AN LMI APPROACH TO ROBUST $H_\infty$ CONTROL FOR
UNCERTAIN CONTINUOUS-TIME SYSTEMS

Shih-Wei Kau, Yung-Sheng Liu, Ching-Hsiang Lee, Lin Hong, and Chun-Hsiung Fang

ABSTRACT

This paper investigates the problems of robust $H_\infty$ control for uncertain continuous-time systems with time-varying, norm-bounded uncertainties in all system matrices. Necessary and sufficient conditions for the above problems are proposed. All conditions are represented in the form of linear matrix inequalities (LMIs). The robust $H_\infty$ controller can be easily designed from the solutions of the LMI conditions. Unlike earlier works, the proposed method does not involve any parameter tuning. Thus the robust $H_\infty$ optimization control problem, which has not been discussed in earlier reports, can be dealt with using this newly proposed method.

KeyWords: Uncertain systems, linear matrix inequality, robust $H_\infty$ control, norm-bounded uncertainty.

I. INTRODUCTION

Robust control problems for uncertain systems have been extensively studied over the past 10 years due to their important applications in practical control systems [1-3]. The design objective of robust control is to ensure system stability as well as to obtain a certain level of performance subject to uncertainties. Robust $H_\infty$ control is a popular approach to solving this problem [4-11]. In [9-11], the uncertainties are assumed to be norm-bounded and to only appear in some of the system matrices. Gu recently extended the results in [9-11] to the case of uncertainties in all system matrices [12]. He reduced the robust $H_\infty$ control problem of uncertain systems to the one-parameter standard $H_\infty$ control problem of uncertainty-free systems. Although the result in [12] is quite neat, it involves the extra work of tuning a scalar parameter. Such parameter tuning makes the optimization of $H_\infty$ performance in uncertain systems quite difficult (see Remark 2 in [12]). A similar result was recently reported in [5], in which the uncertainty is extended to a linear fraction representation, but tuning is still required.

In this paper, a new LMI-based approach is proposed to overcome the above-mentioned drawback. Firstly, a necessary and sufficient LMI condition is given for the robust $H_\infty$ control analysis of uncertain systems. Based on the result, an LMI condition for robust $H_\infty$ control design is proposed. Using the solution of the LMI conditions, one can easily synthesize a robust $H_\infty$ controller for uncertain systems. Under these conditions, the optimal robust controller that minimizes $H_\infty$ performance can also be obtained.

The notation used in this paper is fairly standard. $M < 0$ means that the matrix $M$ is negative definite. $M^T$ represents a transpose of matrix $M$. $I_q$ denotes the $q \times q$ identity matrix. $||T_{w\to y}||_\infty$ stands for the $H_\infty$ norm of transfer function from $w$ to $y$. The symbol ■ stands for the end of a proof.

The paper is organized as follows. Section 2 addresses the problem formulation and then gives some preliminaries needed for development. The main results are presented in section 3. A numerical example is given in section 4 to demonstrate the proposed idea. Section 5 concludes the paper.
II. PROBLEM FORMULATION AND PRELIMINARIES

Consider an uncertain continuous-time system described by
\[
\dot{x}(t) = A_{\Delta}x(t) + B_{nu}w(t) + B_{na}u(t),
\]
\[
y(t) = C_{\Delta}x(t) + D_{na}w(t) + D_{na}u(t),
\]
where \(x(t) \in \mathbb{R}^n\) is the state vector, \(w(t) \in \mathbb{R}^m\) the exogenous input, \(u(t) \in \mathbb{R}^n\) the control input, and \(y(t) \in \mathbb{R}^q\) the controlled output. The system matrices \(A_{\Delta}, B_{nu}, B_{na}, C_{\Delta}, D_{na}\) and \(D_{na}\) are described by
\[
\begin{bmatrix}
A_{\Delta} & B_{nu} & B_{na} \\
C_{\Delta} & D_{na} & D_{na}
\end{bmatrix} \preceq \begin{bmatrix}
A & B_w & B_y \\
C & D_w & D_y
\end{bmatrix} + \begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} \Delta(t) [J_1 J_2 J_3],
\]
where \(A \in \mathbb{R}^{n \times n}, B_w \in \mathbb{R}^{n \times m}, B_y \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{q \times n}, D_w \in \mathbb{R}^{q \times m}\), and \(D_y \in \mathbb{R}^{q \times m}\) are constant matrices representing the nominal system. The constant matrices \(H_1 \in \mathbb{R}^{n \times q}, H_2 \in \mathbb{R}^{n \times q}, J_1 \in \mathbb{R}^{q \times q}, J_2 \in \mathbb{R}^{q \times q}\), and \(J_3 \in \mathbb{R}^{q \times q}\) provide the structural information of uncertainties. \(\Delta(t) \in \mathbb{R}^{n \times q}\) is the norm-bounded uncertain matrix satisfying \(\Delta(t) \Delta(t)^T \leq I_q \forall t\).

The goal of this paper is to design a state feedback gain \(K\) such that when \(u(t) = Kx(t)\), the closed-loop system in (1) is quadratically stable with \(||T_{wu}||_\infty < \gamma\) if and only if there exists a matrix \(P > 0\) and a scalar \(\epsilon > 0\) such that
\[
PA^T + AP + \epsilon H_1 H_1^T + B_w C P + \epsilon H_1 H_2^T + PJ_1^T < 0,
\]
\[
B_w^T - \gamma^2 I_{m_w} D_w^T J_2^T - I_q + \epsilon H_2 H_2^T 0
\]
\[
J_1 P J_2 0 - \epsilon I_q
\]
for all possible uncertainties.

The inequality (3) in Lemma 1 is not solvable since it contains the unknown uncertainty \(\Delta(t)\). To solve (3), the author of [12] transformed it into a new solvable matrix inequality condition without involving \(\Delta(t)\). However, an extra scalar parameter \(\lambda\) was introduced. The parameter \(\lambda\) has to be guessed in advance and then tuned until the inequality is feasible. For design purposes, the new condition is reformulated as a one-parameter (\(\lambda\)) standard \(H_{\infty}\) problem of an uncertainty-free system, where \(\lambda\) still must to be guessed in advance and then tuned for the purpose of controller design.

In this paper, our condition is expressed in LMI directly. It does not require to guess any parameter in advance. Thus, tuning is not necessary. This is very useful for optimal \(H_{\infty}\) control. The following lemmas are necessary to obtain our main results.

**Lemma 2 [5].** Given appropriate dimensional matrices \(X\) and \(Y\) and a symmetric matrix \(Z\),
\[
Z + X\Delta Y + Y^T \Delta^T X^T < 0
\]
for all \(\Delta\) satisfying \(\Delta \Delta^T \leq I\) if and only if there exists a scalar \(\epsilon > 0\) such that
\[
Z + \epsilon XX^T + \epsilon Y^TY < 0.
\]

### III. MAIN RESULTS

In this section, the robust \(H_{\infty}\) control of system (1) is investigated. Two necessary and sufficient LMI-based conditions for robust \(H_{\infty}\) analysis and design are derived.

#### 3.1 Robust \(H_{\infty}\) analysis

**Theorem 1.** The unforced uncertain continuous-time system (1) is quadratically stable with \(||T_{wu}||_\infty < \gamma\) if and only if there exist a matrix \(P > 0\) and a scalar \(\epsilon > 0\) such that
\[
PA^T + AP + \epsilon H_1 H_1^T + B_w C P + \epsilon H_1 H_2^T + PJ_1^T < 0,
\]
\[
B_w^T - \gamma^2 I_{m_w} D_w^T J_2^T - I_q + \epsilon H_2 H_2^T 0
\]
\[
J_1 P J_2 0 - \epsilon I_q
\]
for all possible uncertainties.

The inequality (3) in Lemma 1 is not solvable since it contains the unknown uncertainty \(\Delta(t)\). To solve (3), the author of [12] transformed it into a new solvable matrix inequality condition without involving \(\Delta(t)\). However, an extra scalar parameter \(\lambda\) was introduced. The parameter \(\lambda\) has to be guessed in advance and then tuned until the inequality is feasible. For design purposes, the new condition is reformulated as a one-parameter (\(\lambda\)) standard \(H_{\infty}\) prob-

**Proof.** For simplicity, we will simply use \(\Delta\) instead of \(\Delta(t)\).

Substitution of (2) into (3) yields
\[
\begin{bmatrix}
PA^T + AP + \epsilon H_1 H_1^T & B_w & PC^T + \epsilon H_1 H_2^T & PJ_1^T \\
B_w^T & -\gamma^2 I_{m_w} & D_w^T & J_2^T \\
CP + \epsilon H_2 H_1^T & D_w & -I_q + \epsilon H_2 H_2^T & 0 \\
J_1 P & J_2 & 0 & -\epsilon I_q
\end{bmatrix} < 0.
\]

We can rewrite (5) in a more compact expression:
\[
\bar{A} + \bar{H} \Delta \bar{J} + \bar{J}^T \Delta^T \bar{H}^T < 0,
\]
where
\[
\bar{A} = \begin{bmatrix}
PA^T + AP & B_w & PC^T \\
B_w^T & -\gamma I_{m_w} & D_w^T \\
CP & D_w & -I_q
\end{bmatrix},
\]
\[ \bar{H} = \begin{bmatrix} H_1 \\ 0 \\ H_2 \end{bmatrix}, \quad \bar{J} = [J, P, J_z, 0]. \quad (7) \]

According to Lemma 2, the inequality (6) holds for all \( \Delta \) satisfying \( \Delta^T \Delta \leq I \), if and only if there exists a scalar \( \varepsilon > 0 \) such that
\[ \bar{A} + \varepsilon \bar{H}^T \bar{H} + \varepsilon^{-1} \bar{J}^T \bar{J} < 0. \quad (8) \]

Substituting (7) into (8), we have
\[
\begin{bmatrix}
  P A^T + A P + e^{-1}P J_1 J_z P + e H_1 H_1^T \\
  B_w^e + e^{-1} J_1 J_2 P + e H_1 H_2^T \\
  C P + e H_2 H_1^T \\
  J_1^T \\
  0
\end{bmatrix}
\begin{bmatrix}
  B_w \\
  C P + e H_2 H_1^T \\
  D_w \\
  -I_q + e H_2 H_2^T \\
  0
\end{bmatrix}
< 0,
\]

which can be rewritten as
\[
\begin{bmatrix}
P A^T + A P + e H_1 H_1^T \\
B_w^e + e^{-1} J_1 J_2 P + e H_1 H_2^T \\
C P + e H_2 H_1^T \\
J_1^T \\
0
\end{bmatrix}
\begin{bmatrix}
P J_1 \\
B_w \\
C P + e H_2 H_1^T \\
-D_w \\
0
\end{bmatrix}
+ e^{-1} J_1 J_2 P J_z 0 < 0. \quad (9)
\]

By the Schur complement [13], the above inequality (10) is equivalent to (4). This completes the proof. 

**Remark 1.** In comparison with [12], the proposed result is more suitable for robust controller design. For design controllers, the condition in [12] must be reformulated as a more suitable formulation or parameter tuning. It is described in detail in the following subsection.

### 3.2 Robust \( H_\infty \) design

In this subsection, the above condition is applied to find a state feedback controller, \( u(t) = Kx(t) \), such that the closed-loop system
\[
\begin{align*}
\dot{x}(t) &= ((A + B_w K) + H_1 \Delta(J_1 + J_2 K))x(t) \\
&\quad + (B_w + H_1 \Delta J_2)w(t), \\
y(t) &= ((C + D_w K) + H_2 \Delta(J_1 + J_2 K))x(t) \\
&\quad + (D_w + H_2 \Delta J_2)w(t),
\end{align*}
\]

is quadratically stable with \( \|T_{w y}\|_\infty < \gamma \), where \( T_{w y} \) is the transfer function from \( w \) to \( y \) for the system (11).

**Theorem 2.** Let \( \gamma > 0 \) be given. Then, there exists a state feedback controller such that (11) is quadratically stable with \( \|T_{w y}\|_\infty < \gamma \) if and only if there exist matrices \( P > 0 \) and \( F \), and a scalar \( \varepsilon > 0 \) such that
\[
\begin{bmatrix}
P A^T + A P + e H_1 H_1^T \\
B_w^e + e^{-1} J_1 J_2 P + e H_1 H_2^T \\
C P + e H_2 H_1^T \\
J_1^T \\
0
\end{bmatrix}
\begin{bmatrix}
P J_1 \\
B_w \\
C P + e H_2 H_1^T \\
-D_w \\
0
\end{bmatrix}
+ e^{-1} J_1 J_2 P J_z 0 < 0. \quad (12)
\]

Moreover, the controller can be chosen as \( u(t) = K x(t) = F P^{-1} \dot{x}(t) \).

**Proof.** By Theorem 1, the closed-loop system (11) is quadratically stable with \( \|T_{w y}\|_\infty < \gamma \) if and only if there exist a matrix \( P > 0 \) and a scalar \( \varepsilon > 0 \) satisfying
\[
\begin{bmatrix}
P (A + B_w K)^T + (A + B_w K)P + e H_1 H_1^T \\
B_w^e + e^{-1} J_1 J_2 P + e H_1 H_2^T \\
C P + e H_2 H_1^T \\
(D_w + H_2 \Delta J_2)P \\
0
\end{bmatrix}
\begin{bmatrix}
P J_1 \\
B_w \\
C P + e H_2 H_1^T \\
-D_w \\
0
\end{bmatrix}
+ e^{-1} J_1 J_2 P J_z 0 < 0. \quad (13)
\]

Let \( F = KP \). Then, the equivalence is easily proved. 

**Remark 2.** Based on the result of Theorem 2, the \( H_\infty \) minimization design via state feedback for system (1) is formulated as the following constrained optimization problem:

\[
\text{minimize } \gamma \\
\text{subject to } P > 0, \varepsilon > 0, \text{ and (12).}
\]

With Theorem 2, the problem can be solved efficiently by using the Matlab LMI toolbox. Note that the design method proposed in [12] cannot deal with the optimization problem (see Remark 2 in [12]).

### IV. A NUMERICAL EXAMPLE

Consider a permanent-magnet dc motor system [14, p.107] described by
\[
\begin{bmatrix}
\frac{d i}{d t} \\
\frac{d \omega_m}{d t}
\end{bmatrix}
= \begin{bmatrix}
-\frac{R}{L} & 0 \\
\frac{K_b}{J} & -\frac{B}{J}
\end{bmatrix}
\begin{bmatrix}
i \\
\omega_m
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-1
\end{bmatrix} w + \begin{bmatrix}
1 \\
0
\end{bmatrix} u,
\]

\[
y = \begin{bmatrix}
0 \\
1
\end{bmatrix} \omega_m + 0 w + 0 u,
\]

where
\( i \) = armature current;
\( \omega_m \) = rotor angular velocity;
\( R \) = armature resistance;
\( L \) = armature inductance;
\( K_b \) = back-emf constant;
\( K_i \) = torque constant;
\( B \) = viscous-friction coefficient;
\( J \) = rotor inertia;
\( w \) = disturbance (load torque);
\( u \) = applied voltage.

Due to variation of the operation conditions and the parameter estimation error, it is reasonable to assume that the system parameters contain a nominal part and an uncertain part as follows:

\[
\begin{align*}
R &= \frac{R_n + \delta_1 \Delta_1}{L_n}, & K_b &= \frac{K_{b n} + \delta_2 \Delta_2}{J_n}, & K_i &= \frac{K_{i n} + \delta_3 \Delta_3}{J_n}, \\
B &= \frac{B_n + \delta_4 \Delta_4}{J_n}, & 1 &= \frac{1}{J_n} + \delta_5 \Delta_5, & 1 &= \frac{1}{L_n} + \delta_6 \Delta_6,
\end{align*}
\]

where the subscript \( n \) denotes the nominal value and \( \delta_i \Delta_i \) represents the uncertainties. In the uncertainties, \( \Delta_i, i = 1, 2, \ldots, 6 \) are uncertain but lie within \([-1, 1]\). \( \delta_i \) can be viewed as the bounds of the uncertainties. Using the parameter values of a large dc motor \([15, p.75]\), we set \( R_n = 1, L_n = 1, K_{b n} = 0.1, B_n = 0.5, \) and \( J_n = 2 \), and assume that \( \delta_1 = 0.2, \delta_2 = 0.01, \delta_3 = 0.5, \delta_4 = 0.02, \delta_5 = 0.05, \) and \( \delta_6 = 0.1 \). Then, the dynamic equation of the dc motor system can be rewritten as (2) with

\[
A = \begin{bmatrix} -1 & -0.1 \\ 5 & -0.25 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0 \ 1],
\]

\[
D_w = 0, \quad D_o = 0, \quad H_1 = \begin{bmatrix} \delta_1 & \delta_2 & 0 & 0 & 0 & \delta_3 \\ 0 & \delta_3 & \delta_4 & \delta_5 & \delta_6 \end{bmatrix},
\]

\[
H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta(t) = \text{diag} \{\Delta_1, \Delta_2, \ldots, \Delta_6\},
\]

\[
J_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.
\]

If we set \( \gamma = 0.5 \), then by Theorem 2, we can find \( \varepsilon = 13.6327 \) and the state feedback gain

\[
K = [-9.5470 \ -8.0325].
\]

Figures 1 and 2 show the time response of the closed-loop system for two different kinds of load torque \( w(t) \). As shown in each figure, the total of 1,000 sets of uncertainties, \( \Delta_i \), generated by Matlab’s command \((\text{rand}-0.5)*2\) were used in the simulation. Thus, \( y(t) \) contained 1,000 trajectories. By Remark 2, we obtained \( \gamma = 0.1327 \) and the optimal state feedback robust controller

\[
K = [-10192.0015 \ -16166.0234].
\]

V. CONCLUSION

In this paper, the necessary and sufficient LMI-based conditions have been presented for robust \( H_\infty \) control analysis and the design of uncertain systems. In comparison with previous results, parameter tuning is not required in the proposed approach; thus, minimization of the \( H_\infty \) performance can be achieved easily and efficiently.

REFERENCES


