/or the controller \( C(s) \) can be unstable. Moreover, the product \( G(s)C(s) \) is strictly proper.

It should be noted that the closed-loop identifiability is assured by Assumption 1. From Assumption 2, we can easily obtain the plant input by using the known controller and measurements of the reference and output signals. Assumption 3 is commonly used in closed-loop identification in order to avoid technicalities.

The problem to be considered in this paper is to derive a subspace method of identifying the plant transfer matrix \( G(s) \) under Assumptions 1-3 and using the sampled data \( \{r(t), y(t), u(t)\mid t = 0, \Delta, \ldots, (N - 1) \Delta\} \), where \( \Delta \) is the sampling period and \( N \) is the data length.

**III. DUAL-YOULA PARAMETRIZATION OF PLANT**

We briefly recall the identification method based on the dual-Youla parametrization and apply it to the closed-loop identification problem for a continuous-time system. The basic idea under discrete-time setting is introduced by Hansen et al. [5,6].

Suppose that \( G_0(s) \) is a nominal plant model\(^1\) of \( G(s) \) and is stabilizable by the given controller \( C(s) \). It is assumed that the right and left coprime factorizations of
$C(s)$ and $G_d(s)$ are respectively given by

$$C(s) = \tilde{X}(s) \tilde{Y}^{-1}(s) = Y^{-1}(s) X(s) \quad (3)$$

$$G_d(s) = N_0(s) D_0^{-1}(s) = \tilde{D}^{-1}(s) \tilde{N}_d(s) \quad (4)$$

where $X$, $\tilde{X} \in \mathcal{RH}_{w^p}^\infty$, $N_0$, $\tilde{N}_0 \in \mathcal{RH}_{w^p}^\infty$, $D_0$, $Y \in \mathcal{RH}_{w^p}^\infty$, $\tilde{D}_0$, $\tilde{Y} \in \mathcal{RH}_{w^p}^\infty$. Using the dual-Youla parametrization of (3) and (4), we can express the input-output relation of all the plants stabilized by $C(s)$ as

$$y(t) = (\tilde{D}_0 - RX)^{-1}(\tilde{N}_0 + Ry(t) + (\tilde{D}_0 - RX)^{-1}Q \omega(t))$$

$$= (N_0 + \tilde{Y}R)(D_0 - \tilde{X}R)^{-1}u(t)$$

$$+ [(N_0 + \tilde{Y}R)(D_0 - \tilde{X}R)^{-1}\tilde{X} + \tilde{Y}] Q \omega(t)$$

where $R \in \mathcal{RH}_{w^p}^\infty$, $\tilde{Q} \in \mathcal{RH}_{w^p}^\infty$ are arbitrary transfer matrices.

Now we introduce an auxiliary signal that is computable from the reference signal $r(t)$ as

$$\omega(t) = [D_0(s) + C(s) N_0(s)]^{-1}[u(t) + C(s) y(t)]$$

$$= [Y(s)D_0(s) + X(s)N_0(s)]^{-1}Y(s) r(t) \quad (5)$$

Thus we see that the joint process $(u(t), y(t))$ is expressed as

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \tilde{D}(s) \\ N(s) \end{bmatrix} \omega(t) + \begin{bmatrix} -\tilde{X}Q(s) \\ \tilde{Y}Q(s) \end{bmatrix} \epsilon(t) \quad (6)$$

where $N(s) := N_0(s) + \tilde{Y}(s) R(s)$, $\tilde{D}(s) := D_0(s) - \tilde{X}(s) R(s)$.

Since the open-loop transfer matrix can be computed by $G(s) = \tilde{N}(s) \tilde{D}(s)^{-1}$, the original closed-loop identification problem is transformed into the open-loop one for identifying $\tilde{D}(s)$ and $\tilde{N}(s)$.

Under Assumption 1, it follows from (5) that $\omega(t)$ is a PE signal, and is uncorrelated with the noise $\epsilon(t)$. Therefore, to estimate $\tilde{N}(s)$ and $\tilde{D}(s)$, some open-loop identification techniques can be applied to (6), where the output is the joint process $(u(t), y(t))$ and the input is the auxiliary variable $\omega(t)$. Due to the problems of local minima and the difficult model selection, the application of classical prediction error methods to MIMO systems is quite limited, so that we apply the MOESP-type identification methods to (6).

Suppose that a state space realization of (6) is given by

$$\dot{x}(t) = Ax(t) + B_1 \omega(t) + B_2 \epsilon(t)$$

$$u(t) = C_1 x(t) + D_{11} \omega(t) + D_{12} \epsilon(t)$$

$$y(t) = C_2 x(t) + D_{21} \omega(t) + D_{22} \epsilon(t) \quad (7)$$

where $x(t)$ is the state vector containing the dynamics of the plant, controller and noise model, and $A$ is stable.

Furthermore, eliminating the auxiliary signal $\omega(t)$ from (7), we obtain a state space model relating the input $u(t)$ to the output $y(t)$ as

$$\dot{x}(t) = (A - B_1 D_{11}^{-1} C_1) x(t) + B_2 D_{11}^{-1} u(t) + (B_2 - B_1 D_{11}^{-1} D_{12}) \epsilon(t)$$

$$y(t) = (C_2 - D_{21} D_{11}^{-1} C_1) x(t) + D_{21} D_{11}^{-1} u(t) + (D_{22} - D_{21} D_{11}^{-1} D_{12}) \epsilon(t) \quad (8)$$

It may be noted that $D_{d}(x) + C(x) N_{d}(x)$ is nonsingular in (5), since $D_{d}(s)$ is invertible, and since $C(s) N_{d}(s)$ is strictly proper by Assumption 3. Thus $D_{11}$ is nonsingular. Comparing (8)-(9) with (1), it follows that a realization of the transfer matrix of the open-loop plant is given by

$$G(s) = \begin{bmatrix} A - B_1 D_{11}^{-1} C_1 & B_2 D_{11}^{-1} \\ C_2 - D_{21} D_{11}^{-1} C_1 & D_{21} D_{11}^{-1} D_{12} \end{bmatrix} \quad (10)$$

where we see that the transfer matrix $G(s)$ becomes strictly proper if and only if $D_{21} = 0$.

Since the realization of $G(s)$ above is of higher order, an additional model reduction is necessary to remove irrelevant modes from $G(s)$ of (10), which will be discussed in Section 5.

### IV. δ-OPERATOR SUBSPACE MODEL IDENTIFICATION

In this section, we introduce a δ-operator subspace method based on our recent result [17] in order to identify the open-loop system described by the state space model of (7).

For notational simplicity, we define the joint process as $s(t) \in \mathbb{R}^{p}$, where $p = m + p$. Then the system (7) is expressed as

$$\dot{x}(t) = Ax(t) + B_1 \omega(t) + B_2 \epsilon(t) \quad (11)$$

$$s(t) = C_1 x(t) + D_{11} \omega(t) + D_{12} \epsilon(t) \quad (12)$$

where

$$C \in \mathbb{R}^{p\times m}, D_1 \in \mathbb{R}^{p\times p}, D_2 \in \mathbb{R}^{p\times p}$$
Firstly, we derive a discrete-time δ-operator state space model approximating the continuous-time system (11)-(12).

4.1 δ-operator state space model

By using a simple anti-aliasing filter whose impulse response is given by \( h(t) = \frac{1}{\Delta} (0 \leq t \leq \Delta) \) and the zero-order hold, it can be shown that the sampled input-output behavior of (11)-(12) is given by the following LTI discrete-time state space system

\[
q(x) = A_q x(t) + B_q \omega(t) + \eta_q(t) \tag{13}
\]

\[
s(t) = C_q x(t) + D_q \omega(t) + e_q(t) \tag{14}
\]

where \( q \) is the shift operator with \( q(x) = x(t + \Delta) \). It can be shown [12] that the coefficient matrices in (13)-(14) are given by

\[
A_q = e^{\Delta A}, \quad B_q = \left( \int_0^\Delta e^{\tau \Delta} d\tau \right) B_i = \Delta B_i \tag{15}
\]

\[
C_q = \frac{1}{\Delta} \int_0^\Delta C e^{\tau \Delta} d\tau = C, \quad D_q = D_i \tag{16}
\]

where \( \Delta \) implies that the both sides are nearly equal when \( \Delta \) is sufficiently small. Moreover, \( \eta_q(t) \) and \( e_q(t) \) in (13)-(14) are sampled stochastic disturbances with covariance matrices \( Q_q, R_q, \) and \( S_q, \) respectively, where

\[
Q_q = \text{cov}[\eta_q(t)] = Q_e, \quad R_q = \text{cov}[e_q(t)] = R_e/\Delta,
\]

\[
S_q = \text{cov}[\eta_q(t), e_q(t)] = S_e
\]

and where \( Q_e, R_e, S_e \) are the covariance matrices of \( B\omega(t) \) and \( D\omega(t), \) respectively. It follows from (15)-(16) that \( A_q \rightarrow I, B_q \rightarrow 0, C_q \rightarrow C \) as the sampling period \( \Delta \) tends to zero. Thus the shift operator leads to a model degeneracy at high sampling rates.

We can, however, overcome this problem by introducing the δ-operator defined as

\[
\delta f(t) = \frac{f(t+\Delta) - f(t)}{\Delta}
\]

For a sufficiently small \( \Delta, \) we have \( \delta f(t) \approx f'(t) \). Therefore, by referring to [12], it follows from (13)-(14) that the δ-operator model is given by

\[
\delta x(t) = A_\delta x(t) + B_\delta \omega(t) + \eta_\delta(t) \tag{17}
\]

\[
s(t) = C_\delta x(t) + D_\delta \omega(t) + e_\delta(t) \tag{18}
\]

where

\[
A_\delta = \frac{A_q - I}{\Delta}, \quad B_\delta = \frac{1}{\Delta} B_q, \quad C_\delta = C_q, \quad D_\delta = D_q
\]

\[
\eta_\delta(t) = \frac{1}{\Delta} \eta_q(t), \quad e_\delta(t) = e_q(t)
\]

We see that the covariance matrices of \( \eta_\delta(t) \) and \( e_\delta(t) \) are given by

\[
Q_\delta = \text{cov}[\eta_\delta(t)] = Q_e/\Delta, \quad R_\delta = \text{cov}[e_\delta(t)] = R_e/\Delta,
\]

\[
S_\delta = \text{cov}[\eta_\delta(t), e_\delta(t)] = S_e
\]

Thus \( Q_\delta, R_\delta \) are both order of \( 1/\Delta. \) In order to achieve sampling independence, we define spectral densities as \( \Sigma_\delta = \Delta Q_\delta, \Sigma_e = \Delta R_\delta. \) Hence, for \( \Delta \rightarrow 0, \) we have \( A_\delta \rightarrow A, B_\delta \rightarrow B, C_\delta \rightarrow C, D_\delta = D, \Sigma_\delta \rightarrow Q_e \) and \( \Sigma_e \rightarrow R_e, \) implying that the δ-operator model of (17)-(18) converges to the original continuous-time system of (11)-(12).

It turns out that the optimal filter for (17)-(18) has the following form

\[
\hat{x}(t) = A_\delta \hat{x}(t) + B_\delta \omega(t) + K_\delta \{ s(t) - C_\delta \hat{x}(t) - D_\delta \omega(t) \}
\]

\[
K_\delta = (\Delta A_\delta + I)P_\delta C_\delta^T [\Delta C_\delta P_\delta C_\delta^T + \Sigma_e^{-1}]
\]

where \( \hat{x}(t) \) is the one-step predicted estimate of \( x(t), \) \( K_\delta(t) \) is the Kalman gain, and \( P_\delta(t) \) is the delta form of Riccati equation defined by Eq. (10.5.12) of [12].

Therefore the δ-operator optimal filter converges to the original continuous-time filter for the system (11)-(12) as \( \Delta \rightarrow 0. \) This means that the δ-operator model is a simple good approximation to the continuous-time model when the sampling period is small. Also, since \( P_\delta(t) \) converges to a stationary solution \( P_\delta \) as \( t \) goes to infinity, the Kalman gain \( K_\delta(t) \) also converges to the stationary Kalman gain \( K_\delta. \)

In the following, we consider the stationary case where the Kalman gain \( K_\delta \) is constant. Define the innovation process as

\[
v(t) = s(t) - C_\delta \hat{x}(t) - D_\delta \omega(t)
\]

Then, it follows from (19) that the innovation model for (17)-(18) is given by

\[
\delta \hat{x}(t) = A_\delta \delta \hat{x}(t) + B_\delta \delta \omega(t) + K_\delta \delta v(t)
\]

\[
s(t) = C_\delta \delta \hat{x}(t) + D_\delta \delta \omega(t) + v(t)
\]

It should be noted that the innovation representation above is the basic δ-operator model, to which the subspace identification method of [17] is applied.
4.2 Matrix input-output model

For the innovation model of (22)-(23), the extended observability matrix is defined as
\[
O_k = \begin{bmatrix}
    C_0 & C_0 A_0 & \cdots & C_0 A_0^{k-1}
\end{bmatrix} \in \mathcal{R}^{k+n}
\]
where \( k \) is assumed to be larger than \( n \). We construct the following data matrices
\[
S_{k,N} = \begin{bmatrix}
    s(t) & s(t+(N-1)\Delta) & \cdots & \delta s(t+(N-1)\Delta) & \cdots & \delta^{k-1} s(t+(N-1)\Delta)
\end{bmatrix}
\]
and
\[
W_{k,N} = \begin{bmatrix}
    \omega(t) & \omega(t+(N-1)\Delta) & \cdots & \delta \omega(t+(N-1)\Delta) & \cdots & \delta^{k-1} \omega(t+(N-1)\Delta)
\end{bmatrix}
\]
From (22)-(23), it is straightforward to derive the matrix input-output equation as
\[
S_{k,N} = O_k X_N + \Gamma_k^w W_{k,N} + \Gamma_k^v V_{k,N}
\]
where \( X_N = \begin{bmatrix} \hat{x}(t) & \hat{x}(t+\Delta) & \cdots & \hat{x}(t+(N-1)\Delta) \end{bmatrix} \), and
\[
\Gamma_k^w = \begin{bmatrix}
    I & 0 & \cdots & 0 \\
    C_0 B_0 & I & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    C_0 A_0^{k-2} B_0 & C_0 A_0^{k-3} B_0 & \cdots & I
\end{bmatrix}
\]
\[
\Gamma_k^v = \begin{bmatrix}
    I & 0 & \cdots & 0 \\
    C_0 K_0 & I & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    C_0 A_0^{k-2} K_0 & C_0 A_0^{k-3} K_0 & \cdots & I
\end{bmatrix}
\]
\[
V_{k,N} = \begin{bmatrix}
    v(t) & v(t+(N-1)\Delta) & \cdots & \delta v(t+(N-1)\Delta) & \cdots & \delta^{k-1} v(t+(N-1)\Delta)
\end{bmatrix}
\]
Since the high-order \( \delta \)-domain transform causes high frequency components in (24) and (25), it is necessary to introduce an appropriate preprocessing to get smoothed signals. Therefore, we introduce a stable pre-filter with order \( l \) by
\[
F(\hat{\delta}) = \frac{1}{\hat{\delta}^l + e_1 \hat{\delta}^{l-1} + \cdots + e_l}
\]
where \( e_1, e_2, \ldots, e_l \) are parameters of the pre-filter. Applying the filter to the system (22)-(23) yields
\[
\begin{align*}
\delta \hat{x}^f(t) &= A_0 \hat{x}^f(t) + B_0 \omega^f(t) + K_0 \nu^f(t) \\
\hat{s}^f(t) &= C_0 \hat{x}^f(t) + D_0 \omega^f(t) + \nu^f(t)
\end{align*}
\]
where \( \omega^f(t) = F(\hat{\delta}) \omega(t) \), and \( \hat{s}^f(t), \hat{x}^f(t), \nu^f(t) \) are defined in a similar way. Thus, we see that the matrix input-output equation (26) should be replaced by
\[
S'_{k,N} = O_k X'_N + \Gamma_k^w W'_{k,N} + \Gamma_k^v V'_{k,N}
\]
where
\[
\begin{align*}
S'_{k,N} &= \begin{bmatrix}
    s^f(t) & s^f(t+(N-1)\Delta) & \cdots & \delta s^f(t+(N-1)\Delta) & \cdots & \delta^{k-1} s^f(t+(N-1)\Delta)
\end{bmatrix} \\
W'_{k,N} &= \begin{bmatrix}
    \omega^f(t) & \omega^f(t+(N-1)\Delta) & \cdots & \delta \omega^f(t+(N-1)\Delta) & \cdots & \delta^{k-1} \omega^f(t+(N-1)\Delta)
\end{bmatrix} \\
X'_N &= \begin{bmatrix} \hat{s}^f(t) & \hat{s}^f(t+\Delta) & \cdots & \hat{s}^f(t+(N-1)\Delta) \end{bmatrix}
\]
and \( X'_N, V'_N \) are defined similarly.

In order to construct the data matrices \( S'_{k,N} \) and \( W'_{k,N} \), we define
\[
\Phi_s^f = \begin{bmatrix}
    s^f(t) & \cdots & s^f(t+(N-1)\Delta) \\
    \delta s^f(t) & \cdots & \delta s^f(t+(N-1)\Delta) \\
    \vdots & \vdots & \ddots & \vdots \\
    \delta^{k-1} s^f(t) & \cdots & \delta^{k-1} s^f(t+(N-1)\Delta)
\end{bmatrix}, \quad i = 1, \ldots, \rho
\]
\[
\Phi_w^f = \begin{bmatrix}
    \omega^f(t) & \cdots & \omega^f(t+(N-1)\Delta) \\
    \delta \omega^f(t) & \cdots & \delta \omega^f(t+(N-1)\Delta) \\
    \vdots & \vdots & \ddots & \vdots \\
    \delta^{k-1} \omega^f(t) & \cdots & \delta^{k-1} \omega^f(t+(N-1)\Delta)
\end{bmatrix}, \quad j = 1, \ldots, m
\]
where \( s^f(t) \) and \( \omega^f(t) \) are respectively the \( i \)th and \( j \)th components of \( \hat{s}^f(t) \) and \( \hat{\omega}^f(t) \). From the form of prefilter
of (27),
\[ \delta^l \delta^k = -e_1 \delta^l \delta^k - \cdots - e_{l-1} \delta^l \delta^k + \delta^k \]
\[ \delta^l \omega(t) = -e_1 \delta^l \omega(t) - \cdots - e_{l-1} \delta^l \omega(t) + \omega(t) \]

It follows that the above equations can be expressed as
\[ \delta \Phi^l(t) = \Xi \Phi^l(t) + \Pi \delta \Phi^l(t) \] (31)
\[ \delta \Phi^l(t) = \Xi \Phi^l(t) + \Pi \delta \Phi^l(t) \] (32)

where
\[ \Xi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -e_l & -e_{l-1} & \cdots & -e_1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

Therefore, the elements of data matrices \( S^l \), \( W^l \), can be obtained by solving (31)-(32) with assumed zero initial conditions. It should be noted that the order of the prefilter satisfies \( l \geq k \), because the operator \( \delta^{l-k}F(\delta) \) should be strictly proper.

4.3 Identification algorithm

It is well known that MOESP type subspace state space identification methods generally consist of two steps. The first step is to extract the range space of an extended observability matrix, and the second is to estimate the system matrices by using the extracted range space and the matrix input-output relation [19,20]. More recently, for a general state space model, some conditions are given such that (PO-) MOESP type algorithms provide consistent estimates in [21,22].

It follows from [21] that for the present innovation model of (22), (23), the conditions required for the consistency are given by

1. The innovation process \( \nu(t) \) is white noise.
2. The input \( \omega(t) \) is (quasi-) stationary [1].
3. The system is of minimal phase, \( A_0 \) and \( A_0 - K \) are stable.
4. \( (A_0, C_0) \) is observable and \( (A_0, [B_0 K_0]) \) is reachable.
5. The truncation index \( k \) should satisfy \( k \geq n \), where \( n = \dim \hat{x} \).

We see that in the present situation, all the conditions above are satisfied. If \( r(t) \) has a sufficiently high PE condition, then so does \( \omega(t) \), which is related to the exogenous input \( r(t) \) via (5). Thus the consistency results in [21,22] are useful for getting some insight into the consistency of our algorithm. It should be, however, noted that the consistency result is not directly applicable to a \( \delta \)-operator model based subspace identification algorithm to be developed for continuous-time systems.

In this paper, motivated by the fact that \( \text{Im} O \) can be retrieved from a QR decomposition of an augmented data matrix, formed by the future input-output as well as the past input-output data [20], we employ the following QR decomposition ([17]):

\[ W_{k,N} = \begin{bmatrix} R_{11} & 0 & 0 & 0 \\ I_{10} & R_{21} & R_{22} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ S_{k,N} & R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} \]

where \( I_{10} \) and \( I_{10}^l \) are defined as

\[ I_{10}^l = \begin{bmatrix} \delta^k \omega(t) & \cdots & \delta^k \omega(t+(N-1)\Delta) \\ \vdots & \vdots & \vdots \\ \delta^{2k-1} \omega(t) & \cdots & \delta^{2k-1} \omega(t+(N-1)\Delta) \end{bmatrix} \]

\[ I_{10}^l = \begin{bmatrix} \delta^k \delta^k(t) & \cdots & \delta^k \delta^k(t+(N-1)\Delta) \\ \vdots & \vdots & \vdots \\ \delta^{2k-1} \delta^k(t) & \cdots & \delta^{2k-1} \delta^k(t+(N-1)\Delta) \end{bmatrix} \]

Here the order of the prefilter of (27) should satisfy an even higher-order condition with \( l \geq 2k \), since \( \delta^{2k-1}(\hat{\nu}) = \delta^{2k-1}F(\delta)(\hat{\nu}) \) appearing (34), (35) should be strictly proper.

We observe that the QR decomposition of (33) is a special form used in instrumental variable MOESP type subspace methods. However, \( I_{10}^l \) and \( I_{10}^l \) of (34), (35) are formed by the higher order “derivatives” (or the future) of the input-output data, so that they are different from classical instrumental variable matrices used in [20,14]. Adapting the technique of [20] to the \( \delta \)-operator model setting, we derive two lemmas to be used in the proof of Proposition, which shows consistency of the present algorithm.

Lemma 1. Let the noise matrix \( V^{l,r}_{h,s} \) be defined by

\[ V^{l,r}_{h,s} = \begin{bmatrix} \nu(t) & \cdots & \nu(t+(N-1)\Delta) \\ \delta \nu(t) & \cdots & \delta \nu(t+(N-1)\Delta) \\ \vdots & \vdots & \vdots \\ \delta^{k-1} \nu(t) & \cdots & \delta^{k-1} \nu(t+(N-1)\Delta) \end{bmatrix} \]

and let the QR decomposition be given by (33). Then the following relation holds.
\[
\lim_{N \to \infty} \frac{1}{N} V_{l,N}^f (I_l^f)^T = 0
\] (37)

**Proof.** A proof is given in Appendix A.

**Lemma 2.** For \( N \to \infty \), we have
\[
\lim_{N \to \infty} \frac{1}{N} V_{l,N}^f Q_2 = 0
\] (38)

**Proof.** A proof is given in Appendix B.

Therefore we can obtain the following Proposition.

**Proposition.** Let the QR decomposition be given by (33). Then, for \( N \to \infty \), we obtain
\[
\frac{1}{\sqrt{N}} S_{l,N}^f Q_2 = \frac{1}{\sqrt{N}} O_l X_N^f Q_2
\] (39)
\[
\frac{1}{\sqrt{N}} S_{l,N}^f Q_1 = \frac{1}{\sqrt{N}} O_l X_N^f Q_3
\] (40)

**Proof.** A proof is deferred in Appendix C.

We readily see from Proposition that a consistent estimate of \( \Im(O) \) can be obtained by \( \Im([R_{42}, R_{43}]) \). Thus the corresponding algorithm is summarized as follows.

**IV-MOESP type algorithm**

1. For a given truncation index \( k \), design a filter whose order satisfies \( l \geq 2k \), and then form the data matrices \( S_{l,x}, W_{l,x}^f, I_l^f \) and \( I_l^f \) by (29), (30), (34), and (35), respectively.

2. Compute the QR decomposition of (33).

3. Compute the following SVD
\[
[R_{42}, R_{43}] = \left[ U_1 \ U_2 \right] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 S_1 V_1^T
\]

and put \( O_k = U_1 \), where \( \dim S_1 = n \), and \( S_2 \) contains nearly zero singular values.

4. Determine a quadruple \([A_0, B_0, C_0, D_0]\) by solving the following equations
\[
U_1(1: p(k-1), :) A_0 = U_1(1: p+1: pk,:)
\]
\[
C_0 = U_1(1: p,:)
\]
\[
U_2^T V_{l,k} = U_2^T H
\]

where \( H = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \) and, where the right pseudo-inverse exists due to the PE condition.

By employing the \( \delta \)-operator subspace identification above, we obtain an estimate of the quadruple \([A, B, C, D]\) up to a similarity transformation of the original continuous-time system (11), (12) for small \( \Delta \). Thus it follows from (10) that

\[
\hat{G}(s) = \lim_{\Delta \to 0} \begin{bmatrix} A_0 - B_0 D_0^{-1} C_0 \\ C_0 - D_0^{-1} C_0 \end{bmatrix} B_0 D_0^{-1}
\] (41)

Since the order of the estimated plant model above is higher than that of the true plant, we consider a model reduction procedure in the next section.

**V. MODEL REDUCTION**

For a stable transfer function \( \hat{G}(s) \), we use balance and truncate (B&T) to obtain a low-order approximation model which preserves stability and minimality and satisfies the twice-the-sum-of-the-tail \( H_\infty \) norm error bound. Unfortunately, we cannot apply B&T to the estimated high-order unstable plant model. An alternative approach named fractional balanced reduction (FBR) in [23], however, can handle unstable plants based on normalized fractional representations. A nice property of FBR is that it has a graph-metric error bound that allows a priori robustness analysis using graph-metric theory.

In this paper, we apply the FBR method to high-order possibly unstable estimated plant models to get reduced order models deleting unnecessary modes. We summarize the FBR algorithm as the following procedure [23].

**FBR Algorithm**

**Step 1:** Find a minimal state space triple \( \{A, B, C\} \) of \( \hat{G}(s) \).

**Step 2:** Solve the algebraic Riccati equation
\[
PA + A^T P - PB B^T P + C^T C = 0
\]
and let \( \tilde{A} = A + BK \) where \( K \equiv -B^T P \).

**Step 3:** Obtain the balanced realization \( \left\{ \tilde{A}, \tilde{B}, \left[ \tilde{C} \right] \right\} \) of the stable minimal state space triple \( \left\{ A, B, \left[ C \right] \right\} \).

**Step 4:** Apply the B&T to the above realization, and partition as
\[
\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}
\]

where \( \tilde{A}_{11} \in \mathbb{R}^{n \times n} \).

**Step 5:** The \( r \)-th order reduced model is given by
\[
\hat{G}_r(s) = \left\{ \tilde{A}_r, \tilde{B}_r, \tilde{C}_r \right\} \quad \text{with} \quad \tilde{A}_r = \tilde{A}_{11} - \tilde{B}_1 \tilde{K}_1, \quad \tilde{B}_r = \tilde{B}_1, \quad \tilde{C}_r = \tilde{C}_1.
\]
VI. NUMERICAL EXAMPLES

In this section, we present two numerical examples to show the effectiveness of the δ-operator based IV-MOESP type subspace identification method for continuous-time systems.

Example 1. This model is taken from [13], where an unstable 2nd-order plant is regulated by a PI controller. The configuration of the control system is given by Fig. 1, where the transfer functions of plant and controller are respectively given by

\[ G(s) = \frac{s+1}{s^2+s-2}, \quad C(s) = \frac{10s+15}{s} \]

It is easily seen that the plant has an unstable pole at \( s = 1 \) besides a stable pole at \( s = -2 \), while the zero is \( s = -1 \). The reference signal \( r_1(t) \) is chosen to be a random square signal sampled and held constant for 1 sec, while \( r_2(t) \) is set to zero, and \( v(t) \) is a white noise. The nominal plant \( G_0(s) \) is assumed to be given by

\[ G_0(s) = \frac{s+1.2}{s^2+1.2s-2.4} \]

The prefilter, the model order, the truncation index and the sampling period are respectively chosen as

\[ F(\delta) = \left( \frac{2}{\delta+2} \right)^n, \quad n = 3, \quad k = 4, \quad \Delta = 0.01 \text{ sec} \]

In order to investigate the sensitivity of the proposed method to the noise \( v(t) \), we define the noise-signal ratio (NSR) as ([13])

\[ \text{NSR} = \frac{\sigma(v)}{\sigma(y_1)} \times 100\% \]

where \( \sigma \) denotes the standard deviation and \( y_1(t) = y(t) - y_r(t) \), and where \( y_r(t) \) and \( y(t) \) denote the output components from \( r_1(t) \) and \( v(t) \), respectively. Figure 2 displays the reference, control and output data sampled for 30 sec with NSR about 5%. We observe that the NSR in the control signal \( u(t) \) is 10%, which is much higher than that in the output \( y(t) \).

We have carried out 20 Monte Carlo simulations for different noise realizations, while the same reference signal is used in all experiments. For each input-output sequence, a 3rd-order model is estimated by the proposed subspace identification algorithm. Subsequently, the estimated models are reduced from 3rd-order to 2nd-order by the model reduction procedure described in Section 5.

Figures 3 and 4 respectively depict the Bode plots of the true plant and the estimated plant models for two different NSRs with 5% and 10% in the output signal \( y(t) \), while Figs. 5 and 6 depict the estimates of poles and zero. It is seen from Fig. 3 that the frequency characteristics of the estimated model are very close to the true one. As the NSR becomes larger, the results are still satisfactory, though the estimated Bode plots slightly dispenses near the true value as in Fig. 4. Also, we see that the poles and zero in Figs. 5 and 6 are estimated accurately. The estimated parameters of the plant transfer function are displayed in Table 1, where the mean values of the estimates are quite close to the true values, while the variances increase slightly for larger NSR.

![Fig. 1. The closed-loop system.](image1)

![Fig. 2. Reference input, control and output signals (NSR = 5%).](image2)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>True</th>
<th>Estimated (NSR : 5%)</th>
<th>Estimated (NSR : 10%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Mean</td>
<td>Variance</td>
<td>Mean</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>1</td>
<td>0.9651 0.0006</td>
<td>0.9803 0.0035</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-2</td>
<td>-1.9556 0.0007</td>
<td>-1.9450 0.0038</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>1</td>
<td>0.9952 0.0001</td>
<td>0.9935 0.0005</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>1</td>
<td>0.9893 0.0005</td>
<td>0.9851 0.0016</td>
</tr>
</tbody>
</table>
Example 2. The second example is concerned with a simulation model for a chemical plant with unstable open-loop dynamics depicted in Fig. 7 [24], where the parameters of controller are set as \((K_c, T_i, T_d, \gamma) = (2.8, 45, 2, 0.1)\). The transfer functions of the true and nominal plants are respectively given by

\[
G(s) = \frac{0.0043}{s^2 + 0.0512s - 0.0055}
\]

\[
G_0(s) = \frac{0.0081}{s^2 + 0.08s - 0.008}
\]

Since the response of process dynamics is rather slow, we have selected the sampling period as \(\Delta = 0.2\) min. The reference signal \(r(t)\) is a random signal sampled and held constant for 120 min. A white Gaussian noise \(v(t)\) with zero-mean and variance \(\sigma_v^2 = (0.002)^2\) disturbs the output directly; a typical input-output data for 10 hours is displayed in Fig. 8. It should be noted that even though the NSR in the output \(y\) is quite small (about 1%), the NSR in the control signal \(u\) is about 15.6%; this is due to the fact that the controller shown in Fig. 7 contains an approximate differentiator in the feedback loop.
For the δ-operator subspace identification, the stable prefilter, the model order and the truncation index are respectively chosen as

$$F(\delta) = \left( \frac{0.3}{\delta + 0.3} \right)^{10}, \quad n = 4, \quad k = 5$$

We have done 20 Monte Carlo experiments, where the estimated models by the proposed algorithm are reduced from 4th-order to 2nd-order. From the estimation results shown in Table 2, we see that accurate estimates are obtained, illustrating the effectiveness of the proposed method.

Table 2. Identification results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>True</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Variance</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.0512</td>
<td>0.0507</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.0055</td>
<td>-0.0060</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0043</td>
<td>0.0045</td>
</tr>
</tbody>
</table>

VII. CONCLUSIONS

In this paper, by using the δ-operator approach, we have developed an IV-MOESP type subspace method for closed-loop state space identification of continuous-time systems. First, the dual-Youla parameterization of the plant is applied to obtain an equivalent open-loop identification problem for the joint input-output process. Then, a δ-operator based subspace identification algorithm is developed to give a consistent estimate for the state space model of the joint input-output process, from which a higher-order model of the plant is derived. A lower-order plant model is extracted from the higher-order model by removing the irrelevant modes. It should be noted that the present method, inherited the advantages from the δ-operator model approach and subspace methods, is a natural approach to the continuous-time system identification problem. Furthermore, the simulation results for numerical and realistic process models have shown the feasibility of the present method.

REFERENCES

APPENDIX A

Proof of Lemma 1

In this proof, for simplicity, we assume that the signals are Gaussian and ergodic. For \( i, j = 1, \ldots, k \), it follows from (34) and (36) that the \((i, j)\) block of (37) is represented as

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \delta^{-n} v'(\tau) (\Delta(\delta^{-n} \omega') (\Delta)) = 0 \tag{42}
\]

Define the \(\delta\)-operator transfer functions \( G(\delta) = \delta^{-i} F(\delta) \) and \( H(\delta) = \delta^{j+k-1} F(\delta) \). Then the left hand side of (42) is expressed as

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} v'(\tau) (\Delta(\delta^n \omega')) = E[v'(t)(\omega^k(t))] \tag{43}
\]

where \( v'(t) = G(\delta)v(t), \omega'(t) = H(\delta)\omega(t) \). Since the signals are ergodic, the following equation holds.

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \delta^{-n} v'(\tau) (\Delta(\delta^{j+k-1} \omega') (\Delta)) = E[v'(t)(\omega^k(t))] \tag{44}
\]

where \( E\{\cdot\} \) denotes the mathematical expectation. But we have

\[
E[v'(t)(\omega^k(t))] = \int_{-\infty}^{\infty} \Phi(v'(t), \omega^k(t)) \Phi(v'(t), \omega^k(t)) \, d\omega \tag{45}
\]

where \( \Phi(v', \omega^k) \) denotes the cross power spectral density of \( v(t) \) and \( \omega(t) \).

Now define the Hilbert spaces \( S_\tau = \overline{\text{span}\{s(t), s(t-1), \ldots\}}, \Omega = \overline{\text{span}\{\omega(t), \omega(t-1), \ldots\}} \), and \( \Omega = \overline{\text{span}\{\omega(t-1), \omega(t), \omega(t+1), \ldots\}} \), where \( \text{span} \) denotes the closure of linear spans. It follows from (5) that \( \omega(t) \) is a causal function of the exogenous input \( r(t) \), so that there is no feedback from \( s(t) \) to \( \omega(t) \). Thus, we see from (21) and ([25], Proposition 3) that

\[
v(t) = s(t) - E[s(t) | S_{-1}, \Omega] = s(t) - E[s(t) | S_{-1}, \Omega] \tag{46}
\]

where \( E\{\cdot|\cdot\} \) denotes the conditional expectation. Thus, since \( \omega(t) \in \Omega \) for any \( t \), we have

\[
E[v(t)\omega^k(t)] = E[s(t) - E[s(t) | S_{-1}, \Omega]\omega^k(t)]
\]

where \( E\{\cdot|\cdot\} \) denotes the conditional expectation. Thus, since \( \omega(t) \in \Omega \) for any \( t \), we have

\[
E[v(t)\omega^k(t)] = E[s(t) - E[s(t) | S_{-1}, \Omega]\omega^k(t)]
\]

This implies \( \Phi_{\omega \nu} = 0 \). Hence we have (42).

APPENDIX B

Proof of Lemma 2

For proof, we follow the procedure of [20]. By us-
ing (33), we get
\[
\frac{1}{N} V_{kN}^T (I_n)^T \frac{1}{N} V_{kN} Q R_{22}^T + \frac{1}{N} V_{kN} Q R_{22}^T \tag{43}
\]

Since \( Q_i = (W_{kN}^i)^T (R_{1i}^T) \), (43) is rewritten as
\[
\frac{1}{\sqrt{N}} V_{kN} Q \frac{1}{\sqrt{N}} R_{22}^T = \frac{1}{N} V_{kN} (I_n)^T - \frac{1}{N} V_{kN} (W_{kN}^i)^T \tag{44}
\]

It follows from Lemma 1 that
\[
\lim_{N \to \infty} \frac{1}{N} V_{kN} (I_n)^T = 0, \quad \lim_{N \to \infty} \frac{1}{N} V_{kN} (W_{kN}^i)^T = 0
\]

Thus the left hand side of (44) tends to zero as \( N \to \infty \). But, since \( \omega (t) \) has a sufficiently high PE condition, \( \frac{1}{\sqrt{N}} R_{22} \) is invertible. Hence we have (38).

\[\square\]

**APPENDIX C**

**Proof of Proposition**

It follows from (28) that
\[
\frac{1}{\sqrt{N}} S_{kN} Q_2 = \frac{1}{\sqrt{N}} C \chi_x (I_n)^T Q_2 + \frac{1}{\sqrt{N}} \Gamma^w_{kN} Q_2 + \frac{1}{\sqrt{N}} \Gamma^\nu_{kN} Q_2
\]

Using \( W_{kN}^i = R_0 \), we get
\[
\frac{1}{\sqrt{N}} S_{kN} Q_2 = \frac{1}{\sqrt{N}} C \chi_x (I_n)^T Q_2 + \frac{1}{\sqrt{N}} \Gamma^w_{kN} Q_2 + \frac{1}{\sqrt{N}} \Gamma^\nu_{kN} Q_2
\]

By using Lemma 2, we see that (39) holds. The equation (40) is proved in a similar way.

\[\square\]

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