A SIMPLE CONTROL METHOD COPING WITH
A KINEMATICALLY ILL-POSED INVERSE PROBLEM
OF REDUNDANT ROBOTS: ANALYSIS IN CASE
OF A HANDWRITING ROBOT

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ABSTRACT

In order to enhance dexterity in execution of robot tasks, a redundant number of degrees-of-freedom (DOF) is adopted for design of robotic mechanisms like manipulators and multi-fingered hands. Associated with such DOF redundancy relative to the number of physical variables necessary and sufficient for description of a given task, an extra performance index is introduced for controlling such a redundant robot in order to avoid arising of an ill-posed problem of inverse kinematics from the task space to the joint space. This paper treats a handwriting robot as an illustrative example and shows that such an ill-posedness of DOF redundancy can be resolved in a natural way by using a novel concept named “stability on a manifold”. It is shown theoretically that sensory feedback signals with a simpler form computed on the basis of measurement data of task-description variables render the closed-loop dynamics to converge asymptotically to a target task-description lying on a lower-dimensional manifold of steady states. Computer simulation concerning specified robot tasks verifies the effectiveness of the proposed control scheme, which results in human-like distribution of joint motions.

KeyWords: Redundancy resolution, inverse kinematics, redundant robot, constraint manifold, stability on a manifold.

I. INTRODUCTION

If a robot is designed so as to mimic human limb then its mechanism must be kinematically redundant, that is, its total degrees of freedom is higher than the number of independent physical variables required for description of a given motion task. This kinematic redundancy contributes to enhancement of dexterity and versatility in execution of robot tasks as discussed in a variety of papers and books [1-4]. However, such redundancy of DOFs usually increases the complexity of robot dynamics and therefore makes control problems for execution of given tasks more difficult. It is emphasized in particular that in such a case the inverse kinematics from the operational space (task-description space) to the robot joint space becomes ill-posed. In order to avoid this ill-posedness, many methods have been proposed, most of which are based on two ideas of using either Jacobian pseudoinverse [1] or extended Jacobian matrix by introducing slack variables [5] and introducing some artificial performance criterion for determining uniquely an appropriate joint space trajectory by minimizing it [6] (for further referring to other works concerning previous methods of redundancy resolution, see [2] and [3]).

This paper proposes a simple method for resolving such an ill-posedness problem related to redundancy of DOFs by a natural way without introducing any extra performance criterion. Instead, a novel concept named “stability on a manifold” is introduced and it is shown that there exists a sensory feedback based on measurements of physi-
cal variables of task description and this sensory feedback signal enables the overall closed-loop system naturally and coordinately to converge to a lower-dimensional constraint manifold that describes a set of joint states fulfilling a target given task. The original idea of ”stability on a manifold” was first introduced in control of multi-fingered hands for stable grasp and object-manipulation [7,9]. In this paper, typical robotic tasks of handwriting [10] under DOF redundancy are treated. It is shown that proposed feedback control signals are of a simpler form than those derived by conventional methods of using some extra performance criteria. It is proved on the basis of passivity analysis that the proposed sensory feedback signals render the closed-loop dynamics to converge to a lower-dimensional manifold of steady states in the joint space satisfying a given target task.

The effectiveness of proposed sensory feedback controls is shown by computer simulation in the case of robot handwriting by using a four DOF robot whose end is constrained on a flat surface. The total number of DOFs is redundant relative to the number of physical variables necessary for task description even if holonomic constraints are taken into consideration. It is shown that the proposed sensory feedback control signals yield human-like distribution of joint motions. Further, it is shown that another new concept defined and named as ”asymptotic transferability to a manifold” plays a crucial role in task execution when gravity terms can not be compensated directly but can be controlled by using regressors and estimates for unknown or uncertain link masses.

II. DYNAMICS OF ROBOTICS WITH DOF REDUNDANCY

Robot dynamics like a handwriting robot as shown in Fig.1 can be described by the following Lagrange equation with a holonomic constraint (for example, see [11]).

\[
q^T u = \frac{d}{dt} \left[ \frac{1}{2} \dot{q}^T H(q) \dot{q} + P(q) \right] + \dot{q}^T C_0 \lambda.
\]

(2)

In the case of a handwriting robot shown in Fig. 1 the constraint is described as

\[
\phi(q) = z(q) = l_1 + l_2 \sin q_2 + l_3 \sin(q_2 + q_3) + l_4 \sin(q_2 + q_3 + q_4).
\]

(3)

Hence, the gradient of \( \phi \) in \( q \) is described as

\[
\frac{\partial \phi}{\partial q} = \begin{pmatrix}
-l_2 \cos q_2 - l_3 \cos(q_2 + q_3) - l_4 \cos(q_2 + q_3 + q_4) \\
-l_3 \cos(q_2 + q_3) - l_4 \cos(q_2 + q_3 + q_4) - l_4 \cos(q_2 + q_3 + q_4)
\end{pmatrix}
\]

(4)

At first it is assumed that the gravity term \( g(q) \) is known or can be computed in real-time based on measurement data by optical encoders of joint actuators. The case when \( g(q) \) is uncertain will be treated later. Then, consider a control signal composed of four terms of gravity compensation, angular velocity feedback for damping shaping, feedforward signal of desired pressing force, and position feedback from measured position error in the \( xy \)-plane as described in the following:

\[
u = g(q) - C_i \dot{q} - J_x^T K(x - x_d) - \left(\frac{\partial \phi}{\partial q}\right)^T \lambda_d.
\]

(5)

where \( x = (x, y, z) \), \( x_d = (x_d, y_d, z_d) \) denotes a given desired position on the \( xy \)-plane, \( J_x(q) = \dot{e}x/\dot{e}q \) the \( 2 \times 4 \) Jacobian matrix of \( x \) in \( q \), and \( \lambda > 0 \) a desired pressing force. Substituting this into Eq. (1) yields

\[
H(q) \ddot{q} + \frac{1}{2} H(q) \dot{q} + S(q, \dot{q}) + C(q, \dot{q}) \dot{q} + g(q) + J_x^T(q) K \Delta x - \left(\frac{\partial \phi}{\partial q}\right)^T \Delta \lambda = 0.
\]

(6)

Fig. 1. A handwriting robot with four joints (four degrees of freedom).
where \( C = C_0 + C_1 \), \( \Delta \lambda = \lambda - \lambda_0 \), and \( \Delta x = x - x_0 \). Then, taking an inner product of Eq. (6) with \( \dot{q} \) leads to

\[
\frac{d}{dt} E = -q^T C \dot{q}
\]

where

\[
E = \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{1}{2} \Delta x^T K \Delta x.
\]

It is important to note that \( E \) is neither positive definite in the joint space \( \mathbb{R}^6 = \{(q, \dot{q})\} \) nor positive definite in the constraint 6-dimensional manifold

\[
M_\delta = \{(q, \dot{q}) : \dot{q} = 0 \text{ and } \frac{\partial \phi}{\partial q} = 0\}
\]

because \( E \) includes a quadratic term of only two position variables \( x \) and \( y \). Therefore the scalar quantity \( E \) can not be regarded as a Lyapunov function in verification of stability of the closed-loop dynamics of Eq. (6). However, the quantity \( E \) plays an important role in the proof of convergence of a solution to the Eq. (6). To show this, it is necessary to introduce the following one-dimensional manifold

\[
M_\delta = \{(q, \dot{q}) : \phi(q) = 0 \text{ and } \dot{q} = 0\}
\]

Note that on this manifold the equality \( \lambda = \lambda_0 \) follows from Eq. (6) automatically as far as \( \partial \phi/\partial q \) is non-zero. Then, consider a state \((q^0, \dot{q}^0) = 0\) that belongs to \( M_\delta \) and call it the reference state. At the same time, define a neighborhood \( N(q) \) of the fixed reference state \((q^0, 0)\) with radius \( r \) in such a way that

\[
N^\delta(r) = \left\{(q, \dot{q}) : \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{1}{2} \Delta q^T H(q) \Delta q \leq r^2 \right\}
\]

where \( \Delta q = q - q^0 \) and assume that there exists some positive value \( r_0 > 0 \) such that for any \((q, \dot{q}) \in N^\delta(r_0)\) three Jacobian vectors \( \partial \phi/\partial q \), \( \partial \phi/\partial \dot{q} \), \( \partial \phi/\partial \Delta q \) are independent, and hence \( J_z(q) \) is of full rank (non-degenerate), and therefore \( J_z(q) P_z J_z(q) = 0 \), where \( J_z = \partial \phi/\partial q \),

\[
P_z = I - J_z^T J_z, \quad \text{and} \quad J_z^T = J_z (J_z J_z^T)^{-1}.
\]

As seen from Eq. (4), the Jacobian vector \( J_z \) becomes of zero-vector if and only if all \( a_i, i = 2, 3, 4 \) are equal to zero or \( \pm \pi \), provided that \( 0 \geq q_2 \geq -\pi, |q_3| \leq \pi, \) and \( -\pi \leq q_4 \leq \pi \). Hence it is reasonable to assume that \( r_0 \) can be chosen not so small and must be of order of the square root of the largest eigenvalue of the inertia matrix \( H(q) \) over all \( q \). It is also necessary to introduce a family of neighborhoods of the reference state \((q^0, 0)\) on the constraint manifold \( M_\delta \) in such a manner that

\[
N^\delta_\epsilon(q_0) = \{(q, \dot{q}) : \phi(q) = 0, J_z(q) \dot{q} = 0, \text{ and } E(\Delta x(q), \dot{q}) \leq \epsilon^2\}
\]

where \( E \) stands for the quadratic function of \( \dot{q} \) and \( \Delta x(q) = x(q) - x_0 \). Note that any state lying on \( M_\delta \cap N^\delta(q_0) \) is included in \( N_\epsilon(q_0) \cap N^\delta(q_0) \) for any \( \epsilon \geq 0 \). Now, it is important to introduce the following two concepts:

**Definition (Stability on a manifold).** If for any \( \epsilon > 0 \) there exist \( \delta(\epsilon) > 0 \) depending on \( \epsilon > 0 \) and another constant \( r_1 > 0 \) being less than \( r_0 \) and independent of \( \epsilon \) such that any solution to Eq. (6) starting from an arbitrary initial state lying in \( N_\epsilon(q_0) \cap N^\delta(q_0) \) remains in \( N_\epsilon(q_0) \cap N^\delta(q_0) \) for any \( t > 0 \), then the reference state \((q^0, 0)\) is said to be stable on a manifold (see Fig. 2).

**Definition (Asymptotically transferability to a manifold).** If there exist positive values \( \delta_1 > 0 \) and \( r_1 > 0 \) less than \( r_0 \) such that any solution starting from an arbitrary initial state in \( N_\delta(q_0) \cap N^\delta(q_0) \) remains in \( M_\delta \cap N^\delta(q_0) \) and converges asymptotically to some state in the set \( M_\delta \cap N^\delta(q_0) \) as \( t \rightarrow \infty \), then the neighborhood \( N_\delta(q_0) \cap N^\delta(q_0) \) together with the reference state \((q^0, 0)\) is asymptotically transferable to \( M_\delta \cap N^\delta(q_0) \) (see Fig. 3).
The latter concept named “asymptotic transferability to a manifold” means that even if the state \((q^0, 0)\) jumped to any other state \((q(0), \dot{q}(0))\) at \(t = 0\) affected by some impact disturbance the solution trajectory of the closed-loop dynamics with initial condition \((q(0), \dot{q}(0))\) can transfer asymptotically to some steady state lying on \(M_1 \cap \mathbb{N}^3(r_0)\) that satisfies \(\Phi(q) = 0\) and \(\dot{q} = 0\) without loosing contact between the xy-plane and robot endpoint throughout the transient response.

Now consider stability of the reference state \((q^0, 0)\) on the manifold \(M_1 \cap \mathbb{N}^3(r_0)\). If the reference state \(q^0\) satisfies the condition that \(0 > q_1 > -\pi/2, 0 < q_2 < \pi/2, 0 < q_3 < \pi/2\) and \(\Phi(q) = 0\), then it is easy to check that three 4-dimensional vectors \(\partial \chi/\partial q, \partial \chi/\partial \dot{q},\) and \(\partial \gamma/\partial q\) are independent. Hence it is possible to choose some positive value \(r_0 > 0\) and construct a neighborhood \(\mathbb{N}^3(r_0)\) such that for any \((q, \dot{q}) \in \mathbb{N}^3(r_1)\), \(J_s(q) = \begin{bmatrix} \partial \chi/\partial q & \partial \chi/\partial \dot{q} & \partial \gamma/\partial q \end{bmatrix}\) is nondegenerate (that is, rank \(J_s(q) = 2\)) and there exist two positive constant \(\sigma_0\) and \(\sigma_1\) such that

\[
\sigma_1 I_2 \geq J_s(q) J_s^T(q) \geq J_s(q) P_{\sigma_0} J_s^T(q) \geq \sigma_0 I_2.
\]

Finally, it is reasonable to assume that all physical parameters of the robot of Fig. 1 have their corresponding values of similar orders to those listed in Table 1.

### III. STABILITY ON A MANIFOLD

Now it is possible to state the principal result of the paper in the following theorem:

**Theorem 1.** If the desired force \(\lambda_d > 0\) is chosen within \(2.5 \geq \lambda_d \geq 0.1\) [N], all entries of diagonal damping matrix \(C\) are of a similar order of \(\frac{1}{2} l_i \lambda_d\) \((l_i\) denote link length), and a positive diagonal matrix \(K\) is chosen appropriately so that all diagonal entries are of \(O(10^4)\), then the reference state \((q^0, 0)\) is stable on a manifold.

**Proof.** First it is important to note that \(\Delta \lambda\) can be expressed as

\[
\Delta \lambda = \left( J_s H^{-1} J_s^T \right)^{-1} \left[ J_s H^{-1} \left( \frac{1}{2} \dot{H} + S + C \right) \dot{q} + J_s^T K \Delta x \right] - J_s \ddot{q} = \frac{1}{2} \Delta x \dot{K} \Delta x \leq \delta_1^2
\]

which follows from multiplying Eq. (6) by \(J_s H^{-1}\) from the left and dividing the resultant equation by \(J_s H^{-1} J_s^T\). In this formulation, it should be noted from the constraint \(\dot{\Phi}(q) = 0\) that

\[
J_s \ddot{q} = -J_s \dot{q}.
\]

Since the scalar function \(E\) defined in Eq. (8) does not increase as \(t \to \infty\), it is possible to choose some positive value \(\delta_1\) such that the inequality

\[
E(t) \leq E(0) = \frac{1}{2} \dot{q}^T(0) H(q(0)) \dot{q}(0) + \frac{1}{2} \Delta x(0) K \Delta x(0) \leq \delta_1^2
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Parameter} & \text{Constant} & \text{Value} & \text{Unit} \\
\hline
\text{Link1} & \text{Length} & l_1 & 0.050 \text{ m} \\
& \text{Mass} & m_1 & 40.15 \times 10^{-3} \text{ kg} \\
& \text{Inertia moment} & I_1 & 0.870 \begin{bmatrix} 0 & 0 \\ 0 & 0.067 \end{bmatrix} \times 10^{-3} \text{ kgm}^2 \\
\text{Link2} & \text{Length} & l_2 & 0.080 \text{ m} \\
& \text{Mass} & m_2 & 64.24 \times 10^{-3} \text{ kg} \\
& \text{Inertia moment} & I_2 & 0.107 \begin{bmatrix} 0 & 0 \\ 0 & 3.480 \end{bmatrix} \times 10^{-3} \text{ kgm}^2 \\
\text{Link3} & \text{Length} & l_3 & 0.050 \text{ m} \\
& \text{Mass} & m_3 & 40.15 \times 10^{-3} \text{ kg} \\
& \text{Inertia moment} & I_3 & 0.067 \begin{bmatrix} 0 & 0 \\ 0 & 0.870 \end{bmatrix} \times 10^{-3} \text{ kgm}^2 \\
\text{Link4} & \text{Length} & l_4 & 0.030 \text{ m} \\
& \text{Mass} & m_4 & 24.09 \times 10^{-3} \text{ kg} \\
& \text{Inertia moment} & I_4 & 0.040 \begin{bmatrix} 0 & 0 \\ 0 & 0.201 \end{bmatrix} \times 10^{-3} \text{ kgm}^2 \\
\hline
\end{array}
\]
implies $|\Delta \lambda| < \lambda_m$ because the right hand side of Eq. (15) is linear and homogeneous in $\dot{q}$ and $\Delta x$, i.e., it is written by a function $\lambda(\dot{q}, \Delta x)$ with $\lambda(0, \Delta x)=0$ and $\lambda(\dot{q}, 0)=0$. This means that the pressing force $\lambda$ is always positive as far as the solution to the closed-loop Eq. (6) starting from $(q(0), \dot{q}(0))$ lies in $N_\epsilon(\delta_1) \cap N^\delta(r_0)$. Second, consider the quantity

$$V = E + \gamma \dot{q}^T H(q)P_{\phi}J_x^T K \Delta x$$  \hspace{1cm} (18)

where $\gamma > 0$ is a constant and determined later. Since $J_xP_{\phi}$

$$\dot{V} = E + \gamma \dot{q}^T \left( \frac{1}{2} H + S + C \right) \dot{q} + J_x^T K \Delta x - J_x^T \Delta \lambda \right)^T P_{\phi}J_x^T K \Delta x$$

$$+ \gamma \dot{q}^T \left( (H P_{\phi} J_x^T + H \dot{P}_{\phi} J_x^T + H P_{\phi} J_x^T K \Delta x) + H P_{\phi} J_x^T K \dot{\lambda} \right)$$

$$= E - \gamma \dot{\lambda} x^T (K J_x P_{\phi} J_x^T K \Delta x - \delta \dot{q}^T C P_{\phi} J_x^T K \Delta x$$

$$+ \gamma \dot{q}^T \left( \frac{1}{2} H P_{\phi} J_x^T + H \dot{P}_{\phi} J_x^T + H P_{\phi} J_x^T + S P_{\phi} J_x^T \right) K \Delta x$$

$$+ H P_{\phi} J_x^T K \dot{\lambda} \right).$$  \hspace{1cm} (19)

In the right hand side of this equation, the fourth term is a sum of quadratic forms of $\dot{q}$ because all variable elements in $H$, $P_{\phi}$, $J_x$, $\Delta x$ can appear as sinusoidal functions of components of $q$ and therefore all time derivatives $\dot{H}$, $\dot{\lambda}$, $\lambda$, $x$ and in addition $S(q, \dot{q})$ are homogeneous in $\dot{q}$, that is, they vanish when $\dot{q} = 0$. Thus, there exists a constant $\alpha_0$ such that

$$q^T \left( \frac{1}{2} H P_{\phi} J_x^T + H \dot{P}_{\phi} J_x^T + H P_{\phi} J_x^T + S P_{\phi} J_x^T \right) K \Delta x$$

$$+ H P_{\phi} J_x^T K \dot{\lambda} \right) \leq \frac{\alpha_0}{2} \dot{q}^T H(q) \dot{q}.$$  \hspace{1cm} (20)

Note that, since the maximum singular value of $J_x^T K$ over all $q$, that is, $\lambda_{\text{max}}(J_x^T K)$ over all $q$ is of order $O(1)$ or less than $O(1)$, $\alpha_0$ can be chosen as a value of $O(1)$. Since it follows that

$$\dot{V} \leq E + \gamma \dot{q}^T H(q)P_{\phi}J_x^T K \Delta x$$

$$\leq \frac{1}{2} \dot{q}^T C P_{\phi} C \dot{q} + \Delta \lambda^T K J_x P_{\phi} J_x^T K \Delta x$$

the remaining terms of the right hand side of Eq. (19) can be expressed as

$$\dot{E} - \gamma \Delta \lambda^T K J_x P_{\phi} J_x^T K \Delta x - \gamma \dot{q}^T C P_{\phi} C \dot{q} - \gamma \Delta \lambda^T (K J_x P_{\phi} J_x^T K) \Delta x.$$  \hspace{1cm} (21)

Since $P_{\phi}$ is a projection matrix, $CP_{\phi} C \leq C^2$.  \hspace{1cm} (23)

In order to simplify the argument, we choose $C = cI_1$ and $K = kI_2$ and denote by $h_{\lambda}$ the maximum eigenvalue of $H(q)$ over all $q$. Then, substituting Eqs. (20), (22), and (23) into Eq. (19) yields

$$\dot{V} \leq -\left( c - \frac{\gamma c^2}{2} - \frac{\gamma \alpha_0 h_{\lambda}}{2} \right) \dot{q}^2 - \frac{k^2}{2} \Delta \lambda^T P_{\phi} \Delta x.$$  \hspace{1cm} (24)

Since $c$ is chosen in the same order of $\frac{1}{2} \lambda_{\text{max}}$, the magnitude of $c$ must be in the range $0.004 \leq c \leq 0.1$. Then, by referring to Eq. (14), it is possible to rewrite Eq. (24) into the following:

$$\dot{V} \leq -\left( \frac{2c}{h_{\lambda}} - \frac{\gamma c^2}{h_{\lambda}} - \gamma \alpha_0 \right) \frac{h_{\lambda}}{2} \dot{q}^2 - (\gamma k \alpha_0) \frac{k}{2} \Delta \lambda^2.$$  \hspace{1cm} (25)

At this stage, it is important to evaluate the magnitudes of contents of two brackets ) in the right hand side of Eq. (25). Since $0.004 \leq c \leq 0.1$, $\alpha_0$ is of $O(1)$, and $h_{\lambda}$ is at most of $O(10^{-4})$ as easily predicted from Table 1, and $k$ is of $O(10^4)$, it is possible to set $\gamma = k$ and confirm that

$$\frac{2c}{h_{\lambda}} - \frac{\gamma c^2}{h_{\lambda}} - \gamma \alpha_0 = \alpha_0$$  \hspace{1cm} (26)

because $\sigma_0 < \sigma_1 < \frac{\sum_l l_i^2}{(0.16)^2}$. Hence, it follows from setting $\gamma = k$ and referring to Eqs. (25) and (26) that

$$\dot{V} \leq -\alpha \frac{\frac{h_{\lambda}}{2}}{k} \dot{q}^2 - \frac{k}{2} \Delta \lambda^2 \leq \alpha E.$$  \hspace{1cm} (27)

Next, by using the inequality $|ab| \leq |a/b| |b|^2 + 2b^2$, it is possible to obtain the following inequality:

$$V \leq E + \gamma \dot{q}^T H(q) P_{\phi} J_x^T K \Delta x$$

$$\leq E + \gamma \left( \frac{k}{8} q^T H^2(q) \dot{q}^2 + 2 \Delta \lambda^T J_x P_{\phi} J_x^T \Delta \lambda^2 \right)$$

$$\leq E + \left( \frac{k}{4} \frac{h_{\lambda}}{2} \right) \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{k}{2} \Delta \lambda^2.$$  \hspace{1cm} (28)

Since $\gamma = k$ and $k$ is of $(10^4)$, it follows from Eq. (28) that

$$V \leq E + \frac{1}{3} \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{k}{2} \Delta \lambda^2 \leq \frac{4}{3} E$$  \hspace{1cm} (29)

and similarly

$$V \geq E - \frac{1}{3} \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{k}{2} \Delta \lambda^2 \geq \frac{2}{3} E.$$  \hspace{1cm} (30)
Thus, it follows from inequalities of Eqs. (27), (29), and (30) that
\[
\dot{V}(t) \leq -\frac{3\alpha}{4} V, \quad E \leq \frac{3}{2} V
\]
which leads to
\[
E(t) \leq \frac{3V(0)}{2} e^{-\frac{3\alpha}{4} t} \leq 2E(0)e^{-\frac{3\alpha}{4} t}.
\]  
(32)

Now, for an arbitrarily given \( \varepsilon > 0 \), choose \( r_i = r_i/3 \) and \( \hat{\varepsilon}(\varepsilon) > 0 \) in the following way:
\[
\hat{\varepsilon}(\varepsilon) = \begin{cases} \frac{1}{\sqrt{2}} \varepsilon & \text{if } \varepsilon \geq \varepsilon \\ \frac{1}{\sqrt{2}} & \text{if } 0 < \varepsilon < \varepsilon \end{cases}
\]  
(33)
where \( \varepsilon \) is defined as
\[
\varepsilon = \min \left\{ \frac{r_0}{3}, \delta_1, \left( \frac{h_m}{h_L} \right)^{1/2} \alpha \frac{r_0}{8} \right\}.
\]  
(34)
To prove that for any \( (q(0), \dot{q}(0)) \in N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \cap N^8(\varepsilon) \) the solution with this initial condition remains in \( N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \cap N^8(\varepsilon) \), define
\[
\begin{align*}
\|q(t) - q(0)\|_\eta & = \left( \frac{h_M}{h_L} \right)^{1/2} \|q(t) - q(0)\|^2 \\
\|q(t) - q(0)\|_{\mu_H} & = \left( \frac{h_M}{h_L} \right)^{1/2} \|q(t) - q(0)\|^2 \\text{H} \left( H(q(t))(q(t) - q(0)) \right)^{1/2} \\
\text{H} & = \left( \frac{h_M}{h_L} \right)^{1/2} \|q(t)\|_{\mu_H}.
\end{align*}
\]  
(35)
Then, it follows that
\[
\frac{d}{dt} \|q(t) - q(0)\|_\eta = \frac{h_M}{h_L} \left( \frac{\hat{\varepsilon}(\varepsilon)^2}{2} \|q(t) - q(0)\|^2 \right)^{1/2} \\text{H} \left( H(q(t))(q(t) - q(0)) \right)^{1/2} \|q(t) - q(0)\|_{\mu_H}
\leq \left( \frac{h_M}{h_L} \right)^{1/2} \|q(t)\|_{\mu_H}.
\]  
(36)
Hence, it follows from Eqs. (36), (8), and (32) that
\[
\begin{align*}
\|q(t) - q(0)\|_{\mu_H} & \leq \|q(t) - q(0)\|_\eta \leq \int_0^t \left( \frac{h_M}{h_L} \right)^{1/2} \|\hat{q}^{\varepsilon}(\varepsilon)\|_{\mu_H} \, d\tau \\
& \leq \left( \frac{h_M}{h_L} \right)^{1/2} \int_0^t \left( \frac{h_M}{h_L} \right)^{1/2} \|\hat{q}^{\varepsilon}(\varepsilon)\|_{\mu_H} \, d\tau \\
& \leq \left( \frac{h_M}{h_L} \right)^{1/2} \int_0^t \left( \frac{h_M}{h_L} \right)^{1/2} \|\hat{q}^{\varepsilon}(\varepsilon)\|_{\mu_H} \, d\tau \leq \left( \frac{h_M}{h_L} \right)^{1/2} \|\hat{q}^{\varepsilon}(\varepsilon)\|_{\mu_H} \left( 2E(0)^{1/2} \frac{8}{3\alpha} \right) .
\end{align*}
\]  
(37)

In the case that \( \varepsilon \) is given as \( \varepsilon \geq \varepsilon \), \( \hat{\varepsilon}(\varepsilon) \) is set as
\( \hat{\varepsilon}(\varepsilon) = \frac{h_M}{h_L} \|\hat{q}^{\varepsilon}(\varepsilon)\|_{\mu_H} \) and therefore substitution of the initial condition \( E(0) \leq \delta(\varepsilon)^2 \) into Eq. (37) leads to
\[
\|q(t) - q(0)\|_{\mu_H} \leq \left( \frac{h_M}{h_L} \right)^{1/2} \left( 2 \sqrt{\frac{\varepsilon}{\sqrt{2}}} \right)^{1/2} \frac{8}{3\alpha} .
\]  
(38)
in which the last inequality follows from Eq. (34). In the case that \( 0 < \varepsilon < \varepsilon \), \( \hat{\varepsilon}(\varepsilon) = \varepsilon / \sqrt{2} < \varepsilon / \sqrt{2} \) and hence, according to Eq. (38), it follows that
\[
\|q(t) - q(0)\|_{\mu_H} \leq \frac{r_0}{3}.
\]  
(39)
Thus, if \( (q(0), \dot{q}(0)) \in N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \cap N^8(\varepsilon) \), then, by noting from the definition of \( N^8(\varepsilon) \) that
\[
\|q(0) - q^0\|_{\mu_H} < \frac{r_0}{3},
\]  
(40)
it follows from Eqs. (17), (38), (39), and (40) that
\[
\begin{align*}
\|q(t) - q^0\|^2_{\mu_H} + \|q(t)\|^2_{\mu_H} & \leq \|q(t) - q(0)\|^2_{\mu_H} + \|q(t)\|^2_{\mu_H} \\
& \leq \|q(t) - q(0)\|^2_{\mu_H} + \|q(0) - q^0\|^2_{\mu_H} + \|\dot{q}(t)\|^2_{\mu_H} \\
& \leq \frac{r_0}{3} + \frac{r_0}{3} + \hat{\varepsilon}(\varepsilon) \leq \frac{2r_0}{3} + \frac{\varepsilon}{\sqrt{2}} \leq r_0.
\end{align*}
\]  
(41)
where the last inequality follows from the first part of Eq. (34). Since it is evident that \( E(t) \leq \delta(\varepsilon)^2 \leq \varepsilon^2 \) implies that \( E(t) \leq \varepsilon^2 \) and therefore \( (q(t), \dot{q}(t)) \in N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \) it is concluded that \( (q(t), \dot{q}(t)) \) remains in \( N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \cap N^8(\varepsilon) \). Thus, the proof is completed.

**Theorem 2.** Under the same conditions as in Theorem 1, the neighborhood \( N_\varepsilon(\hat{\varepsilon}(\varepsilon)) \cap N^8(\varepsilon) \) with \( \hat{\varepsilon} = \varepsilon \) and \( r_i = r_i/3 \) of the reference state \( (q^0, 0) \in M_1 \cap N^8(r_i) \) is asymptotically transferable to the manifold \( M_1 \cap N^8(r_i) \), where \( \varepsilon \) is defined in Eq. (34).

The proof is straightforward from Eq. (32) which shows that \( E(t) \to 0 \) and thereby \( \dot{x}(t) \to 0 \) and \( \dot{q}(t) \to 0 \) as \( t \to \infty \). At the same time, this proves from Eq. (6) that \( \Delta x(t) \to 0 \) as \( t \to \infty \).

**IV. TRANSFERABILITY TO A STABLE EQUILIBRIUM MANIFOLD**

In this section, the case that physical parameters appearing in the gravity term \( g(q) \) are uncertain is treated. First, note that the gravity term is described as
\[ g(q) = g \left( \begin{array}{l} m_s s_2 \cos q_2 + m_1 [l_2 \cos q_2 + s_3 \cos (q_2 + q_4)] + m_s [l_2 \cos q_2 + l_3 \cos (q_2 + q_3) + s_4 \cos (q_2 + q_3 + q_4)] \\ m_s s_2 \cos (q_2 + q_3) + m_1 [l_3 \cos q_2 + q_4] + s_4 \cos (q_2 + q_3 + q_4)] \\ m_s s_4 \cos (q_2 + q_3 + q_4) \end{array} \right) \]  

(42)

where \( s_i \) denotes the length from the \( i \)th joint center to the mass center of the \( i \)th link for \( i = 2, 3, 4 \). Then, by introducing the regressor \( Z(q) \) and the vector of uncertain parameters \( \Theta [12-14] \) as

\[
\begin{bmatrix}
Z(q) = g \\
\Theta = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
m_s s_2 + m_n l_3 + m_n l_3 \\
m_n s_2 + m_n l_3 \\
m_n s_4
\end{bmatrix}
\end{bmatrix}
\]

(43)

the gravity term can be expressed as

\[ g(q) = Z(q) \Theta. \]  

(44)

The uncertain parameters \( \Theta \) can be approximated by an estimate \( \hat{\Theta}(t) \) which is calculated by the following parameter update law:

\[
\hat{\Theta}(t) = \hat{\Theta}(0) - \int_0^t \Gamma^{-1} Z^T(q(\tau)) \\
\cdot \hat{\dot{q}}(\tau) + \alpha P_0(q(\tau)) J^T_k(q(\tau)) K \Delta \Theta(\tau) d \tau
\]

(45)

where \( \alpha \) is an appropriate positive constant and \( \Gamma \) is a \( 3 \times 3 \) positive constant diagonal matrix. Then, the control signal can be composed of the following expression

\[ u = Z(q) \hat{\Theta} - C \dot{q} - J^T_k K \Delta x - J^T_k \Delta \lambda_d \]  

(46)

where the gravity compensation \( g(q) \) can be replaced with the computable one \( Z(q) \hat{\Theta} \) and remaining other terms are the same as those defined in Eq. (5). Substitution of Eq. (46) into Eq. (1) yields the closed-loop dynamics

\[
H(q) \ddot{q} + \frac{1}{2} H(q) q + S(q, \dot{q}) + C \dot{q} \\
+ Z(q) \Delta \Theta + J^T_k K \Delta x + J^T_k \Delta \lambda = 0
\]

(47)

where \( \Delta \Theta = \Theta - \hat{\Theta} \). It is easy to see that an inner product of Eq. (47) with \( \ddot{q} \) is reduced to the following equality

\[
\frac{d}{dt} E_0 = \dot{q}^T C \dot{q} + \alpha \Delta x^T K J P_0 Z(q) \Delta \Theta
\]

(48)

where

\[
E_0 = \frac{1}{2} \dot{q}^T H(q) \dot{q} + \frac{1}{2} \Delta x^T K \Delta x + \frac{1}{2} \Delta \Theta^T \Gamma \Delta \Theta.
\]

(49)

Next it is necessary to introduce another scalar quantity of the form

\[
R_0 = \frac{1}{2} \ddot{q}^T H(q) P_0 \{ J^T_k K \Delta x + Z(q) \Delta \Theta \}.
\]

(50)

The time-derivative of this quantity is expressed as

\[
\dot{R}_0 = \frac{1}{2} \ddot{q}^T H(q) P_0 \{ J^T_k K \Delta x + Z(q) \Delta \Theta \} \\
+ \frac{1}{2} \dot{q}^T \{ (H P_0 + H \dot{P}_0) (J^T_k K \Delta x + Z(q) \Delta \Theta) \\
+ H P_0 (J^T_k K \Delta x + J^T_k K x + \dot{Z}(q) \Delta \Theta - \dot{Z} \hat{\Theta}) \}.
\]

(51)

Since the second term of the right hand side is quadratic in \( \dot{q} \) and the order of coefficients is of \( \hbar \), the maximum of \( \text{spectral radius of } H(q) \) over all \( q \) can be reduced to

\[
R_0 \leq - \frac{1}{2} [ \frac{1}{2} (C \dot{q} + Z \Delta \Theta + J^T_k K \Delta x) \] \\
\times P_0 (J^T_k K \Delta x + Z \Delta \Theta) + O(\| \dot{q} \|^2) (\rho_m(J^T_k K) \Delta x) \\
+ \rho_m(Z) \| \Delta \Theta \| + \rho_m(J (J^T_k K x + Z \Delta \Theta)) \]

(52)

where \( \rho_m(A) \) denotes the maximum of spectral radius of matrix \( A \) over all \( q \) if \( A \) is dependent on \( q \). Since \( P_0, J^T_k = 0, \{H/2 + S\} \dot{q} \) is also quadratic in \( \dot{q} \), and it follows that

\[
\frac{1}{2} \ddot{q}^T C \dot{q} + \alpha \Delta x^T K J P_0 Z(q) \Delta \Theta \\
\leq \frac{1}{2} (C \dot{q} + Z \Delta \Theta + J^T_k K \Delta x) \] \\
\times P_0 (J^T_k K \Delta x + Z \Delta \Theta) - \frac{1}{2} \| \Delta \Theta^T \| Z P_0 Z \Delta \Theta \\
+ \Delta x^T K J \rho_m(J^T_k K \Delta x) - \Delta \Theta^T \| Z \Delta \Theta \\
\leq \frac{1}{2} \ddot{q}^T C \dot{q} + \frac{1}{4} \| \Delta \Theta^T \| Z P_0 Z \Delta \Theta + \Delta x^T K J \rho_m(J^T_k K \Delta x)
\]

(53)
Eq. (52) is further reduced to the form
\[ \ddot{R}_b \leq \frac{1}{2} \dot{q}^T C \dot{q} - \frac{1}{4} \{\Delta \Theta^T Z \dot{P}_q Z \Delta \Theta + \Delta x^T K J_x \dot{J}_x^T K \Delta x\} \]
\[ - \Delta x^T Z \dot{P}_q \dot{J}_x^T K \Delta x + \eta (\| \Delta x \| \| \Delta \Theta \|) \theta (\| \dot{q} \| \theta) \]
(54)
where the quadratic term in \( \dot{q} \) is put into \( \eta \theta (\| \dot{q} \| \theta) \) and
\[ \eta = \rho_m (J_x^T K J_x + Z \Gamma^{-1} Z) \]
\[ + \rho_m (J_x^T K) \| \Delta x \| + \rho_m (Z) \| \Delta \Theta \|. \]
(55)
Thus, it follows straightforwardly from Eqs. (48) and (54) that
\[ \frac{d}{dt} (E_0 + \alpha R_0) \leq -\dot{q}^T \left( C - \frac{\alpha}{2} \dot{Z}^T \right) \dot{q} - \frac{\alpha}{4} \{\Delta \Theta^T Z \dot{P}_q Z \Delta \Theta \}
\[ + \Delta x^T K J_x \dot{J}_x^T K \Delta x\} + \alpha \eta \theta (\| \dot{q} \| \theta). \]
(56)
Similarly to derivation of Eq. (15) it follows from Eq. (47) that
\[ \Delta \lambda = \left[ J_x H^{-1} J_q^{-1}\right]^{-1} \left[ -\dot{J}_q \dot{q} + J_q H^{-1}\right. \]
\[ \left. \{ \frac{1}{2} \dot{H} + \dot{S} \} \dot{q} + (C \dot{q} + \dot{Z} \Delta \Theta + J_q^T K \Delta x) \} \right]. \]
(57)
This means that, if all initial values of \( \dot{q}, \Delta \Theta \), and \( \Delta x \) at \( t = 0 \) are sufficiently small, the scalar-valued function \( E_0 + \alpha R_0 \) is positive definite in \( \dot{q}, \Delta \Theta \), and \( \Delta x \), and its time derivative \( \frac{d}{dt} (E_0 + \alpha R_0) \) is non-positive definite, then all magnitudes of \( \dot{q}(t), \Delta \Theta(t) \) and \( \Delta x(t) \) remain small for all \( t > 0 \) and thereby it will be possible to show that \( \Delta \lambda(t) = \lambda(t) - \lambda_\sigma > -\lambda_\sigma \) for all \( t > 0 \). Hence, it is necessary to show firstly that \( E_0 + \alpha R_0 \) with \( \alpha = \alpha_\sigma \) becomes positive definite. In order to simplify the argument without loosing the generality, let us assume that all gain matrices \( C, K \), and \( \Gamma \) are \( cl_\alpha \), \( kl_\alpha \) and \( \gamma_\alpha \) respectively. Then, by choosing \( \alpha = 1/\epsilon \) it is easy to check from Table 1 that the following inequalities are satisfied provided that \( \gamma \) is at least of \( O(10^\epsilon) \):
\[ 0 \leq 8 \alpha h_{m} K J_x(q) J_x^T(q) K \leq \alpha^{-1} K, \]
\[ 0 \leq 8 \alpha h_{m} Z^T(q) Z(q) \leq \alpha^{-1} \Gamma \]
(58)
This helps to verify that
\[ R_0 = \frac{1}{2} \dot{q}^T H(q) P_q \{ J_x^T K \Delta x + Z(q) \Delta \Theta \} \]
\[ \geq -\frac{1}{8\alpha} \dot{q}^T H(q) \dot{q} - 2 \alpha h_{m} \Delta \Theta^T Z J_x \dot{P}_q Z \Delta \Theta \]
\[ - \frac{1}{8\alpha} \dot{q}^T H(q) \dot{q} - 2 \alpha h_{m} \Delta \Theta^T Z \dot{P}_q Z \Delta \Theta \]
\[ \geq -\frac{1}{4\alpha} \{ \dot{q}^T H(q) \dot{q} + \Delta \Theta^T Z \Delta \Theta + \Delta \Theta^T \Gamma \Delta \Theta \} \]
(59)
Hence, it follows that
\[ E_0 + \alpha R_0 \geq E_0 - \frac{1}{2} E_0 = \frac{1}{2} E_0 \]
and similarly to this it follows that
\[ E_0 + \alpha R_0 \leq \frac{3}{2} E_0. \]
(60)
Next it is necessary to verify the non-positive definiteness of time-derivative of \( E_0 + \alpha R_0 \). If the damping constant \( \epsilon \) is chosen as \( \epsilon = 0.02 \) as in Table 2 in the next section, then the choice \( \alpha = 1/\epsilon \) leads to \( C - \alpha \epsilon \Delta \Theta^T Z \geq C/2 \) and the magnitude of \( \alpha \epsilon \Delta \Theta \) is by far less than \( \epsilon/4 \) provided \( \gamma \) is chosen as in Table 2 and \( \Delta \Theta(0) \) and \( \Delta x(0) \) are chosen sufficiently small. Thus, for small \( t > 0 \) it follows from Eq. (56) that
\[ \frac{d}{dt} (E_0 + \alpha R_0) \]
\[ \leq -\frac{c}{2} \| \dot{q} \|^2 - \frac{\alpha}{4} \{\Delta \Theta^T Z \dot{P}_q Z \Delta \Theta + \Delta \Theta^T K J_x \dot{J}_x^T K \Delta x\}
\[ + \alpha \eta \theta (\| \dot{q} \| \theta) \]
\[ \leq -\frac{1}{4} \epsilon \| \dot{q} \|^2 \]
\[ + \alpha \{\Delta \Theta^T Z \dot{P}_q Z \Delta \Theta + \Delta \Theta^T K J_x \dot{J}_x^T K \Delta x\} \]
(61)
This shows that for small \( t > 0 \) the positive definite scalar quantity \( E_0 + \alpha R_0 \) does not increase with increasing \( t \) and therefore the magnitudes of \( \dot{q}(t), \Delta \Theta(t) \), and \( \Delta x(t) \) are bounded from some constants respectively depending on magnitudes of chosen \( \dot{q}(0), \Delta \Theta(0) \), and \( \Delta x(0) \). Further, integration of the quadratic term of the content of bracket [ ] in the right hand side of Eq. (62) over \( t \in [0, \infty) \) must be bounded because \( E_0 + \alpha R_0 \geq 0 \). This shows that \( \dot{q}(t) \rightarrow 0 \), \( P_q \Delta \Theta(t) \rightarrow 0 \) and \( \Delta x(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Thus, it can be expected that
\[ \dot{J}_x^T J_x Z \Delta \Theta(t) \rightarrow -J_x^T J_x \Delta \lambda(t) \rightarrow 0 \] as \( t \rightarrow \infty \).
(63)
However, it seems quite difficult to prove that the position vector \( q(t) \) remains in the vicinity of \( q(0) \), that is, in \( (q(t), \dot{q}(t)) \in M_q \cap N^\epsilon(r_0) \). According to computer simulations as shown in the next section, solution trajectories of the closed-loop dynamics of Eq. (50) converge to the lower-dimensional manifold \( M_l \) even if the initial position \( x(0) \) on the plane \( z = 0 \) is not so close to the target position \( x_c \). Therefore, we postulate the following.

<table>
<thead>
<tr>
<th>Table 2. Choice for gains.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant [Unit]</td>
</tr>
<tr>
<td>( k ) [N/m]</td>
</tr>
<tr>
<td>( c ) [smN]</td>
</tr>
<tr>
<td>( \alpha ) [1/smN]</td>
</tr>
<tr>
<td>( \gamma )</td>
</tr>
<tr>
<td>( \epsilon_0 )</td>
</tr>
</tbody>
</table>
Postulate. Under the choices of kinematic parameters of the robot and feedback gains $\Gamma = \gamma, K = kl, C = cl,$ and $\alpha = 1/c$ with their values given in Table 2, then there exist positive numbers $\delta_1 > 0$ and $r_1 > 0$ such that a neighborhood $N_6(\delta_1) \cap N_8(r_1)$ of the reference state $(q^0, 0) \in M_1$ is asymptotically transferable to $M_1 \cap N_6(\delta_0) \cap N_8(r_0)$, provided that the initial guesses $\Theta(0)$ of the uncertain parameters $\Theta$ satisfies $||{\Delta \Theta}(0)|| \leq \epsilon_0 ||\Theta||$ with sufficiently small $\epsilon_0 > 0$ (say, $0 \leq \epsilon_0 \leq 1/5$ as in Table 2).

V. COMPUTER SIMULATION RESULTS

In order to show the validity of Theorems 1 and 2, computer simulations based on physical parameters of the robot and feedback gain parameters as shown in Tables 1 and 2 and initial and desired conditions as shown in Table 3 are carried out. Numerical solutions of the differential equations under the geometric constraint are obtained by using Baumgarte’s method [15], in which a coefficient $\gamma_f$ for highly over-damped second order differential equations corresponding to the constraint $z = 0$, that is,

$$\dot{Q} + \gamma_f \dot{Q} + (\gamma_f^2/4)Q = 0$$

(64)

is chosen as $\gamma_f = 1000$. In the first simulation when the control of Eq. (5) is used, a better choice for the damping gain $C = cl$ with $c = 0.02$ is chosen firstly after several trials of simulation by changing values of $c$ together with $k$ for $K = kl$. Then, for a fixed $c = 0.02$ [msN] transient responses of $x, y,$ and constraint force $\lambda$ are shown in Figs. 4 to 6 with changing the position feedback gain $k$. According to the figures, the best choice for $k$ is around $k = 15$ [N/m].

Table 3. Initial conditions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>$x_0$</td>
<td>0</td>
<td>[m]</td>
</tr>
<tr>
<td></td>
<td>$y_0$</td>
<td>$9.12 \times 10^{-2}$</td>
<td>[m]</td>
</tr>
<tr>
<td>Force</td>
<td>$\lambda_0$</td>
<td>0.20</td>
<td>[N]</td>
</tr>
</tbody>
</table>

It is quite interesting to note that in transient behaviours of robot motion the projected value of $q$ to the subspace orthogonal to the image space of $J^T(q) \cdot J(q)$ is almost fixed as shown in Fig. 7, where $\Delta q$ is defined as

$$\Delta q = \{I - J^T(q) \cdot J(q)\} \cdot q.$$  

(65)
This shows that the control signal defined by Eq. (5) does little affect joint motion of the robot irrelevant to movement of the endpoint of the robot in the xy-plane and control of the constraint force, which looks like a human-like skilled motion. In order to examine the validity of inequality (14), transient behaviours of two eigenvalues of the $2 \times 2$ matrix $J_x P_y J_y^T$ are computed as shown in Fig. 8. This figure shows the existence of constants $\sigma_1$ and $\sigma_0$ satisfying Eq. (14).

In the case that the gravity term is uncertain, computer simulation based on the control signal of Eq. (46) is conducted, which results in Figs. 9 to 11, where $\gamma$, $\alpha$, and $\varepsilon_0$ are set as in Table 2 and initial and desired conditions are given in Table 3. As seen from Figs. 9 and 10, transient behaviour of position variable $\mathbf{x}$ is smooth and converges to the desired one but the constraint force $\lambda$ is rather oscillatory though it eventually converges to a constant value. It is interesting to notice from Fig. 11 that the constraint torque term $J_y J_x \lambda$ minus the effect of the gravity force estimation error relative to the image subspace of $J_y J_x$ converges to $J_y J_x \lambda_d$, that is,

$$J_y \Delta \lambda(t) - J_y J_x \Delta \Theta(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (66)$$

**VI. MOMENTUM FEEDBACK BY USING A FORCE SENSOR**

In order to control the constraint force $\lambda$ faithfully to the desired and specified value $\lambda_d$, it is necessary to use a force sensor that can detect the magnitude $\lambda(t)$ of the constraint force arising in the direction $z$ normal to the xy-plane. Then, it is possible to compute the momentum error in real-time in such a manner as

$$\Delta \Lambda(t) = \int_0^t (\lambda(t) - \lambda_d) \, d\tau. \quad (67)$$

It is possible to consider a control signal based on all measured data and described as

$$u = Z(q) \tilde{\Theta} - C \dot{q} - P_y J_y^T K \Delta \mathbf{x} - J_y^T \lambda_d + \beta J_y^T \Delta \Lambda \quad (68)$$

where the estimator $\tilde{\Theta}$ for uncertain gravity parameters is updated in the following way

$$\tilde{\Theta}(t) = \tilde{\Theta}(0) - \Gamma^{-1} \int_0^t Z(q) \dot{\Theta}(t) \, d\tau \cdot \{\dot{q}(t) + \alpha P_y (q(t)) J_y^T (q(t)) K \Delta \mathbf{x}(t)\} \, d\tau. \quad (69)$$

![Fig. 8. Eigenvalues of the $2 \times 2$ matrix $J_x P_y J_y^T$ for various values of $k$.](image)

![Fig. 9. Transient response of the x-component of robot endpoint position when the control of Eq. (49) is exerted.](image)

![Fig. 10. Transient response of the constraint force $\lambda$, which converges to some fixed value $\lambda_\infty$ different from the desired value $\lambda_d$.](image)

![Fig. 11. Transient responses of four components of torque vector $\tau = J_x J_y Z \Delta \Theta - J_y \lambda$, which converges to zero as $t \to \infty$.](image)
The closed-loop equation when Eq. (68) is substituted into Eq. (1) is written as

\[
H(q) \ddot{q} + \left( \frac{1}{2} \dot{H}(q) + S(q, \dot{q}) + C \right) \dot{q} + Z(q) \Delta \Theta \\
+ P_q J_q^T K \Delta x - J_q^T \left[ \Delta \lambda + \beta \Delta \Lambda \right] = 0. 
\] (70)

It is then possible to show that

\[
\frac{d}{dt} E \leq -\frac{1}{2} \dot{q}^T H(q) \dot{q} - \frac{1}{2} \Delta \Theta^T \Gamma \Delta \Theta + \frac{1}{2} + \frac{\alpha c}{2} \Delta x^T K \Delta x \\
+ \alpha \dot{q}^T H(q) P_q J_q^T K \Delta x
\] (71)

and it is assumed that \( C = c l_f \). If all constant gains \( c, \alpha, \) and \( \Gamma = \gamma f \) are chosen as in Table 2, it is possible to show that \( E \geq 0 \) and \( E \leq 0 \), which means that \( E \) is bounded and non-increasing with increasing \( t \). Then, it is possible to verify that, for adequately small \(|q(0) - q_f|\), \( |\dot{q}(0)|\), \( ||\Delta x(0)||\), \( ||\Delta \Theta(0)||\), all \( ||\dot{q}(t)||, ||\Delta x(t)||, \) and \( ||\Delta \Theta(t)||\) remain small. Then, it follows from Eq. (70) that both \( \Delta \lambda \) and \( \dot{q} \) become uniformly bounded and hence both \( \dot{q}(t) \) and \( x(t) \) are uniformly continuous. Since \( L^1(0, \infty) \)-norms of \( \dot{q}(t) \) and \( \Delta x(t) \) are bounded from Eq. (71), the uniform continuity of \( \dot{q}(t) \) and \( \Delta x(t) \) implies that \( \dot{q}(t) \rightarrow 0 \) and \( \Delta x(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Then, by using a similar argument to that given in section 3.3 of the text book [11] it is possible to show that \( \Delta \Lambda \) tends to a constant value as \( t \rightarrow \infty \) and thereby \( \Delta \Theta(t) \) converges to zero as \( t \rightarrow \infty \). Thus, the following result is obtained:

**Theorem 3.** Under the same assumptions as in Postulate and an appropriate choice of constant \( \beta > 0 \), it follows that \( \ddot{q}(t) \rightarrow 0, \Delta x(t) \rightarrow 0, \Delta \Theta(t) \rightarrow 0, \Delta \lambda(t) \rightarrow 0 \) as \( t \rightarrow \infty \) provided that \( ||\Delta x(0)||, ||\dot{q}(0)||, ||\Delta \Theta(0)||, \) and \( ||q(0) - q_f|| \) are chosen adequately and sufficiently small.

It can be postulated that in this case the position vector \( q(t) \) itself remains in a neighborhood of \( M_{c} \cap N_r(\bar{r}_0) \) with some \( \delta_1 > 0 \) and \( r_1 > 0 \). However, it seems also difficult to prove this in a mathematically rigorous way.

**VII. CONCLUSION**

A natural way of redundancy resolution for robot tasks is proposed on the basis of a new concept “stability on a manifold” and a sensory feedback by using an illustrative example of handwriting robots. Another examples of redundancy resolution based on “stability on a manifold” have been presented in recent papers [7-9] that treat problems of stable grasp and object manipulation by a pair of multi-DOF fingers with rolling contacts. Simulation results verify the validity of the proposed method and show that a human-like robot motion can be realized in the case of handwriting robot tasks without introducing any extra performance index.

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