LMI APPROACH TO ROBUST FILTERING FOR DISCRETE TIME-DELAY SYSTEMS WITH NONLINEAR DISTURBANCES

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ABSTRACT

This paper investigates the problem of robust filtering for a class of uncertain nonlinear discrete-time systems with multiple state delays. It is assumed that the parameter uncertainties appearing in all the system matrices reside in a polytope, and that the nonlinearities entering into both the state and measurement equations satisfy global Lipschitz conditions. Attention is focused on the design of robust full-order and reduced-order filters guaranteeing a prescribed noise attenuation level in an $H_{\infty}$ or $l_2-l_\infty$ sense with respect to all energy-bounded noise disturbances for all admissible uncertainties and time delays. Both delay-dependent and independent approaches are developed by using linear matrix inequality (LMI) techniques, which are applicable to systems either with or without a priori information on the size of delays.

KeyWords: Linear matrix inequality, nonlinear disturbance, robust filtering, state-delayed systems.

I. INTRODUCTION

In recent years, much attention has been devoted to time-delay systems, due to the fact that time delay, which is frequently encountered in various engineering systems, is often the main cause of instability and poor performance of systems. A great number of analysis and synthesis results for time-delay systems have been reported (see, for instance, [1-5] and the references therein). On the other hand, over the past four decades, state estimation has attracted considerable attention from the control community, since it can extract useful information about the inner state variables of a dynamic system, where the states can not be directly measured. However, when a priori information about the external noises is not precisely available, the celebrated Kalman filtering scheme is no longer applicable. For such cases, $H_{\infty}$ filtering was introduced in 1989 [6], where the input signal is assumed to be energy bounded and the main objective is to minimize the $H_{\infty}$ norm of the filtering error system. Other performance indexes introduced for systems with partially known noise information are $l_2-l_\infty$ (energy-to-peak) [7] and $l_1$ (peak-to-peak) [8], which have different physical meanings when used as performance indexes for filtering error systems.

The filtering problem for time-delay systems has been studied in some papers [3,5,9]. Very recently, some of the obtained results were further extended to time-delay systems with nonlinear disturbances [4,10]. It is worth noting that the filters designed in [4,10] are of the observer type and are independent of the delay size. To the best of the authors’ knowledge, no general-structured filtering result, whether dependent on or independent of the delay size, has been reported for discrete time-delay systems in the simultaneous presence of polytopic parameter uncertainties and nonlinear disturbances.

This paper is concerned with the problem of robust filtering for a class of uncertain nonlinear discrete-time systems with multiple delays in the state. More specifically, the parameter uncertainties entering into all the matrices of the system state space model are assumed to reside in a polytope, and the nonlinearities appearing in both the state and measurement equations are supposed to satisfy global Lipschitz conditions. Two performance indexes ($H_{\infty}$ and $l_2-l_\infty$) are adopted, and we present both delay-dependent and delay-independent approaches to solving the filter de-
sign problem. A numerical example is presented to illustrate the applicability of the developed filter design methods.

**Notations.** The superscript $T$ stands for matrix transposition; $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space; $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices; the notation $P > 0$ means that $P$ is symmetric and positive definite; and $0$ and I represent the identity matrix and zero matrix, respectively; $\ell_2[0, \infty)$ is the space of the square-summable vector function over $[0, \infty)$; the notation $\| \cdot \|$ refers to the Euclidean vector norm. In addition, in symmetric block matrices or long matrix expressions, we use $*$ to denote an ellipsis for vector norm. In addition, in symmetric block matrices or dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

### II. PROBLEM FORMULATION

Consider a stable discrete-time uncertain nonlinear system with multiple delays in the state

$$x(t+1) = A_0 x(t) + \sum_{j=1}^{q} A_j x_{dj} + Ff(t) + Bu(t),$$

$$y(t) = C_0 x(t) + \sum_{j=1}^{q} C_j x_{dj} + Gg(t) + Do(t),$$

$$z(t) = Hx(t),$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^n$ is the measured output; $z(t) \in \mathbb{R}^n$ is the signal to be estimated; $\omega(t) \in \mathbb{R}^n$ is the noise input belonging to $\ell_2[0, \infty)$; $x_{dj}$ denotes $x(t-d_j)$, and $d_j > 0, j = 1, \ldots, q$, are constant time delays; $\phi(t)$ is the given initial condition sequence with $\bar{d} = \max(d_j)$; and $f(t) \triangleq f(x_0, x_{d_1}, \ldots, x_{d_q})$ and $g(t) \triangleq g(x_0, x_{d_1}, \ldots, x_{d_q})$ are nonlinear functions. Throughout the paper, we make the following assumptions:

**Assumption 2.1.** The nonlinear functions satisfy the following:

(a) $f(0, 0, \ldots, 0) = 0$, $g(0, 0, \ldots, 0) = 0$;

(b) (Lipschitz conditions) there exist known real appropriately dimensioned matrices $M_j, N_j, j = 1, \ldots, q$ such that, for all $x_0, x_1, \ldots, x_q, y_0, y_1, \ldots, y_q \in \mathbb{R}^n$,

$$\| f(x_0, x_1, \ldots, x_q) - f(y_0, y_1, \ldots, y_q) \| \leq \sum_{j=0}^{q} M_j \| x_j - y_j \|,$$

$$\| g(x_0, x_1, \ldots, x_q) - g(y_0, y_1, \ldots, y_q) \| \leq \sum_{j=0}^{q} N_j \| x_j - y_j \|.$$

**Assumption 2.2.** The system matrices are appropriately dimensioned with partially unknown parameters. We assume that $\Omega = (A_0, \ldots, A_q, F, B, C_0, \ldots, C_q, G, D, H) \in \mathbb{R}$, where $\mathbb{R}$ is a given convex bounded polyhedral domain described by the $s$ vertices

$$\Omega = \{ \Omega(\lambda) \mid \Omega(\lambda) = \sum_{i=1}^{s} \lambda_i \Omega_i, \sum_{i=1}^{s} \lambda_i = 1, \lambda_i \geq 0 \},$$

and $\Omega_i = (A_{i_0}, \ldots, A_{i_q}, F, B, C_{i_0}, \ldots, C_{i_q}, G, D, H)$.

**Remark 2.1.** The system under investigation in this paper contains both parameter and nonlinear disturbance. As can be seen in Assumption 2.2, the parameter uncertainties are assumed to be of the polytopic type, entering into all the matrices of the system model. The polytopic uncertainty has been widely used in studies on the problems of robust control and filtering for uncertain systems (see, e.g., [5,7,9]), and many practical systems possess parameter uncertainties which can be either exactly modeled or overbounded by the polytope $\mathbb{R}$. In addition, the nonlinear disturbance in Assumption 2 has also been widely used in the literature (see, e.g., [4,10,11]).

Here we are interested in estimating the signal $z(t)$ by using a linear filter with the general structure described by

$$x(t+1) = A_1 x(t) + B_1 y(t),$$

$$z(t) = C_1 x(t),$$

where $x(t) \in \mathbb{R}^d$ is the filter state vector. It is important to note that, here, we are interested not only in the full-order filtering problem (when $k = n$) but also in the reduced-order filtering problem (when $1 \leq k < n$). As can be seen in the following, these two filtering problems can be solved in a unified framework.

Augmenting the model of (1) to include the states of the filter, we obtain the filtering error system

$$\xi(t+1) = \overline{A} \xi(t) + \sum_{j=1}^{q} \overline{A}_j \xi_{dj} + \overline{F} \eta(t) + \overline{B} \omega(t),$$

$$\eta(t) = [\phi'(t), 0]^T, \quad t = -\bar{d}, -\bar{d}+1, \ldots, 0,$$

where $\xi(t) \in \mathbb{R}^q$ is the filter state vector. As can be seen in the literature, generally $k$ is not less than $n$. In the present paper, we are interested not only in the full-order filtering problem but also in the reduced-order filtering problem.
for all admissible uncertainties and time delays: (a) the filtering error system in (3) is asymptotically stable (denoted as A.S. for simplicity); (b) the filtering error system in (3) guarantees a prescribed noise attenuation level in an $H_\infty$ or $l_2-l_\infty$ sense. More specifically, under zero-initial conditions and for all nonzero $\omega \in [0, \infty)$, it should be guaranteed that $\| e \| < \gamma^' | \omega |$, for the $H_\infty$ filtering problem, and that $\| e \| < \gamma^' | \omega |$, for the $l_2-l_\infty$ filtering problem, where $\| \omega \| = \sum_{i=0}^{\infty} v_i^r(t) o(t)$, $\| e \| = \sum_{i=0}^{\infty} e_i^r(t) e(t)$ and $\| e \| = \sup_i |e_i^r(t) e(t)|$.

Remark 2.2. The formulation of the above filtering problem is somewhat similar to that of the output feedback control problem (see, for instance, [11,12]) in the sense that both problems can be dealt with using the state-augmentation method. However, it should be noted that the filtering problem has its own features and, therefore, should be treated separately from the output feedback problem. A typical example is that the matrices of the filtering error system composed of a polytopic system and a filter also belong to a polytope, but this is not the case for the output feedback problem.

Throughout this paper, $(\overline{A}_0, \cdots, \overline{A}_q, \overline{F}_i, \overline{B}_i, \overline{C}_i)$ denotes matrices $\Omega$, evaluated at each of the vertices of polytope $\mathfrak{P}$, and we use the following notations to facilitate our presentation: $\overline{A}_d \triangleq [\overline{A}_i \cdots \overline{A}_q]$, $\overline{A}_a \triangleq [\overline{A}_i \cdots \overline{A}_q]$, $A_d \triangleq [A_i \cdots A_q]$, $A_a \triangleq [A_i \cdots A_q]$, $C_d \triangleq [C_i \cdots C_q]$, $C_a \triangleq [C_i \cdots C_q]$, $M_d \triangleq \text{diag}(M_i, \cdots, M_q)$, and $N_d \triangleq \text{diag}(N_i, \cdots, N_q)$. In addition, all the proofs of the theorems are provided in appendices.

### III. DELAY-DEPENDENT APPROACH

In this section, we will present a delay-dependent approach to the full-order and reduced-order robust $H_\infty$ and $l_2-l_\infty$ filtering problems. To obtain delay-dependent conditions for the filtering analyses, we first apply the following transformations to system (3). Since it holds that

$$
\xi_d = \xi(t) - \sum_{m=-d}^{t-1} [\xi(m+1) - \xi(m)] = \xi(t) - \sum_{m=-d}^{t-1} \delta(m),
$$

where $\delta(m) = \xi(m+1) - \xi(m) = (\overline{A}_0 - I) \xi(m) + \sum_{j=-d}^{q} \overline{A}_j \xi(m-d_j)$ and scalars $\xi_1, \xi_2$ such that

(i) LMIs (7) hold, then system (3) is A.S. with an $H_\infty$ noise attenuation level bound $\gamma$;

(ii) if they exist such that LMIs (8) hold, then system (3) is A.S. with an $l_2-l_\infty$ noise attenuation level bound $\gamma$.

Based on the transformed system model in (6), we have the following theorem for filtering analyses.

**Theorem 3.1.** Consider system (1) with an exactly known $\Omega$ and suppose that the filter matrices in (2) are given. If there exist matrices $P > 0$, $Q_j > 0$, $\overline{X}_j, \overline{Y}_j, \overline{Z}_j > 0$, $R_j, S_j, T_j > 0$ and scalars $\epsilon_1, \epsilon_2$ such that

(i) LMIs (7) hold, then system (3) is A.S. with an $H_\infty$ noise attenuation level bound $\gamma$;

(ii) if they exist such that LMIs (8) hold, then system (3) is A.S. with an $l_2-l_\infty$ noise attenuation level bound $\gamma$.
\[
\begin{bmatrix}
X_j & Y_j \\
* & Z_j
\end{bmatrix} \succeq 0, \quad \begin{bmatrix}
R_j & S_j \\
* & T_j
\end{bmatrix} \succeq 0, \quad \forall j = 1, \ldots, q, \quad (9a,b)
\]

where
\[
\begin{align*}
\Gamma & \triangleq \sum_{j=1}^{q} d_j (Z_j + T_j), \\
Y_d & \triangleq [Y_1 \ldots Y_q], \quad S_d \triangleq [S_1 \ldots S_q], \\
\{4, 4\} & \triangleq -P + (\varepsilon_1 + \varepsilon_2) (q+1) K^T (M_0^T M_0 + N_0^T N_0) K \\
& \quad + \sum_{j=1}^{q} (d_j X_j + Y_j K + K^T Y_j^T + K^T Q_{j} K), \\
\{5, 5\} & \triangleq -Q_d + (\varepsilon_1 + \varepsilon_2) (q+1) (M_0^T M_d + N_0^T N_d). \quad (10)
\end{align*}
\]

Then, by introducing the parameter-dependent stability idea [13], we can obtain the following theorem, which provides sufficient conditions of robust $H_\infty$ and $l_2$-$l_\infty$ performance for the filtering error system in (3). The way in which the proof of the theorem can be carried out is similar to that for the proof of Theorem 4 in [7] and is therefore, omitted here.

**Theorem 3.2.** Consider system (1) with $\Omega \in \mathbb{R}$ representing uncertain system matrices, and suppose the filter matrices in (2) are given. If there exist matrices $V, W, P_i > 0, Q_{ji} > 0, X_{ji}, Y_{ji}, Z_{ji} > 0, R_{ji}, S_{ji} > 0$ and scalars $\varepsilon_1, \varepsilon_2$ such that

(i) for $i = 1, \ldots, s$, LMIs (11) hold, then system (3) is A.S. with an $H_\infty$ noise attenuation level bound $\gamma$;

(ii) if they exist such that for $i = 1, \ldots, s$, LMIs (12a) hold, then system (3) is A.S. with an $l_2$-$l_\infty$ noise attenuation level bound $\gamma$;

where
\[
\begin{align*}
\Gamma_i & \triangleq \sum_{j=1}^{q} d_j (Z_{ji} + T_{ji}), \quad Y_{di} \triangleq [Y_{i1} \ldots Y_{iq}], \quad S_{di} \triangleq [S_{i1} \ldots S_{iq}], \\
\{4, 4\}_i & \triangleq -P_i + (\varepsilon_1 + \varepsilon_2) (q+1) K_i^T (M_0^T M_0 + N_0^T N_0) K_i \\
& \quad + \sum_{j=1}^{q} (d_j X_{ji} + Y_{ji} K_i + K_i^T Y_{ji}^T + K_i^T Q_{ji} K_i), \\
\{5, 5\}_i & \triangleq -Q_{di} + (\varepsilon_1 + \varepsilon_2) (q+1) (M_0^T M_d + N_0^T N_d). \quad (14)
\end{align*}
\]
Remark 3.1. Instead of directly extending Theorem 3.1 to polytopic uncertain systems based on the notion of quadratic stability, here, we incorporate a new result for the parameter-dependent stability issue [13]. The sufficient conditions formulated in Theorem 3.2 are both delay-dependent and parameter-dependent (in the sense that different positive definite matrices are needed for each vertex of the uncertain polytope \( \mathcal{P} \)); therefore, Theorem 3.2 has great potential for reducing the conservatism by the delay-independent approach in the quadratic framework.

Now, the filtering synthesis problems can be solved immediately on the basis of Theorem 3.2.

\[
\begin{bmatrix}
-\epsilon_i I & Y_{i1} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & Y_{i2} & 0 & Y_{i3} & Y_{i4} & Y_{i5} & 0 & 0 \\
* & * & -I & Y_{i6} & 0 & 0 & 0 & 0 \\
* & * & * & Y_{i7} & Y_{i8} & Y_{i9} & Y_{i10} & 0 \\
* & * & * & * & \{S, S\}_i & -S_0^T \Lambda_0^T \Lambda_0 W & 0 \\
* & * & * & * & * & \gamma F_1 & W^T F_1 & 0 \\
* & * & * & * & * & * & -\epsilon_2 I & \\
\end{bmatrix} < 0, \quad \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_3 \end{bmatrix} > 0, \quad (15a,b)
\]

\[
\begin{bmatrix}
-\epsilon_i I & Y_{i1} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & Y_{i2} & 0 & Y_{i3} & Y_{i4} & Y_{i5} & 0 & 0 \\
* & * & Y_{i7} & Y_{i8} & Y_{i9} & Y_{i10} & 0 & 0 \\
* & * & * & \{S, S\}_i & -S_0^T \Lambda_0^T \Lambda_0 W & 0 \\
* & * & * & * & -I + \sum_{j=1}^{q} d_j R_{ji} & B_j^T W & 0 \\
* & * & * & * & * & Y_{i11} & W^T F_1 \\
* & * & * & * & * & * & -\epsilon_2 I & \\
\end{bmatrix} \geq 0, \quad \begin{bmatrix} R_{ji} & S_{ji} \\ T_{ji} & \end{bmatrix} \geq 0, \quad \forall j = 1, \ldots, q, \quad (17a,b)
\]

where

\[
E \triangleq \begin{bmatrix} I_{k,s} \\ 0_{(a-1)\times k} \end{bmatrix}, \quad \gamma_{ii} \triangleq \begin{bmatrix} F_1^T \bar{P}_1 & F_1^T \bar{P}_2 \\ G_1^T \bar{B}_1^T E T & G_1^T \bar{B}_1^T F \end{bmatrix}, \quad \gamma_{2i} \triangleq \begin{bmatrix} \bar{P}_1 - \bar{P}_1^T & \bar{P}_2^T - \bar{P}_2 - \gamma F_1 \\ \bar{P}_3^T - \bar{P}_3 & \end{bmatrix},
\]

\[
Y_{ii} \triangleq \begin{bmatrix} \bar{P}_1 A_0 + E \bar{B}_1 C_{ii} + E \bar{A}_0 \\ \bar{P}_2 A_0 + E \bar{B}_1 C_{ii} \end{bmatrix} A_0, \quad \gamma_{ii} \triangleq \begin{bmatrix} \bar{P}_1 A_0 + E \bar{B}_1 C_{ii} \\ \bar{P}_2 A_0 + E \bar{B}_1 C_{ii} \end{bmatrix}, \quad \gamma_{3i} \triangleq \begin{bmatrix} \bar{P}_1 B + E \bar{B}_1 D_i \\ \bar{P}_2 B + E \bar{B}_1 D_i \end{bmatrix} + \begin{bmatrix} \bar{P}_3 - \gamma F_1 \\ \end{bmatrix}, \quad \gamma_{4i} \triangleq \begin{bmatrix} \bar{P}_1 A_0 + E \bar{B}_1 C_{ii} \\ \bar{P}_2 A_0 + E \bar{B}_1 C_{ii} \end{bmatrix} A_0, \quad \gamma_{ii} \triangleq \begin{bmatrix} \bar{P}_1 B + E \bar{B}_1 D_i \\ \bar{P}_2 B + E \bar{B}_1 D_i \end{bmatrix} + \begin{bmatrix} \bar{P}_3 - \gamma F_1 \\ \end{bmatrix},
\]

\[
\begin{bmatrix} \bar{P}_1 (I_{k,s} + (\epsilon_{ii} + \epsilon_{2i})(q+1)(M_0^T M_0 + N_0^T N_0) + \sum_{j=1}^{q} d_j \bar{X}_{i,j} + \bar{Y}_{i,j} + \bar{F}_1^T + \bar{Q}_{ji}) \\ \bar{P}_2 (I_{k,s} + (\epsilon_{ii} + \epsilon_{2i})(q+1)(M_0^T M_0 + N_0^T N_0) + \sum_{j=1}^{q} d_j \bar{X}_{i,j} + \bar{Y}_{i,j} + \bar{F}_1^T + \bar{Q}_{ji}) \end{bmatrix} \geq 0,
\]

\[
Y_{ii} \triangleq \begin{bmatrix} \sum_{j=1}^{q} S_{ji} \\ 0 \end{bmatrix}, \quad \gamma_{ii} \triangleq \begin{bmatrix} (A_0^T - I) W \\ 0 \end{bmatrix}, \quad \gamma_{ii} \triangleq \begin{bmatrix} \gamma F_1 - W - W^T \\ \end{bmatrix}.
\]

Theorem 3.3. Consider system (1) with \( \Omega \in \mathcal{R} \) representing uncertain system matrices, and let \( \gamma > 0, d_j > 0, j = 1, \ldots, q \), be given scalars. If there exist matrices \( \bar{P}_1, \bar{P}_2, \bar{F}_1, \bar{F}_2, \bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{D}_i, \bar{Q}_{ji} > 0 \), \( \bar{X}_{i,j}, \bar{Y}_{i,j}, \bar{F}_1, \bar{F}_2 > 0 \), \( R_{ji}, S_{ji}, T_{ji} > 0 \), and scalars \( \epsilon_{ii}, \epsilon_{2i} \) such that

(i) for \( i = 1, \ldots, s \), LMI (15) and (17) hold, then an admissible robust \( H_{\infty} \) filter of the form (2) exists;
(ii) if they exist such that for \( i = 1, \ldots, s \), LMI (16) and (17) hold, then an admissible robust \( l_2-l_\infty \) filter of the form (2) exists:
Moreover, under the above conditions, the matrices of an admisible $H_\infty$ or $l_2-l_\infty$ filter can be given by
\[
A_F = \bar{F}_5^{-1} \bar{A}_F, \quad B_F = \bar{F}_5^{-1} \bar{B}_F, \quad C_F = \bar{C}_F.
\] (18)

**Remark 3.2.** To obtain certain LMI conditions for the existence of desired filters, some linearization procedures have to be adopted. Since the standard linearization methods adopted in [9] assume the off-diagonal entry of certain matrix (the matrix to be partitioned, in this paper, it is $V$ in Theorem 3.2) to be square and nonsingular, they can only be used to deal with the full-order filtering problem. To keep the reduced-order filter design tractable, here, we seek a different linearization procedure, which can be used to solve both the full-order and reduced-order filtering synthesis problems in a unified framework. It is worth noting that the matrix $E$ defined in Theorem 3.3 plays an instrumental role. For full-order filtering, the matrix $E$ becomes an identity matrix of dimension $n$, and for the reduced-order case, we impose a structural restriction on the $(2,1)$ block entry of matrix $V$, which introduces some overdesign into filter design.

**IV. DELAY-INDEPENDENT APPROACH**

It should be noted that the derivation of the delay-dependent performance in the above section exploits the most efficient bounding technique (Lemma 1), which has been used in many problems related to time-delay systems. It is also interesting to note that the delay-dependent performance (Theorem 3.1) is most powerful in the sense that it also implies a delay-independent result based on the choice of certain matrices. If we choose the matrices
\[
X_j = \frac{a_{j(j+k)(j+k)}}{d_j}, \quad Y_j = 0, \quad Z_j = \frac{a_{j(j+k)}}{d_j},
\]
\[
R_j = \frac{a_{j(j+k)}}{d_j}, \quad S_j = 0, \quad T_j = \frac{a_{j(j+k)}}{d_j},
\]
for a sufficiently small positive constant $\sigma$, the conditions in Theorem 3.1 imply delay-independent performance conditions, as presented in the following theorem without proof.

**Theorem 4.1.** Consider system (1) with an exactly known $\Omega$ and suppose the filter matrices in (2) are given. If there exist matrices $P > 0, \Omega > 0$ and a scalar $\varepsilon$ such that

(i) LMI (19) holds, then system (3) is A.S. with an $H_\infty$ noise attenuation level bound $\gamma$;

(ii) if they exist such that LMI (20) hold, then system (3) is A.S. with an $l_2-l_\infty$ noise attenuation level bound $\gamma$:

\[
\begin{bmatrix}
-\varepsilon I & \bar{F}^T P & 0 & 0 & 0 \\
* & -P & \bar{P}_A & \bar{P}_B \\
* & * & -I & \bar{C} & 0 \\
* & * & * & [4,4] & 0 \\
* & * & * & * & [5,5] \\
\end{bmatrix} < 0,
\]
(19)

\[
\begin{bmatrix}
-\varepsilon I & \bar{F}^T P & 0 & 0 \\
* & -P & \bar{P}_A & \bar{P}_B \\
* & * & -I & \bar{C} & \gamma^2 I \\
* & * & [4,4] & 0 & 0 \\
* & * & * & [5,5] & 0 \\
* & * & * & * & -I \\
\end{bmatrix} < 0,
\]
(20a,b)

where

\[
[4,4] = -P + \sum_{j=1}^{q} K_j^T Q_j K + \varepsilon(q+1) K^T (M_j^T M_j + N_j^T N_j) K ,
\]
\[
[5,5] = -Q d + \varepsilon(q+1) (M_j^T M_j + N_j^T N_j),
\]
\[
Q d = \text{diag} \{Q_1, \ldots, Q_s\}.
\]

Then as byproduct of Theorem 3.3, we can readily obtain delay-independent sufficient conditions for the existence of robust $H_\infty$ and $l_2-l_\infty$ filters for system (1).

**Theorem 4.2.** Consider system (1) with $\Omega \in \Re$ representing uncertain system matrices. If there exist matrices $\bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{A}_F, \bar{B}_F, \bar{C}_F, \bar{P}_1 > 0, \bar{P}_2 > 0, \bar{P}_3 > 0, Q_i > 0$ and scalars $\varepsilon_i$ such that

(i) for $i = 1, \ldots, s$, LMIs (21) hold, then an admissible robust $H_\infty$ filter of the form (2) exists;

(ii) if they exist such that for $i = 1, \ldots, s$, LMIs (22) hold, then an admissible robust $l_2-l_\infty$ filter of the form (2) exists:

\[
\begin{bmatrix}
-\varepsilon I & \bar{Y}_i & 0 & 0 & 0 \\
* & \bar{Y}_i & 0 & 0 & 0 \\
* & * & -I & \bar{A}_i & 0 \\
* & * & * & [5,5] & 0 \\
* & * & * & * & -I \\
\end{bmatrix} < 0,
\]
(21a,b)

\[
\begin{bmatrix}
-\varepsilon I & \bar{Y}_i & 0 & 0 \\
* & \bar{Y}_i & 0 & 0 & 0 \\
* & * & -I & \bar{A}_i & 0 \\
* & * & * & [5,5] & 0 \\
* & * & * & * & -I \\
\end{bmatrix} < 0,
\]
(22a,b)
where
\[
\Delta_i \triangleq \begin{bmatrix}
-\overline{P}_{i1} + e_i(q+1)(M_0^T M_0 + N_0^T N_0) + \sum_{j=1}^{\gamma} Q_{ji} \\
-\overline{P}_{i2} & -\overline{P}_{i3}
\end{bmatrix}
\]
and
\[
[5,5] \triangleq -Q_{ii} + e_i(q+1)(M_0^T M_0 + N_0^T N_0).
\]
Moreover, under the above conditions, the matrices of an admissible \( H_\gamma \) or \( l_2-l_\infty \) filter can be given by (18).

V. ILLUSTRATIVE EXAMPLE

For simplicity, we will only consider the \( H_\gamma \) filtering problem. Consider the simplified longitudinal flight system in [4]. It is easy to see that this system has the structure of system (1) with the following matrices:
\[
A_0 = \begin{bmatrix}
0.9944 & -0.1203 & -0.4302 \\
0.0017 & 0.9902 & -0.0747 + 0.01\alpha \\
0 & 0.8187 & 0
\end{bmatrix}, \quad A_1 = 0_{3\times 3},
\]
\[
F = 0_{3\times 1}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix},
\]
\[
C_\gamma = [0.2 \ 0.1 \ 0.1 + 0.01\alpha], \quad C_\gamma^* = [0.1 \ 0.1 + 0.01\alpha \ 0],
\]
\[
G = 0.2, \quad D = 0.1, \quad \text{and} \quad H = [0 \ 0.1 \ 0.2].
\]
In the nonlinear functions satisfy Assumption 2.1 with \( M_0 = M_1 = 0_{3\times 3}, N_0 = [0 \ 0 \ 0.2], \) and \( N_1 = [0 \ 0.1 \ 0] \).

In [4], an \( H_\gamma \) filter with an observer-type structure is designed for the above system with a prespecified noise attenuation level of \( \gamma = 1.5 \). Here we are interested in designing a linear filter of the general form (2) to estimate signal \( z(t) \). By adopting the delay-independent approach, we can obtain the minimum guaranteed cost \( \gamma^* = 0.0196 \) with full-order filter matrices given by
\[
A_F = \begin{bmatrix}
0.3663 & 0.9960 & 0.1130 \\
-0.0221 & 0.9828 & 0.0143 \\
-0.0211 & 0.2357 & 0.0210
\end{bmatrix}, \quad B_F = \begin{bmatrix} -2.8867 \\ -0.1073 \\ -0.1953 \end{bmatrix},
\]
\[
C_F = [0.0002 \ -0.0999 \ -0.2056],
\]
and \( \gamma^* = 0.0225 \), and with second-order filter matrices given by
\[
A_F = \begin{bmatrix}
0.9663 & 0.0128 \\
0.0001 & 0.9343
\end{bmatrix}, \quad B_F = \begin{bmatrix} -0.0872 \\ -0.0122 \end{bmatrix},
\]
and \( C_F = [-0.0005 \ -0.1357] \).

In addition, assuming \( d = 1 \) and by solving Problem 2 (the delay-dependent approach), we can obtain \( \gamma^* = 0.0171 \) for full-order filtering and \( \gamma^* = 0.0224 \) for second-order filtering. From these obtained minimum guaranteed costs, it can be seen that the delay-dependent approach can yield less conservative results for systems with small delays.

VI. CONCLUDING REMARKS

The full-order and reduced-order robust \( H_\gamma \) and \( l_2-l_\infty \) filtering problems have been solved for linear discrete time-delay systems in the simultaneous presence of both polytopic parameter uncertainties and nonlinear disturbance. Both delay-independent and dependent approaches have been presented. They are suitable for systems either with or without a priori information about the size of the delays. LMI-based sufficient conditions have been obtained for the existence of admissible filters, thus allowing the problems to be solved using convex optimization procedures. Finally, since the results obtained in this paper are all sufficient conditions, the failure to satisfy these conditions does not imply that such filters do not exist.

REFERENCES

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**APPENDIX A**

**PROOF OF THEOREM 3.1**

By applying congruence transformation to (7) and (8a) using \( \text{diag} \{ I, P^{-1}, I, I, I, I, \Gamma^{-1}, I \} \) and \( \text{diag} \{ I, P^{-1}, I, I, I, \Gamma^{-1}, I \} \), respectively, together with some Schur complement operations, we obtain

\[
\Theta = \begin{bmatrix}
\Theta_{11} + \Xi^T \Xi & \Theta_{12} & \Theta_{13} \\
* & \Theta_{22} & \Theta_{23} \\
* & * & -\gamma I + \Theta_{33}
\end{bmatrix} < 0, \\
\tilde{\Theta} = \begin{bmatrix}
\tilde{\Theta}_{11} & \tilde{\Theta}_{12} & \tilde{\Theta}_{13} \\
* & \tilde{\Theta}_{22} & \tilde{\Theta}_{23} \\
* & * & -I + \Theta_{33}
\end{bmatrix} < 0,
\]

where

\[
\Phi = (P^{-1} - \varepsilon_i^{-1} F F^T)^{-1}, \quad \Psi = \Gamma^{-1} - \varepsilon_i^{-1} K F F^T K^T, \\
\Theta_{11} = \{4, 4\} + \tilde{\Theta}_{11} \Phi \tilde{\Theta}_{11} + (\bar{A}_0 - I)^T \Xi \Psi K (\bar{A}_0 - I), \\
\Theta_{12} = -Y_d + \tilde{\Theta}_{12} \Phi \tilde{\Theta}_{12} + (\bar{A}_0 - I)^T \Xi \Psi K \tilde{\Theta}_{12}, \\
\Theta_{22} = \{5, 5\} + \tilde{\Theta}_{22} \Phi \tilde{\Theta}_{22} + \tilde{\Theta}_{22} \Phi \Psi K \tilde{\Theta}_{22}, \\
\Theta_{23} = \sum_{j=1}^q K^T S_j + \tilde{\Theta}_{23} \Phi \tilde{\Theta}_{23} + (\bar{A}_0 - I)^T \Xi \Psi K \tilde{\Theta}_{23}, \\
\Theta_{33} = \sum_{j=1}^q d_j R_j + \tilde{\Theta}_{33} \Phi \tilde{\Theta}_{33} + \tilde{\Theta}_{33} \Phi \Xi^T \Xi K \tilde{\Theta}_{33}.
\]

Now, we can construct a Lyapunov functional candidate as

\[
\chi_{2(t)} := \chi_1 + \chi_2 + \chi_3,
\]

\[
\chi_i := \bar{z}_i^T(t) P \bar{z}_i(t), \quad \chi_2 := \sum_{i=1}^n \sum_{j=1}^{m_i} \bar{z}_i^T(j) K^T Q_i K \bar{z}_i(j), \\
\chi_3 := \sum_{j=1}^q \sum_{i=1}^{r_i} \bar{d}_i^T(t) K^T (Z_j + T_j) K \bar{d}_i(t),
\]

where \( P > 0, Q_i > 0, Z_j > 0, T_j > 0 \). Define \( \Delta \chi_{2(t)} := \chi_{2(t+1)} - \chi_{2(t)} \); then, along (6), we have

\[
\Delta \chi_1 = \bar{z}_i^T(t+1) P \bar{z}_i(t+1) - \bar{z}_i^T(t) P \bar{z}_i(t).
\]

Since both (7) and (8) imply \( \epsilon_i > 0 \) and \( \epsilon_i I - F F^T P F > 0 \), by defining \( \Pi = P, \quad \Phi_2 = F, \quad \Phi_1 = \eta(t), \) and \( \Phi_1 = (\bar{A}_0 + \sum_{j=1}^q \bar{A}_j K)(\bar{z}_i(t) - \sum_{j=1}^{r_i} \bar{d}_i(t) K \bar{d}_i(t) + \tilde{\Theta}_{23}(t)) \)

in Lemma 1 of [14], we have

\[
\Delta \chi_1 \leq \bar{z}_i^T(t) \left[ (\bar{A}_0 + \sum_{j=1}^q \bar{A}_j K)^T \Phi (\bar{A}_0 + \sum_{j=1}^q \bar{A}_j K) - P \right] \bar{z}_i(t) \\
+ \left[ \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{A}_j K \delta_i(m) \right] \Phi \left[ \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{A}_j K \delta_i(m) \right] \\
+ 2 \bar{z}_i^T(t) \left( \bar{A}_0 + \sum_{j=1}^q \bar{A}_j K \right) \Phi \tilde{\Theta}_{23} \bar{z}_i(t) + \omega_i(t) \bar{B}^T \Phi B \omega(t) + \epsilon_i \eta_i(t) \eta(t)
\]

\[
-2 \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{A}_j K \delta_i(m) \bar{z}_i^T(t) + \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{A}_j K \delta_i(m) \bar{z}_i^T(t) \\
-2 \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{d}_i(t) \bar{B}^T \Phi \tilde{\Theta}_{23} \delta_i(m),
\]

where \( \Phi \) is defined in (A1). In order to derive an upper bound of \( \Delta \chi_1 \), we have to find upper bounds for the last two terms in the above equation. First, by using Lemma 1 of [15], we can obtain

\[
-2 \bar{z}_i^T(t) (\bar{A}_0 + \sum_{j=1}^q \bar{A}_j K)^T \Phi \tilde{\Theta}_{23} \delta_i(m) \leq \bar{z}_i^T(t) X_i \bar{z}_i(t) + 2 \bar{z}_i^T(t) Y_i \bar{A}_0 + \sum_{j=1}^q \sum_{i=1}^{m_i} \bar{A}_j K \delta_i(m) \Phi \tilde{\Theta}_{23} \delta_i(m) + \bar{z}_i^T(t) K^T Z \delta_i(m)
\]

\[
+ 2 \bar{z}_i^T(t) \bar{B}^T \Phi \tilde{\Theta}_{23} \delta_i(m) + \bar{z}_i^T(t) K^T T \delta_i(m)
\]

with \( X_i, Y_i, Z_i \) satisfying (9a), and \( R_i, S_i, T_i \) satisfying (9b). In addition, we have

\[
\Delta \chi_2 = \sum_{j=1}^q \left[ \bar{z}_i^T(t) K^T Q_i \bar{z}_i(t) - \bar{z}_i^T(t) K^T (Z_j + T_j) K \bar{z}_i(t) \right],
\]

\[
\Delta \chi_3 = \sum_{j=1}^q \left[ \bar{z}_i^T(t) K^T Q_i \bar{z}_i(t) - \bar{z}_i^T(t) K^T (Z_j + T_j) K \bar{z}_i(t) \right].
\]
\[ \Delta x_3 = \sum_{j=1}^q \sum_{l=0}^{j-1} \left[ \delta_l \left( t \right) K^T \left( Z_j + T_j \right) K \delta_l \left( t \right) \right] \]

\[ = \delta_l \left( t \right) K^T \Gamma K \delta_l \left( t \right) \]

\[ - \sum_{j=1}^{r+1} \delta_j \left( m \right) K^T \left( Z_j + T_j \right) K \delta_j \left( m \right) , \quad (A9) \]

where \( \Gamma \) is defined in (10). Both (7) and (8) imply \( \varepsilon_2 > 0 \) and \( \varepsilon_2 I - F^T \Gamma K F > 0 \), by defining \( \Pi \triangleq \Gamma \), \( \Phi_1 \triangleq K \left( A_0 - I \right) \xi \left( t \right) + \sum_{j=1}^{r+1} \tilde{A}_j K \xi_{\delta j} + \tilde{B} \omega (t) \), \( \Phi_2 \triangleq K F \), and \( \Phi_3 \triangleq \eta^\prime \left( t \right) \) Lemma 1 of [14], we have an upper bound on the first term of \( \Delta x_3 \):

\[ \delta_l \left( t \right) K^T \Gamma K \delta_l \left( t \right) \leq \left[ \left( A_0 - I \right) \xi \left( t \right) + \sum_{j=1}^{r+1} \tilde{A}_j K \xi_{\delta j} + \tilde{B} \omega (t) \right]^T \]

\[ \cdot K^T \Psi K \left( A_0 - I \right) \xi \left( t \right) + \sum_{j=1}^{r+1} \tilde{A}_j K \xi_{\delta j} + \tilde{B} \omega (t) \]

\[ + \varepsilon_2 \eta^\prime \left( t \right) \eta \left( t \right) , \quad (A10) \]

\[ \quad \text{where } \psi \text{ is defined in (A1).} \]

In addition, using Assumption 2.1 and arguments similar to those in [14], we have

\[ \eta^\prime \left( t \right) \eta \left( t \right) \leq \left( q + 1 \right) \xi \left( t \right) K^T \left( M_0^T M_0 + N_0^T N_0 \right) K \xi \left( t \right) \]

\[ + \sum_{j=1}^{r+1} \xi_{\delta j} K^T \left( M_j^T M_j + N_j^T N_j \right) K \xi_{\delta j} \].

\[ \quad \text{(A11)} \]

Therefore, assuming a zero disturbance input, from (5), (A5)-(A11), we find that

\[ \Delta x_{\delta (t)} = \Delta x_1 + \Delta x_2 + \Delta x_3 \leq \xi \left( t \right) \Theta \xi \left( t \right) \]

\[ \quad \text{where } \Theta \triangleq \Theta_{11} \star \Theta_{22} \].

Using the Schur complement, we find that both (A1) and (A2) imply \( \Theta < 0 \). Then from the Lyapunov-Krasovskii stability theory, we can conclude that the filtering error system in (4) is asymptotically stable.

The next task is to establish the \( H_\infty \) and \( l_2-l_\infty \) performance for the filtering error system. Assuming the zero initial condition, we have \( \chi_{\delta (t)} \left| t=0 \right. = 0 \). First, we consider the index \( J_1 \triangleq \sum_{l=0}^{\infty} \left[ e^l \left( t \right) e^l \left( t \right) - \gamma^l \omega^l \left( t \right) \omega (t) \right] ; \text{then, for any nonzero } \omega \left| t_0 = 0 \right. \text{, we have} \]

\[ J_1 \leq \sum_{l=0}^{\infty} \left( e^l \left( t \right) e^l \left( t \right) - \gamma^l \omega^l \left( t \right) \omega (t) + \Delta x_{\delta (t)} \right) \]

\[ \leq \sum_{l=0}^{\infty} \tilde{e}^l \left( t \right) \Theta \tilde{e}^l \left( t \right) , \]

\[ \quad \text{where } \tilde{e}^l \left( t \right) \triangleq \left[ \xi \left( t \right) \xi_{\delta j} \star K^T \right] \}

\[ \quad \text{and } \Theta \text{ is defined in (A1).} \]

Therefore, we have \( J_1 \leq 0 \) and \( \left\| e \right\|_2 \leq \gamma \left\| \omega \right\|_2 ; \text{thus, the } H_\infty \text{ performance for the filtering error system is established.} \]

To establish the \( l_2-l_\infty \) performance for the filtering error system, consider another index \( J_2 \triangleq \chi_{\delta (t)} \left| - \sum_{l=0}^{\infty} \omega^l \left( t \right) \omega (t) \right. \text{. Then, for any nonzero } \omega \in l_2 \left[ 0, \infty \right) \text{ and } t > 0 \text{, we have} \]

\[ J_2 \leq \sum_{l=0}^{\infty} \left( \chi_{\delta (t)} \right) \leq \sum_{l=0}^{\infty} \tilde{e}^l \left( t \right) \Theta \tilde{e}^l \left( t \right) , \]

\[ \text{where } \tilde{\Theta} \text{ is defined in (A2). Then, we have } J_2 \leq 0 \text{, which implies that } \tilde{e}^l \left( t \right) \tilde{e}^l \left( t \right) \leq \gamma^l \sum_{l=0}^{\infty} \omega^l \left( t \right) \omega (t) \text{.} \]

On the other hand, \( (8b) \) is equivalent to \( C^T C < \gamma^l P \).

Thus, we can infer that, for all \( t > 0 \),

\[ e^l \left( t \right) e^l \left( t \right) \leq \gamma^l \sum_{l=0}^{\infty} \omega^l \left( t \right) \omega (t) \text{.} \]

Taking the supremum over \( t > 0 \) yields \( \left\| e \left( t \right) \right\|_2 < \gamma^l \left\| \omega \right\|_2 \) for all nonzero \( \omega \in l_2 \left[ 0, \infty \right) ; \text{thus, the filtering error system (3) is asymptotically stable with an } l_2-l_\infty \text{ noise attenuation level bound } \gamma. \]

\section*{APPENDIX B}

\section*{PROOF OF THEOREM 3.3}

\subsection*{(H. filtering case)}

Since LMI (15) imply \( \bar{V}_3 + \bar{F}_3^T - \bar{F}_3 \), \( V_3 > 0 \) and \( \bar{P}_3 > 0 \), we can infer that \( \bar{V}_3 + \bar{F}_3^T > 0 \); therefore, \( \bar{V}_3 \) is nonsingular. In addition, we can always find square and nonsingular matrices \( U \) and \( L \) satisfying \( \bar{V}_3 = U^T L^U \). Now, we introduce the following matrix variables:

\[ J_3 \triangleq \begin{bmatrix} I & 0 \\ 0 & L^U U \end{bmatrix} , \quad V \triangleq \begin{bmatrix} \bar{V}_3 & \bar{V}_3 U^{-1} L \\ U E^T & 0 \end{bmatrix} , \quad P \triangleq \begin{bmatrix} P_0 \star P_0 \\ P_3 \star P_3 \end{bmatrix} , \quad X_3 \triangleq \begin{bmatrix} X_{1,\beta} & X_{2,\beta} \\ * & X_{3,\beta} \end{bmatrix} , \quad Y_3 \triangleq \begin{bmatrix} Y_{1,\beta} & Y_{2,\beta} \\ * & Y_{3,\beta} \end{bmatrix} \]

\[ J_3 = \begin{bmatrix} X_{1,\beta} & X_{2,\beta} \\ * & X_{3,\beta} \end{bmatrix} , \quad Y_3 = \begin{bmatrix} Y_{1,\beta} & Y_{2,\beta} \\ * & Y_{3,\beta} \end{bmatrix} \]

\[ \begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} \triangleq \begin{bmatrix} 0 & I \\ 0 & 0 \\ I \end{bmatrix} , \quad \bar{A}_F \bar{B}_F \bar{U} \quad 0 \quad L \\ 0 \quad 0 \quad I \].

Note that the above matrices \( \left( A_F, B_F, C_F \right) \) are uniquely defined. Now, through some tedious matrix manipulations, we can establish that (15) and (17) are equivalent to
Applying congruence transformation to (B2a) and (B2b) by means of \( \text{diag}\{I, J_2^{-1}, l, 1, 1, 1\} \) and \( J_2^{-1} \), respectively, yields LMI (11) and \( P_i > 0 \). Applying congruence transformation to (B3a) by means of \( \text{diag}\{J_2^{-1}, l\} \) yields (13a). In addition, (B3b) is the same as (13b). Therefore, we conclude from Theorem 3.2 that the filter with a state-space realization \( (A_F, B_F, C_F) \) as defined in (B1) guarantees that the filtering error system in (3) is asymptotically stable with an \( H_\infty \) noise attenuation level bound \( \gamma \).

If we can find admissible robust filters in the light of LMIs (11) and (13), then the filter matrices can be calculated from the definition in (B1). Now, let us denote the filter transfer function from \( y(t) \) to \( z(t) \) by \( T_{xy} = C_F(zI - A_F)^{-1}B_F \).

Substituting the filter matrices with (B1) and considering the relationship \( \bar{F}_2 = U^T L U \) results in \( T(z) = C_F(zI - \bar{F}_2 A_F)^{-1} F_1^{-1} B_F \). Therefore, an admissible filter can be given by (18), and the proof is completed.

\[ (l, l) \text{-filtering case}. \] The proof is similar to that for the \( H_\infty \) filtering case. ■

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