DISCRETIZING CONTINUOUS-TIME CONTROLLERS WITH FUZZY LOGIC SYSTEMS AND ITS STABILITY ANALYSIS

Chen-Chia Chuang, Jin-Tsong Jeng, and Yung-Cheng Lee

ABSTRACT

In this paper, a new method, applying the fuzzy logic system, is proposed to discretize the continuous-time controller in computer-controlled systems. All the continuous-time controllers can be reconstructed by the proposed method under the Sampling Theorem. That is, the fuzzy logic systems are used to add nonlinearity and to approximate smooth functions. Hence, the proposed controller is a new smooth controller that can replace the original controller, independent of the sampling time under the Sampling Theorem. Consequently, the proposed controller not only can discretize the continuous-time controllers, but also can tolerate a wider range of sampling time uncertainty. Besides, the input-output stability is proposed for discretizing the continuous-time controller of the fuzzy logic systems. Finally, computer simulation shows that the proposed method can easily reconstruct the continuous-time controller and has very good robustness for different sampling times.

KeyWords: Discretizing continuous-time controller, input-output stability, fuzzy logic systems, computer control systems.

I. INTRODUCTION

When a continuous-time controller is discretized to become a discrete-time controller using the conventional methods, such as the design of discrete equivalents by numerical integration, zero-pole mapping equivalents, and hold equivalents, an undesirable feature that the discrete-time controller is dependent on the systems sampling time $T_s$ [1]. For control of linear systems using generalized sampled-data hold function [2], the discrete-time controller does depend on the sampling time $T_s$. For the nonlinear continuous-time systems, the discrete-time controller also depends on the sampling time $T_s$ [3]. For the input delay systems, the discrete-time controller also depends on the sampling time $T_s$ [4]. For the open-loop continuous-time systems, the discrete-time controller also depends on the sampling time $T_s$ [5]. Therefore, it is difficult to design a digital controller, which has robustness to different sampling time $T_s$ based only on conventional methods. Recently, Kabbamba and Hara [6], Keller and Anderson [7], and Markazi and Hori [8], applied $H_\infty$ control theory to the worst-case analysis on the sampling time $T_s$ for linear systems respectively; however, all their methods also depend on the sampling time. Besides, the sampling time effect on of high-order digitization’s has been
studied in [9]. In general, if the sampling time changes, previous methods require that the digital controller must be redesigned. On the other hand, if we want to obtain a similar analog controller, one approach to overcome this difficulty is to increase the sampling frequency. However, as indicated in [10], there are losses on reachability and observability when using the high sampling frequency. In this paper, the problem of discretizing a continuous-time controller by fuzzy logic system (FLS) in a computer-controlled system is proposed to improve the above drawbacks. That is, the FLS is used as the algorithm in this computer-controlled system. At the same time, the FLS controller is a continuous-time controller. However, the FLS with A/D and D/A is a discrete-time controller. That is, this computer-controlled system uses discrete-time controller to control continuous-time system. Hence, using the FLS to approximate \( C(s) \) can obtain better performance. At the same time, the sampling values of \( C(s) \) under the sampling time are regarded as target data for the FLS. Hence, one can view the proposed method as approximations of analog-control systems.

In general, the FLS consists of four important parts; namely, the fuzzification interface, knowledge base, decision-making unit, and defuzzification [11]. Besides, the FLS have been proved to be a university approximator [12-14]. That is, the FLS is capable of approximating any real continuous function on a compact set to an arbitrary accuracy. Hence, the FLS has many applications in control and identification of nonlinear systems. In this paper, a method using the FLS to overcome the above drawbacks in a discrete-time system is proposed first. The FLS controller is independent of the sampling time under the Sampling Theorem. That is, the FLS, as a new controller, can add nonlinearity and approximate a smooth function; consequently, the FLS not only can discretize the continuous-time controllers, but also can tolerate a wider range of sampling time uncertainty. Besides, we also show that the new method has the input-output stability in the monotone stable sense for the discretization of continuous-time controller. That is, if the error is less than a specified small value, then this new controller satisfies the monotone stability. If the error is nearly equal to zero, then this new controller is an optimal controller in the sense of discretization. Besides, we will also show that the FLS controller is independent of the sampling time under the Sampling Theorem. As a result, it is easy to obtain a similar analog controller. Hence, applying FLS to discretization of continuous-time controller is a practical approach.

II. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Problem formulation

In this paper, the problem of discretizing a continuous-time controller by the FLS in the computer-controlled system is considered as shown in Fig. 1, where \( u_1(t) \) is the disturbance input, \( u_2(t) \) is the input, \( C \) is a continuous-time controller, A/D is an analog to digital converter, D/A is a digital to analog converter, \( T_s \) is a sampling time, \( e[t_k] \) and \( u[t_k] \) are discrete-time signal, \( u(t) \) and \( y(t) \) are continuous-time signal. From Fig. 1, the plant is a continuous-time system and the computer controller is a discrete-time term. Our paper uses the FLS as a learning algorithm to approximate continuous-time controller. Because the learning procedure use point vs. point data, the A/D and D/A converter in the learning procedure is only used for the sampling function. Hence, the FLS controller in the learning procedure is a continuous-time controller. That is, the FLS use the discrete-point to reconstruct smooth continuous systems under the Sampling Theorem. At the same time, the FLS is used to add nonlinearity and to approximate smooth functions. Hence, the proposed controller is a new smooth controller that can replace the original controller. Besides, the sampling and hold functions in these converters are used in the control procedure. Hence, the FLS in the control procedure with A/D and D/A is a discrete-time controller.

![Fig. 1. Discretizating the continuous-time controller by the FLS controller.](image-url)
It is assumed that
(i) continuous-time controller $C$ is the input-output
put stability in the monotone stable sense
(monotone stable) in the $L_p$ norm which has been
designed for the control plant,
(ii) controller $C$, a nonlinear operator, will stabilize
the plant, and
(iii) the training data sets are obtained under the
Sampling Theorem [15].

It is desired that the FLS in Fig. 1 will not alter
the conventional computer-controlled implementation
structure. It is also required that the proposed FLS
satisfies the input-output stability of discretization of
the continuous-time controller in the learning proce-
dure and overcomes the difficulty of sampling time
uncertainty in the control procedure.

2.2 Preliminaries

Definition 1 [16]. For each real $p \in [1, \infty)$, the set $L_p \subset [0, \infty] = L_\infty$ consists of all measurable function $f(\cdot): \mathbb{R} \to \mathbb{R}$ such that
\[
\int_0^\infty | f(t) |^p dt < \infty ,
\]
where $R_+$ denote the set of nonnegative real numbers.
The set $L_\infty [1, \infty] = L_\infty$ consists of all measurable
function $f(\cdot): \mathbb{R} \to \mathbb{R}$ that are essentially bounded on
$[0, \infty)$.

Definition 2 [16]. For $p \in [1, \infty)$, the function $\| \cdot \|_p: L_p \to \mathbb{R}$ is defined by
\[
\| f(\cdot) \|_p = \left( \int_0^\infty | f(t) |^p dt \right)^{1/p}.
\]

Fact 1 (Hölder’s Inequality) [16]. Let $p, q \in [1, \infty)$, and suppose
\[
\frac{1}{p} + \frac{1}{q} = 1 .
\]
Suppose $f_i(\cdot) \in L_p$ and $f_j(\cdot) \in L_q$. Then the function $h: \mathbb{R} \to \mathbb{R}$ defined by
\[
h(t) = f_1(t) f_2(t) ,
\]
belongs to $L_1$. That is
\[
\| f_1(\cdot) f_2(\cdot) \|_1 \leq \| f_1(\cdot) \|_p \| f_2(\cdot) \|_q .
\]
The sub-multiplicative property of the $H_\infty$-norm satisfies
\[
\| F \|_\infty \leq \| F_1 \|_\infty \| F_2 \|_\infty ,
\]
where $H_\infty$-norm is defined by
\[
\| F \|_\infty = \max_{\varepsilon \in L_2([0,\infty])} \| F(\varepsilon) \|_2 .
\]

and $F_1$ and $F_2$ are nonlinear operator from $L_2$ to $L_2$.

Theorem 1 [16]. (Universal Approximation Theorem) Suppose that the input universe of discourse $U$ is a
compact set in $\mathbb{R}^n$. Then, for any given real continuous
function $g(x)$ on $U$ and arbitrary $\varepsilon > 0$, there exists a
FLS $f(x)$ such that
\[
\sup \left| f(x) - g(x) \right| < \varepsilon .
\]

That is, the FLS with product inference engines, single-
ton fuzzifier, center-of-gravity defuzzifiers, and Gaussian
membership function are universal approximations.

According to Theorem 1, the FLS can approximate
continuous functions arbitrarily well. That is, Theorem 1
shows that the FLS can approximate continuous functions
to arbitrary accuracy; the following corollary extends the
result to discrete functions.

Corollary 1 [16]. For any square integrable function $g(x)$
on the compact set $U \subset \mathbb{R}^n$, that is, for any $g \in L_2(U)$, there exists a FLS $f(x)$ such that
\[
\sqrt{\int_U | f(x) - g(x) |^2 dx} < \varepsilon .
\]
Theorem 1 and Corollary 1 provide a justification for
using the FLS in a variety of applications. Specifically,
they show that for any kind of nonlinear operations the
problem may require, it is always possible to design a
FLS that performs the required operation with any de-
gree of accuracy. They also provide a theoretical expla-
nation for the success of FLS in practical applications.

A typical FLS is as

Rule $i$

IF $z_i(t)$ is $M_{li}$ and \ldots and $z_r$ is $M_{ri}$,
THEN $y = y_i$ \hspace{1cm} $i = 1, 2, \ldots, r$.

(10)

where $M_j$ is fuzzy set, $r$ is the number of IF-THEN rules,
$z_i (t)$ is the premise variables, $Z = [z_1 z_2 \ldots z_r]$
y is output.

Given a pair $(Z, y)$, the final output of the FLS $f(Z)$
is inferred as follows:
\[
f(Z) = \frac{\sum_{i=1}^{r} S_i y_i}{\sum_{i=1}^{r} S_i} = \sum_{i=1}^{r} y_i h_i
\]

(11)

where $S_i = \prod_{j=1}^{r} M_j(z_j)$, $h_i = \frac{S_i}{\sum_{i=1}^{r} S_j}$, and $M_j(z_j(t))$ is
the grade of membership of $z_i(t)$ in $M_{i}$. In this paper, we assumed that $S_i(Z) \geq 0$, $i = 1 \sim r$ and $\sum S_i(Z) > 0$. Besides, the membership function is the Gaussian function in this paper. Finally, one of the most important advantages of FLS for discretizing continuous-time controller is that the FLS have the capability to approximate nonlinear mappings.

III. MAIN RESULTS

3.1 The stability analysis of the proposed structure

**Theorem 2** [17]. Consider the finite dimensional system
\[
\dot{x} = f(x, u); \quad x(0) = x_0; \quad t \geq 0, \quad y = g(x, u),
\]
where $u \in R^m, x \in R^n$ and functions $f$ and $g$ satisfy conditions:

C1. For each initial condition $x(0) \in R^n$ and each input $u: R_+ \to R^m$ regulated, Eq. (12) has a unique solution defined on $R_+$.

C2. $z = 0$ is a globally exponentially stable solution of $\dot{z} = f(z, 0)$ i.e. any solution $z(t, x_0)$ of $\dot{z} = f(z, 0)$ satisfies $\|z(t, x_0)\| \leq K \|x_0\| e^{-\beta t}$; $t \geq 0$ for some $K > 1$ and $\beta > 0$.

C3. $\exists k_1, k_2, p_1, q_1, l_1, l_2, R$, and $\exists p_1 \in [0, 1]$ s.t.
\[
\|f(x, 0) - f(x_0, 0)\| \leq k_1 \|x - x_0\|_2,
\]
\[
\|f(x, u_1) - f(x, u_2)\|_2 \leq (k_2 \|x\|_2^p + k_3) \|u_1 - u_2\|_2^p ,
\]
\[
\|g(x, u)\|_2 \leq l_1 \|x\|_2^p + l_2 \|u\|_2^p ,
\]
\[
g(0, 0) = 0 .
\]

$\forall x_1, x_2, x \in R^n, \forall u_1, u_2, u \in R^m$ and $\|\cdot\|_2$ denotes the Euclidean norm.

If the constant $k_2$ in C3 is zero, then the system Eq. (12) is input-output stability in the monotone stable sense (monotone stable) in the following sense:

R1. Let $u \in L_{p_1} \cap L_{p_2}$, then $y \in L_{p}$ where $p \geq \max (p_{1} q_{1}^{-1}, q_{2}^{-1}, 1)$ and $\|v\| \leq g (\|x\|_{p_1}, \|u\|_{p_2})$ for some $g$ such that $g(0, \cdot), g(\cdot, 0) \in M_0$, where $M = \{f: R_+ \to R_+\}$ and $f$ monotone increasing homeomorphism of $R_+$ onto $R_+$.

**Remark 1.** The C1, C2, C3, and R1 are the conditions of monotone stable for the finite dimensional system. That is, Theorem 2 guarantees that a nonlinear operator is input-output stability in the monotone stable sense. Also, from Eq. (12), Theorem 2 can directly be used in multi-input case. Besides, from Theorem 2, $\|y\|_{p} \leq g (\|u\|_{p_1}, \|u\|_{p_2})$ can be obtained. In addition, under Theorem 2, we can obtain
\[
\|x\|_{p_0} \leq \frac{Lk^3}{b^3} \|u\|_{p_0, p_2}^p ,
\]
where $L = \exp \left( (k_1 + \beta') \frac{\ln K}{\beta - \beta'} \right)$, and $\beta' < \beta$. However, this result cannot be directly used to derive the induced norm. Hence, we must simplify this result as Corollary 2 given below.

**Corollary 2.** Consider the finite dimensional system Eq. (12) where functions $f$ and $g$ satisfy conditions C1, C2, C3, and

C4. $p_1 = 0, p_2 = q_1 = q_2 = 1$,

then the system Eq. (12) is monotone stable in the $L_p$ norm.

**Proof.** Because $u \in L_{p(p,p)} \subset L_1 \cap L_{p_0}$. The results R1 then follows immediately from Eq. (15), Eq. (16), and Eq. (17) as
\[
\|y\|_p \leq l_1 \left( \frac{Lk^3}{b^3} \right) \|u\|_{p_0, p_2}^p + l_2 \|u\|_{p_0, p_2}^p ; \quad t \in R_+ ,
\]
which holds for $p \geq \max (1, (p_2 q_1^{-1}, q_2^{-1}))$.

Substituting C4 into Eq. (18) yields
\[
\|y\|_p \leq \sigma \|u\|_{p_0, \sigma} , \quad \sigma = l_1 \left( \frac{Lk^3}{b^3} \right) + l_2 , \quad p \geq 1 .
\]

From Eq. (19) it follows that the system Eq. (12) is monotone stable in the $L_p$ norm.

**Assumption 1.** Let $v_1$ and $v_2$ be two signal spaces. The operators $F_1: v_1 \to v_2$ and $F_2: v_2 \to v_1$ are such that for any input signals $u_1 \in v_1$ and $u_2 \in v_2$ there exists unique signals $u_1, y_2 \in v_1$ and $u_2, y_1 \in v_2$.

**Definition 3** [17]. The feedback system (Fig. 2) under Assumption 1 is called monotone stable if there exist functions $f_1: R_+ \times R_+ \to R_+$ and $f_2: R_+ \to R_+$ and constants $\beta_1, \beta_2 \in R$.
\[ y_1 \leq g_1(||u||_1, ||u||_2) + \beta_1, \quad (20) \]
\[ y_2 \leq g_2(||u||_1, ||u||_2) + \beta_2, \quad (21) \]
\[ \forall u_1 \in \mathbb{U}_1, \forall u_2 \in \mathbb{U}_2 \quad \text{and} \quad g_1(0,0), g_2(0,0), g_1(\cdot, g_2(\cdot, 0)) \in M_0, \quad \text{where} \quad M_0 = M \cup \{0\}, \quad 0 \text{ denotes the zero function } f = 0. \]

**Theorem 3** [17]. Consider the feedback system of Fig. 2, where \( F_1 \) and \( F_2 \) are nonlinear operators. Let \( F_1 \) and \( F_2 \) are monotone stable with gain functions as in \[ ||y||_p \leq \sigma_i \cdot ||u||_p + \gamma_i, \quad \sigma_i \geq 0, p_i > 0, \quad p = [1, \infty) \] with \( \gamma_i = 0, i = 1, 2 \). Suppose Assumption 1 holds. The feedback system is monotone stable if

\[ p_1p_2 = 1, \sigma_i\sigma_j^p < 1 \quad \text{or} \quad \sigma_i\sigma_j^p < 1 \quad \text{holds}. \]

**Corollary 3.** Consider the feedback system of Fig. 2. Let \( F_1 \) and \( F_2 \) be monotone stable in the \( L_p \) norm as in \[ ||y||_p \leq \sigma_i \cdot ||u||_p + \gamma_i, \quad \sigma_i \geq 0, p_i > 0, \quad p = [1, \infty) \] with \( \gamma_i = 0, i = 1, 2 \). Suppose Assumption 1 holds. The feedback system is monotone stable in the \( L_p \) norm if \( \sigma_1\sigma_2 < 1 \).

**Proof.** Because \( F_1 \) and \( F_2 \) are monotone stable in the \( L_p \) norm as in \[ ||y||_p \leq \sigma_i \cdot ||u||_p + \gamma_i, \quad \sigma_i \geq 0, p_i > 0, \quad p = [1, \infty) \] with \( \gamma_i = 0, i = 1, 2 \). Therefore, \( p_2 = 1 \). From Theorem 3, we can show that \( p_3 = 1 \). Therefore, the feedback system is monotone stable in the \( L_p \) norm if \( \sigma_1\sigma_2 < 1 \).

By a simple transformation of the system given in Fig. 2, the range of applicability of Corollary 3 can be expanded. The idea is to introduce an additional monotone stable in the \( L_p \) norm operator \( C \) by first subtracting it and then adding it to \( F_2 \), as shown in Fig. 3. After some simple transformations of Fig. 3, Fig. 4 can be obtained.

**Theorem 4.** Consider the system in Fig. 4, and suppose that \( F_1 \) and \( F_2 \) are monotone stable in the \( L_p \) norm. The system in Fig. 3 is monotone stable in the \( L_p \) norm if there exists an operator \( C \) which is monotone stable in the \( L_p \) norm, and

(i) \( F'(F_1, C, u_2) \) is monotone stable in the \( L_p \) norm,

(ii) \( \sigma_1(F_2 - C) \sigma_2(F'(F_1, C, u_2)) < 1 \).

where \( F'(F_1, C, u_2) \) denotes the operator between \( e_2 \) and \( \tilde{e}_1 \) in Fig. 4.

**Proof.** The proof is the same as Theorem 3, and is omitted.

Note that the Fig. 1 can be Added a \( C \) and subtracted a \( C \) to obtain Fig. 5 in the learning procedure, and can be further expressed by Fig. 6 after loop transformation, where \( P(P, C, u_2) \) denotes the operator between \( e_2 \) and \( \tilde{e}_1 \) in Fig. 6. Besides, the operator \( C \) can be a linear or nonlinear controller. From Corollary 2 and Theorem 4, we have the following Theorem 5.
Theorem 5. Let $C$ be an operator, which can stabilize $P$. If the following conditions hold,
(i). there exists an operator $C$, which is monotone stable in the $L_p$ norm,
(ii). there exists an operator $FLS$, which is monotone stable in the $L_p$ norm,
(iii). $P'$ is monotone stable in the $L_p$ norm, and
(iv). $\sigma_1(FLS)\sigma_2(P') < 1$, where $FLS = FLS - C$
where $P' = P^*(C, P)$, then the system in Fig. 4 is monotone stable in the $L_p$ norm.

Theorem 6. Assume the system in Fig 6 satisfies the conditions for monotone stable given in Theorem 5. In particular, if $p = 2$, then
\[
\sigma_1(FLS)\sigma_2(P') = \|FLS\|_\infty \|P'\|_\infty,
\]
and when $\|FLS\|_\infty \|P'\|_\infty < 1$, it implies that
\[
\|FLS - P'\|_\infty < 1,
\]
Proof. It is shown in Appendix A.

Remark 2. When $u_2$ is equal to zero, from Fig. 6, we conclude that a state feedback controller structure or a output feedback controller structure can be discretized via neural networks.

Based on above results, if the error between output of $C$ and FLS is less than a small constant, and Eq. (23) $\|FLS - P'\|_\infty < 1$ holds, then this new controller satisfies the monotone stability. If the error is nearly equal to zero and Eq. (23) holds, then this new controller is an optimal controller in the sense of discretization. That is, the Eq. (23) ensures the system's stability robustness in Fig. 1.

3.2 The robustness of FLS controller with different sampling time

In order to train the FLS, a modify learning structure is proposed in Fig. 7. That is, Fig. 6 on the FLS can be modified as Fig. 7 for the learning procedure, where $\tilde{e}[t_k]$ is the controller input, $O[t_k]$ is the controller output, $\hat{O}[t_k]$ is the output of FLS controller, and $\tilde{e}[t_k]$ is the error between $O[t_k]$ and $\hat{O}[t_k]$. The sampling time $Ts$ must satisfy the Sampling Theorem. Then the FLS can approximate smooth functions $C(\cdot)$ arbitrarily well. Therefore, we can obtain the following result from the Theorem 1
\[
|C(\tilde{e}(T_i), T_i) - \hat{F}(\tilde{e}(T_i), T_i)| < \sum_{n=1}^{N} \tilde{e}^2(n) = \varepsilon,
\]
and
\[
C(\tilde{e}(T_i), T_i) = \hat{F}(\tilde{e}(T_i), T_i) + \varepsilon/N,
\]
where $T_i$ is a constant in the interval $[T_{i-1}, T_{i+1}]$, $N$ is the number of sampling points, $T_i$ is represent the whole interval, and $\varepsilon$ is an arbitrary small constant. Therefore, we conclude
\[
C(\tilde{e}(T_i), T_i) \approx \hat{F}(\tilde{e}(T_i), T_i).
\]
Since there is no restriction on the step size, we can conclude that the FLS controller $\hat{F}(\cdot)$ is independent of the sampling time $Ts$.

IV. COMPUTER SIMULATIONS

The proposed method of using the FLS to discretize the continuous-time controller is simulated for following three examples. The simulation programs are running under the Matlab software. The simulation results of the proposed method and some conventional methods are compared for several different sampling times. The learning algorithm in this paper is the recursive least square method [18].

Example 1. (Unstable system) Consider the plant $G(s)$ [19] is
\[
G(s) = \frac{(s - 1)}{s(s - 0.5)},
\]
the continuous-time controller $C(s)$ is
\[
C(s) = \frac{-11.5s - 1}{s + 15},
\]
the conventional discretized controller is

\[ C(z) = \frac{-11.5z + 11.499}{(z - 0.9851)} \quad \text{with} \quad T_s = 0.01 \text{ sec.} \quad (29) \]

The number of the proposed FLS rule is equal to 22, the membership function is Gaussian function, and sum square error is 0.01. In this example, it needs 985 epochs. Fig. 8 shows the target data and approximated results with FLS. Figure 9 shows the close-loop step responses with digital controller \( C(z) \) for three different sampling times. Because this plant is an unstable case, the response is easily affected by the different sampling time. Figure 10 shows the close-loop step responses with FLS controller for different sampling times. The results show that the new method has very good robustness to different sampling times. Figure 11 shows the norm values in Eq. (23) at the last two epochs. The norm values satisfy Eq. (23) in the last epoch. It means that the training procedure is successfully to approximate controller and the proposed structure satisfies the input-output stability.

**Example 2.** (Nonlinear-case) Consider the robot dynamics are modeled as a second-order system \( G(s) \) with two poles at \( s = -1 \) and include a time-delay of \( \pi/4 \) seconds.

**Fig. 9.** Step responses obtained from the conventional controller with different sampling times in example 1 with unstable systems. (\( T = 0.01 \text{sec.} \))

**Fig. 10.** Step responses obtained from the FLS controller with different sampling times in example 1 with unstable systems. (\( T = 0.01 \text{sec.} \))

**Fig. 11.** Shown the norm values in Eq. (23) for example 1.
the continuous-time controller $C(s)$ is

$$C(s) = \frac{0.188s + 0.0164}{s},$$

(31)

the conventional discretized controller is

$$C(z) = \frac{(0.188z - 0.1879)}{(z - 1)}$$

with $T_s = 0.05$ sec. (32)

The number of the proposed FLS rule is equal to 25, the membership function is Gaussian function, and sum square error is 0.01. In this example, it needs 1021 epochs. Figure 12 shows the target data and approximated results with FLS. Figure 13 shows the close-loop sin responses with digital controller $C(z)$ for three different sampling times. Figure 14 shows the close-loop sin responses with FLS controller for different sampling times. The results show that the new method has very good robustness to different sampling times. The number of the proposed FLS rule is equal to 26, the membership function is Gaussian function, and sum square error is 0.01. In this example, it needs 1400 epochs. Fig. 16 shows the responses for stabilization via the proposed FLS controller for three different sampling times. The results show that the proposed method has very good robustness to different sampling times. Besides, the target controller can stabilize the nonlinear system and be found [20]. At the same time, the stabilization time is much longer than that of target controller. Hence, we only show the time to 20 with the FLS in Fig. 16.

**Example 3.** (Nonlinear case [20]) To stabilize a nonlinear system whose normal form is

$$\dot{z} + z^3 - z^5 + yz = 0, \quad \dot{y} = u,$$

(33)

where $u$ is controller, $u = 8z^3\dot{z}$ and the sampling-time $T$ equals 0.01. The state vector is $[z \quad \dot{z}]^T$. Note that the system is nonminimum phase, Let the Lyapunov function be $V = \frac{1}{2}z^2 + \frac{1}{6}\dot{z}^6$ and that the internal dynamics would be asymptotically stable if $y = 2z^4$. The number of the proposed FLS rule is equal to 26, the membership function is Gaussian function, and sum square error is 0.01. In this example, it needs 1400 epochs. Fig. 16 shows the responses for stabilization via the proposed FLS controller for three different sampling times. The results show that the proposed method has very good robustness to different sampling times. Besides, the target controller can stabilize the nonlinear system and be found [20]. At the same time, the stabilization time is much longer than that of target controller. Hence, we only show the time to 20 with the FLS in Fig. 16.
Fig. 14. Sin responses obtained from the FLS controller with different sampling times in example 2 (T = 0.05 sec).

Fig. 15. Shown the norm values in Eq. (23) at the last two epochs for example 2.

Fig. 16. To stabilize a nonlinear system with different sampling times by the FLS controller in example 3.

Remark 3. The conditions for the conventional discretized controller (fixed sampling period against different sampling systems) used in the examples. If the conventional discretized controller is modified by different sampling rate, the performance will be similar to the proposed FLS controller.

V. CONCLUSION

This paper develops a new method to discretize the continuous-time controller by the FLS. Results show that the proposed method has very good robustness to different sampling times. All continuous-time controllers can be reconstructed by the proposed method and without changing the computer-controlled structure under the Sampling Theorem. That is, the FLS are used to add nonlinearity and to approximate smooth functions. Consequently, the FLS not only can discretize the continuous-time controllers, but also can tolerate a wider range of sampling time uncertainty. Besides, the input-output stability of the system is proposed for discretization of continuous-time controller with FLS. Hence, using the FLS to the discretization of continuous-time controller is more practical in some applications.

APPENDIX A

Proof of Theorem 6.

First, we shall derive \( \sigma_1 \left\{ \tilde{F}_S \right\} \). Since

\[
\sigma_1(F) = \inf \{ \sigma_1: \| F(e) \|_p \leq \sigma_1 \| e \|_p \},
\]

and from Fig. 6, we obtain

\[
\| \tilde{F}_S(\tilde{e}_1) \|_p \leq \sigma_1 \| \tilde{e}_1 \|_p,
\]

or

\[
\frac{\| \tilde{F}_S(\tilde{e}_1) \|_p}{\| \tilde{e}_1 \|_p} \leq \sigma_1.
\]

From Eq. (A1) and Eq. (A2), we have

\[
\inf \sigma_1 = \max_{\tilde{e}_1 \in L_1(0, \infty)} \frac{\| F S(\tilde{e}_1) \|_p}{\| \tilde{e}_1 \|_p}.
\]

Now let \( p = 2 \)

\[
\inf \sigma_1 = \max_{\tilde{e}_1 \in L_1(0, \infty)} \frac{\| F S(\tilde{e}_1) \|_2}{\| \tilde{e}_1 \|_2} = \| \tilde{F}_S \|_2.
\]

Next, the same result can be obtained for \( \sigma_2(P) \) under \( p = 2 \).
\[
\inf \sigma_2 = \max_{e_2 \in [-0, \sigma]} \| P'(e_2) \|_2 = \| P \|_\infty . \tag{A5}
\]

Therefore,
\[
\inf \sigma_2 = \| P' \|_\infty = \sigma_2(P'). \tag{A6}
\]

From Eq. (A4) and Eq. (A6), we can obtain
\[
\sigma \left( \frac{\| FLS \|_\infty}{\sigma} \right) \sigma(P') = \| FLS \|_\infty \| P' \|_\infty . \tag{A7}
\]

From Theorem 5 and Eq. (A7), it follows that
\[
\| FLS \|_\infty \| P' \|_\infty < 1. \tag{A8}
\]

We know that when (A8) holds, it implies that
\[
\| FLS \cdot P' \|_\infty < 1 \quad \text{from the sub-multiplicative property of the } H_\infty \text{-norm.}
\]

ACKNOWLEDGMENT

The author wishes to thank the referees for their constructive comments, which have improved the results of the paper.

REFERENCES