GENERALIZED QUADRATIC STABILIZATION FOR DISCRETE-TIME SINGULAR SYSTEMS WITH TIME-DELAY AND NONLINEAR PERTURBATION

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ABSTRACT

This paper discusses a generalized quadratic stabilization problem for a class of discrete-time singular systems with time-delay and nonlinear perturbation (DSSDP), which satisfies Lipschitz condition. By means of the S-procedure approach, necessary and sufficient conditions are presented via a matrix inequality such that the control system is generalized quadratically stabilizable. An explicit expression of the static state feedback controllers is obtained via some free choices of parameters. It is shown in this paper that generalized quadratic stability also implies exponential stability for linear discrete-time singular systems or more generally, DSSDP. In addition, this new approach for discrete singular systems (DSS) is developed in order to cast the problem as a convex optimization involving linear matrix inequalities (LMIs), such that the controller can stabilize the overall system. This approach provides generalized quadratic stabilization for uncertain DSS and also extends the existing robust stabilization results for non-singular discrete systems with perturbation. The approach is illustrated here by means of numerical examples.

KeyWords: Discrete-time singular system, time-delay, perturbation, state feedback, generalized quadratic stabilization.

I. INTRODUCTION

The quadratic stabilization theory for non-singular discrete-time systems has been extensively developed [1-6]. The problem of robust quadratic stabilization is that of finding a feedback controller such that the closed-loop sys-
sufficient conditions were presented to guarantee robust pole location within a specified disk. Unfortunately, the robust pole location technique developed in [19] cannot be applied to study the robust stabilization problem for singular systems with time-varying uncertainty or, more generally, nonlinear perturbation. It is worth pointing out that few efficient algorithms are available for constructing state feedback for relative high-order DSS; examples are algorithms based on nonlinear matrix inequalities [14,17]. To the best of the authors’ knowledge, the issue of robust stabilization for discrete-time singular system with time-delay and nonlinear perturbation (DSSDP) has not been fully investigated and remains important and challenging.

In this paper, a type of DSSDP is discussed, which contains nonlinear time-varying perturbations in both the state and control inputs and perturbation that satisfies the Lipschitz constraint. The objective of this paper is to present generalized quadratic stability and stabilization controller designs based on linear static state feedback. It is well known that quadratic stability for non-singular systems usually implies exponential stability based on the Lyapunov stability theory. However, a nontrivial question needs to be addressed: “Does generalized quadratic stability also imply exponential stability for linear discrete-time singular systems with time-delay or, more generally, DSSDP?” This question has not yet been answered in the literature. One of the related difficulties lies in the fact that the standard Lyapunov stability theory cannot be applied to solve the above problems. The reason is that the addressed systems are singular and the quadratic Lyapunov function or functional for singular systems may not be positive definite. In this paper, we will attempt to solve the above problem and present an affirmative answer to the question. The answer to this question will lead to some interesting results with respect to DSSDP.

By means of the S-procedure approach, the global exponential stability result for unforced DSSDP is first established via LMIs. The delay-dependent convergent rate is also estimated for the system state. Then a necessary and sufficient condition is presented for generalized quadratic stabilization without constraints on system matrices and where the control design procedure provides more freedom in choosing parameter matrices. The advantage of this extra freedom is that it can be used to improve the system performance. Furthermore, an LMI stabilization approach is used to solve the generalized quadratic stabilization problem for DSSDP. In this case, the static state feedback controller can be obtained efficiently by means of interior-point optimization algorithms. Therefore, this paper also presents an efficient algorithm for relative high-order DSS.

Notations. $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$: the set of real numbers, $n$-vectors, and $n$ by $m$ matrices, respectively; $\mathbb{Z} = \{0, 1, 2, \ldots\}$; $W^T$: the transpose of matrix $W \in \mathbb{R}^{m\times n}$; $\|W\|$: $\lambda_{\max}(W^TW)^{1/2}$, i.e. the square root of the maximal eigenvalue of $W^TW$; $X^T$: the transpose of matrix $X^{-1}$, $I$ ($I_n$); the identity matrix of appropriate dimensions (of $\mathbb{R}^{n\times n}$); $\|x\| = \sqrt{x^T x}$, where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Throughout this paper, for symmetric matrices $X$ and $Y$, $X \geq Y$ respectively, $X > Y$: $X - Y$ is positive semi-definite (respectively, positive definite); $X \leq Y$ respectively, $X < Y$: $X - Y$ is negative semi-definite (respectively, negative definite). Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. PRELIMINARIES

Consider the following unforced discrete-time nonlinear singular systems:

$$\begin{align*}
Ex(k+1) &= Ax(k) + A_d x(k-d) + Gg[k, x(k), x(k-d)],
\end{align*}$$

(1)

where $x(k) \in \mathbb{R}^n$ is the system state; $A, A_d \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{m\times n}$ are constant matrices; $d > 0$ is a known positive integer time-delay of the system; the matrix $E \in \mathbb{R}^{m\times m}$ may be singular, and it is assumed that $0 \leq \text{rank}(E) = r \leq n$; $g = [g[k, x(k), x(k-d)] \in \mathbb{R}^r$ is a vector-valued nonlinear time-varying perturbation with $g(k, 0, 0) = 0$ for all $k \in \mathbb{Z}$ and satisfies the following Lipschitz condition for all $(k, x(k), x(k-d))$, $(\tilde{x}(k), \tilde{x}(k-d)) \in \mathbb{Z} \times \mathbb{R}^r \times \mathbb{R}^r$:

$$\begin{align*}
&\|g[k, x(k), x(k-d)] - g[k, \tilde{x}(k), \tilde{x}(k-d)]\| \\
&\leq \|M_x[x(k) - \tilde{x}(k)] + M_{x_k}[x(k-d) - \tilde{x}(k-d)]\|,
\end{align*}$$

(2)

where $M_x, M_{x_k}$ are constant matrices with appropriate dimensions. From (2), we have

$$\begin{align*}
&\|g[k, x(k), x(k-d)]\| \leq \|M_x x(k) + M_{x_k} x(k-d)\|.
\end{align*}$$

(3)

For convenience, $g$ denotes (tolerable) Lipschitz perturbation, or tolerable perturbation, in this paper.

Remark 2.1. The structure in the form of (3) has been extensively discussed in the literature for both continuous and discrete time systems [3, 20 and the references therein]. It is worth mentioning that the matched uncertainty can be regarded as a special case of (3). However, robust stabilization for this type of DSSDP has seldom been investigated. It should be noted that the Lipschitz condition (2) is used to show the existence and uniqueness of the solution for DSSDP; it can be replaced by a less conservative constraint (3) if nonsingular systems are considered.

Definition 2.1. (1) The pair $(E, A)$ is said to be regular if det $(sE - A)$ is not identical zero. (2) The pair $(E, A)$ is said to be casual if det $(sE - A) = \text{rank}(E)$.

Motivated by the definitions in [3, 14, 21], we give the following definition of generalized quadratic stability.

Definition 2.2. System (1) is said to be generalized quadratic stable if there exist a symmetric matrix $P$ and a positive definite matrix $Q$ such that $E^TP E \geq 0$ and
\[ \Delta_k = \left[ Ax(k) + A_j x(k-d) + G g[k, x(k), x(k-d)] \right]^T \]
\[ P \left[ Ax(k) + A_j x(k-d) + G g[k, x(k), x(k-d)] \right] \]
\[ - x^T(k) E^T P E x(k) + x^T(k) Q x(k) \]
\[ - x^T(k-d) Q x(k-d) \]
\[ < 0 \quad (4) \]
for all tolerable perturbations (3) and
\[ (k, x(-d), x(-d+1), \ldots, x(0)) \]
\[ \in Z \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n - \{0\} . \]

If there exist a symmetric matrix \( P \) and a positive definite matrix \( Q \) such that (4) holds, then \( A^T P A - E^T P E < 0 \). It follows from [14] that \( A^T P A - E^T P E < 0 \) and \( E^T P E \geq 0 \), which imply that \( (E, A) \) is regular and causal.

**Remark 2.2.** The above definition of the regularity and causality applies only to linear singular systems. If perturbation \( g[k, x(k), x(k-d)] \) is taken as linear uncertainty, for example, \( g[k, x(k), x(k-d)] = F I (M x(k) + M_j x(k-d)) \), where \( F I (\sigma) \) is a time-invariant uncertainty (independent of \( k \) with \( F I (\sigma) \leq I \) and \( \sigma \) is in a compact set \( \mathcal{F} \)), then (4) implies \( (A + G F(\sigma) M) P (A + G F(\sigma) M) - E^T P E < 0 \). It follows from [14] that \( (E, A + G F(\sigma) M) \) is regular and causal for any \( \sigma \in \mathcal{F} \).

The following lemma will be used in the sequel.

**Lemma 2.1.** (S-procedure lemma) Let \( \Omega_0(x) \) and \( \Omega_1(x) \) be two arbitrary quadratic forms over \( \mathbb{R}^n \); then \( \Omega_0(x) < 0 \) for all \( x \in \mathbb{R}^n - \{0\} \) satisfying \( \Omega_1(x) \leq 0 \) if and only if there exists a constant scalar \( \tau \geq 0 \) such that
\[ \Omega_0(x) - \tau \Omega_1(x) < 0 , \quad \forall x \in \mathbb{R}^n - \{0\} . \]

Based on the above S-procedure lemma, we have the following preliminary result in the sense of Definition 2.2.

**Lemma 2.2.** If system (1) is generalized quadratically stable, then for all tolerable perturbations (3), the solution \( x(k) \) of system (1) is globally exponentially stable.

**Proof.** If there exist a symmetric matrix \( P \) and a positive definite matrix \( Q \) such that (4) holds, then choose the Lyapunov functional candidate as follows:
\[ V_k = x^T(k) E^T P E x(k) + \sum_{i=1}^{d} x^T(k-i) Q x(k-i) . \]
\[ (5) \]
Thus, the difference of \( V_k \) along system (1) yields
\[ V_{k+1} - V_k = \Delta_k < 0 , \quad \forall (x(k), x(k-d)) \in \mathbb{R}^n \times \mathbb{R}^n - \{0\} \]
\[ (6) \]
under constraint (3). If follows from Lemma 2.1 that there exists a scalar \( \tau \geq 0 \) such that
\[ \Gamma_{k,t} := V_{k+1} - V_k - \tau \left( \begin{array}{c} g^T g - (M x(k) + M_j x(k-d))^T \\ [M x(k) + M_j x(k-d)] \end{array} \right) \]
\[ < 0 , \quad \forall (x(k), x(k-d), g) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n - \{0\} . \]

It is clear that \( \tau > 0 \) in (7). In fact, if \( \tau = 0 \), then it is impossible for (7) to hold without any constraint on \( g \). In this case, let
\[ \Omega_1 := \left( \begin{array}{ccc} A^T P A - E^T P E + Q + \tau M^T M & A^T P A + \tau M^T M_j & A^T P G \\ A^T P A + \tau M^T M_j & A^T P A - Q + \tau M^T M_j & A^T P G \\ G^T P A & G^T P A_j & G^T P G - \tau I \end{array} \right) ; \]
\[ (8) \]
then, we have
\[ \Gamma_{k,t} = (x^T(k) - x^T(k-d)) g^T \Omega_1 \left( \begin{array}{c} x(k) \\ (x(k-d) \end{array} \right) . \]
\[ (9) \]
Thus, (7) implies that
\[ \Omega_1 < 0 . \]

Next we will prove the exponential stability of the system. Let \( \lambda_{\max}(\Omega_1) = \lambda_0; \) then \( \lambda_0 > 0 \), and for any \( x(k) \neq 0 \), we have
\[ \Gamma_{k,t} \leq \lambda_0 \left( \| x(k) \|^2 + \| x(k-d) \|^2 \right) + \| g[k, x(k), x(k-d)] \|^2 < 0 . \]
\[ (10) \]
It follows from constraints (3) and (11) that
\[ V_{k+1} - V_k < -\lambda_0 (\| x(k) \|^2 + \| x(k-d) \|^2) \]
\[ + \| g[k, x(k), x(k-d)] \|^2 ) \leq -\lambda_0 (\| x(k) \|^2 , \]
\[ (12) \]
which also implies that \( 0 \leq V_k \leq V_0 \) and \( V_{k+1} \leq V_k \) for all \( k \geq 0 \). Denote
\[ \lambda_m = \max \{ \lambda_{\max}(E^T P E), \lambda_{\min}(Q) \} ; \]
then, \( \lambda_m > 0 \). In this case, (5) implies that for any \( k \geq 0 \),
\[ V_k \leq \lambda_m \sum_{i=0}^{d} \| x(k-i) \|^2 . \]
\[ (13) \]
In addition, (12) and (13) imply that for any \( k \geq d \),
\[ V_{k+1} - V_{k-d} = (V_{k+1} - V_k) + (V_k - V_{k-1}) + \cdots + (V_{k-d} - V_{k-d}) \]
\[ < -\lambda_0 \sum_{i=0}^{d} \| x(k-i) \|^2 \]
\[ \leq -\frac{\lambda_0}{\lambda_w} V_k \]
\[ \leq -\frac{\lambda_0}{\lambda_w} V_{k+1}. \tag{14} \]

Let \( \beta = 1 + \frac{\lambda_0}{\lambda_w} \); then \( \beta > 1 \). Rewriting (14), we have
\[ V_{k+1} < \beta^{-1} V_{k-d}. \tag{15} \]

Note that for any positive integer \( k \), there exist non-negative integers \( s \) and \( t \) such that \( k = s(d+1) + t \), where \( 0 \leq t \leq d \). Then by using (15) repeatedly, we have
\[ V_{k} = V_{s(d+1)+t} < \beta^{-1} V_{(t-1)d+t} < \cdots < \beta^{-t} V_t \leq \beta^{-t} \beta^{-s} V_0, \tag{16} \]

where \( c = \frac{\beta^d}{\beta^t} > 0 \). In addition, (12) also implies that
\[ \| x(k) \|^2 < -\lambda_0^{-1} V_{k+1} + \lambda_0^{-1} V_k \leq \lambda_0^{-1} V_k. \tag{17} \]

Therefore, we have
\[ \| x(k) \|^2 \leq \lambda_0^{-1} \beta^{-d-k} V_0 \tag{18} \]

for any initial condition \( x(-d), x(-d+1), \ldots, x(0) \), and (18) implies that the solution of system (1) is exponentially stable. This completes the proof. \( \blacksquare \)

**Remark 2.3.** In the above proof, the standard Lyapunov stability theory cannot be applied directly. The reason is that the quadratic Lyapunov functional (5) is not positive definite and the discussed system is singular, which fail to satisfy the standard Lyapunov stability theory requiring that the system be non-singular, and the related Lyapunov function (or functional) be positive definite in order to show the stability.

**Remark 2.4.** From (18), the convergent rates of system state \( x(k) \) can be estimated as follows:
\[ \| x(k) \| \leq \frac{V_0}{\lambda_0} \exp \left[ -\frac{\ln \beta}{2(d+1)} k \right], \tag{19} \]

which implies that the convergent rate of \( x(k) \) is dependent on the system time-delay, \( d \). Usually, the smaller the time-delay, \( d \), is, the larger the convergent rate of \( x_k \) is. On the other hand, no exponential stability for DSS is discussed in [14,17].

The following result is fundamental to this paper, which presents a necessary and sufficient condition for generalized quadratic stability of system (1) in the sense of Definition 2.2.

**Theorem 2.1.** Singular system (1) is generalized quadratically stable if and only if there exist a symmetric matrix \( P \) and a positive definite matrix \( Q \) such that the following LMIs on \( P \) and \( Q \) are solvable:
\[ E^T P E \geq 0 \]
\[ \Omega_k := \left( \begin{array}{cc} \mathbf{A}^T \mathbf{P} - E^T P E + Q + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \\ \mathbf{A}^T \mathbf{P} + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \end{array} \right) < 0. \tag{20} \]

**Proof.**

**Sufficiency:** From (20) we have
\[ \Delta_k := \left( \begin{array}{cc} x^T(k) & x^T(k-d) \\ x^{T}(k-d) & g^T \end{array} \right) \mathbf{Q_k} \left( \begin{array}{cc} x(k) \\ g \end{array} \right) < 0, \]

\[ \forall (x(k), x(k-d), g) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{-} - \{0\}. \tag{21} \]

It follows from Lemma 2.1 that for all \( (x(k), x(k-d)) \neq 0 \), \( \Delta_k < 0 \) under constraint (3), which completes the proof of sufficiency.

**Necessity:** Similar to the proof of Lemma 2.2, we have that (10) holds for some \( \tau > 0 \). That is, there exist two symmetric matrices \( P, Q \) with \( Q > 0 \) such that
\[ \left( \begin{array}{cc} \mathbf{A}^T \mathbf{P} - E^T P E + Q + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \\ \mathbf{A}^T \mathbf{P} + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \end{array} \right) \mathbf{Q} < 0. \tag{22} \]

Let \( \hat{P} = \frac{1}{\tau} P \) and \( \hat{Q} = \frac{1}{\tau} Q \); then it follows from the Schur Complement Lemma that the LMIs in (20) hold if \( P, Q \) is replaced with \( \hat{P}, \hat{Q} \), respectively, which completes the proof. \( \blacksquare \)

In order to use the LMI Toolbox [23], nonstrict LMIs (20) must be converted into strict ones by means of the following theorem.

**Theorem 2.2.** Singular system (1) is generalized quadratically stable if and only if there exist two positive definite matrices \( \mathbf{X}, \mathbf{Q} \in \mathbb{R}^{n+q \times n} \) and symmetric matrix \( \mathbf{Y} \in \mathbb{R}^{(n+q) \times (n+q)} \) such that the following LMI on \( \mathbf{X}, \mathbf{Q}, \mathbf{Y} \) is solvable:
\[ \left( \begin{array}{cc} \mathbf{A}^T \mathbf{P} - E^T P E + \mathbf{Q} + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \\ \mathbf{A}^T \mathbf{P} + \mathbf{M}^T \mathbf{M} & \mathbf{A}^T \mathbf{P} + \mathbf{M} \mathbf{M}^T \mathbf{M} \end{array} \right) \mathbf{Q} < 0. \tag{23} \]

where \( \mathbf{P} = \mathbf{X} + \mathbf{E} \mathbf{Y} \mathbf{E}^T \), \( \mathbf{E} \in \mathbb{R}^{(n+q) \times n} \) is a full column rank matrix with \( \mathbf{E}^T \mathbf{E} = 0 \).
Proof.

Sufficiency: It is clear that the LMIs in (20) hold if \( P = X + E_1^T Y E_1 \) and (23) holds.

Necessity: If the LMIs in (20) hold, then the pair \((E, A)\) is regular and causal, so there exist two nonsingular matrices \(U\) and \(V\) such that

\[
UEV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad UAV = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n_r} \end{pmatrix}.
\] (24)

For convenience, decompose \(U\) as \(U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}\) with \(U_1 \in \mathbb{R}^{r \times r}\) and \(U_2 \in \mathbb{R}^{(n-r) \times r}\). Let \(U^{-T}PU^{-1} = \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}\), where \(P_1 \in \mathbb{R}^{r \times r}\), \(P_2 \in \mathbb{R}^{(n-r) \times r}\), and \(P_3 \in \mathbb{R}^{(n-r) \times (n-r)}\). It is easy to show from (20) that \(A_1^TP_1A_1 - P_1 < 0\). In addition, \(E_1^TP_3E_1 \geq 0\) implies that \(P_1 \geq 0\), so we have that \(P_1 > 0\). Let

\[
X = U^T \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix} U; \quad \text{then it follows from}
\]

the Schur Complement that \(X > 0\). Furthermore, \(U_2E = 0\), that is, \(U_2E = 0\), which implies that there exists a nonsingular matrix \(\Lambda \in \mathbb{R}^{(n-r) \times (n-r)}\) such that \(U_2 = \Lambda E_1\). Then let

\[
Y = \Lambda^T \begin{pmatrix} P_3 - P_2 P_1^{-1} P_2^T & \Lambda \end{pmatrix} \Lambda. \quad \text{Hence,}
\]

\[
P = U^T \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix} U = X + E_1^T Y E_1,
\] (25)

which completes the proof. \(\square\)

Remark 2.5. Since (1) is generalized quadratically stable for all perturbation, the exponential stability of the system

\[
Ex(k+1) = Ax(k) + A_d x(k-d) + Bu(k)
\] (26)

implies that \(\{z \mid \det(zE - A - z^{-d} A_d) = 0\} \subset D_{in}(0, 1)\), where \(D_{in}(0, 1)\) is the interior of the unit disk with the center at the origin. It follows from Lemma 1 in [19] and Theorem 2.1 that (26) is also regular and causal in the sense of Definition 3 in [19].

Remark 2.6. From Lemma 2.2 and Theorem 2.1, we can see that the global exponential stability for system (1) can be guaranteed by means of an LMI procedure. It is clear that system (1) with \(A_d = 0\) and \(G = 0\) is the same as that in [14]. Similarly, if \(A_d = 0\), \(G = M\) and nonlinear time-varying function \(g\) is chosen as a linear time-invariant uncertainty, that is, \(g = F(\sigma)N\), then system (1) is the same as that in [17]. Therefore Theorem 2.1 also can be regarded as an extension of Lemma 1 [17].

Remark 2.7. Theorem 2.1 also can be regarded as an extension of Theorem 1 [3], where a type of non-singular system, that is, system (1) with \(E = I, A_d = 0\), and \(G = 1\), was considered.

Remark 2.8. For LMIs (20), the software for semidefinite programming has been developed to solve the nonstrict LMIs [24].

III. STATE FEEDBACK

Consider the following singular control system:

\[
Ex(k+1) = Ax(k) + A_d x(k-d) + Bu(k)
\] (27)

\[+ Gf[k, x(k), x(k-d), u(k)],\]

where \(E, A\), and \(x\) are the same as those in (1). \(u(k) \in \mathbb{R}^m\) is the system input, and \(B \in \mathbb{R}^{n \times m}\) is a constant matrix. \(f = f[k, x(k), x(k-d), u(k)]\) is a time-varying vector-valued Lipschitz perturbation; that is, for all \((k, x(k), x(k-d), u(k)), (k, \tilde{x}(k), \tilde{x}(k-d), u(k)) \in Z \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m\),

\[
|| f[k, x(k), x(k-d), u(k)] - f[k, \tilde{x}(k), \tilde{x}(k-d), u(k)] || 
\]

\[\leq || M[x(k) - \tilde{x}(k)] + M_d[x(k-d) - \tilde{x}(k-d)] || ,
\] (28)

where \(M, M_d\), and \(N\) are constant matrices with appropriate dimensions. Then for all \([k, x(k), x(k-d), u(k)] \in Z \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m\), we have

\[
|| f[k, x(k), x(k-d), u(k)] || 
\]

\[\leq || Mx(k) + M_d x(k-d) + Nu(k) || .
\] (29)

Remark 3.1. System (27) contains both state and control input perturbation, which can be regarded as an extension of the singular system discussed in [17], where only structured time-invariant uncertainty was considered. That is, (27) with \(G = M\) and \(f(k, x, u) = F(\sigma)N\) is identical to system (1) in [17]. In addition, system (27) also can be regarded as an extension of non-singular system (14) discussed in [3].

In this paper, we construct the following state feedback controller:

\[
u(k) = Kx(k).
\] (30)

In this case, the closed-loop systems of (27) and (30) are as follows:

\[
Ex(k+1) = (A + BK) x(k) + A_d x(k-d)
\]

\[+ Gf[k, x(k), x(k-d), Kx(k)],
\] (31)

where

\[
|| f[k, x(k), x(k-d), Kx(k)] || 
\]

\[\leq || (M + NK)x(k) + M_d x(k-d) || .
\] (32)
If the closed-loop system (31) is generalized quadratically stable in the sense of Definition 2.2, then system (27) is said to be generalized quadratically stabilized via the static state feedback controller (30), and it is called a generalized quadratic stabilizing controller.

For the sake of simplicity, let

\[
\Gamma := \begin{pmatrix}
A_f^T P A_d - Q + M_d^T M_d & A_f^T P G \\
G^T P A_d & G^T P G - I
\end{pmatrix},
\]

\[
\Gamma_1 := \begin{pmatrix}
A_f^T P A + M_f^T M & A_f^T P B + M_f^T N \\
G^T P A & G^T P B,
\end{pmatrix}, \quad \Gamma_2 := \begin{pmatrix}
A_f^T P B + M_f^T N & G^T P B,
\end{pmatrix},
\]

\[
\Psi_0 := B^T P B + N^T N + \Gamma_1 \Gamma_2^{-1} \Gamma_2,
\]

\[
\Psi_1 := A_f^T P B + M_f^T N + \Gamma_1 \Gamma_2^{-1} \Gamma_2,
\]

\[
\Psi_2 := A_f^T P A + M_f^T M - E^T P E + Q + \Gamma_2 \Gamma_1^{-1} \Gamma_2.
\]

The following theorem presents a necessary and sufficient condition under which system (27) is generalized quadratically stabilizable via (30), and it also provides a control gain design by means of parameterization.

**Theorem 3.1.** Singular system (27) with constraint (29) is generalized quadratically stabilizable if and only if there exist a symmetric matrix \(P\) and a positive definite matrix \(Q\) such that

\[
E^T P E \geq 0, \quad \Gamma < 0,
\]

and if there exist matrices \(\Phi_1, \Phi_2,\) and a scalar \(\epsilon > 0\) such that \(\Psi_0 + \epsilon I\) is nonsingular and

\[
\Phi_1^T \Phi_1 - \Phi_2^T \Phi_2 - \Psi_1 (\Psi_0 + \epsilon I)^{-1} \Psi_1^T + \Psi_2 < 0.
\]

In this case, the corresponding controller gain \(K\) is as follows:

\[
K = D \begin{pmatrix}
\Lambda_1^+ \Phi_1 \\
\Lambda_2^+ \Phi_2
\end{pmatrix},
\]

where parameters \(\Phi_1\) and \(\Phi_2\) satisfy (35). \(\Lambda_1\) and \(\Lambda_2\) are defined based on the decomposition of \(\Psi_0 + \epsilon I\) as follows:

\[
D^T (\Psi_0 + \epsilon I) D = \begin{pmatrix}
\Lambda_1 & 0 \\
0 & -\Lambda_2
\end{pmatrix},
\]

where \(D\) is orthogonal and \(\Lambda_1\) and \(\Lambda_2\) are positive and diagonal.

**Proof.** From Theorem 2.1, we have that the generalized quadratic stability of the closed-loop systems (31) with constraint (32) is equivalent to saying that there exist a symmetric matrix \(P\) and a positive definite matrix \(Q\) such that \(E^T P E \geq 0\) and

\[
\begin{pmatrix}
A_f^T P A_d - E^T P E + Q + M_d^T M_d \\
A_f^T P A_d + M_d^T M_d \\
A_f^T P A_d - Q + M_d^T M_d \\
G^T P A_d
\end{pmatrix} < 0,
\]

where \(A_e = A + BK\) and \(M_e = M + NK\). Using the Schur Complement Lemma again, we have that (38) is equivalent to \(\Gamma > 0\) and

\[
\begin{pmatrix}
A_f^T P A_d - E^T P E + Q + M_d^T M_d \\
A_f^T P A_d + M_d^T M_d \\
A_f^T P A_d - Q + M_d^T M_d \\
G^T P A_d
\end{pmatrix} < 0.
\]

After some manipulations, we have that (39) is equivalent to

\[
K^T \Psi_0 K + K^T \Psi_1^T + \Psi_1 K + \Psi_2 < 0,
\]

where parameters \(\Psi_i (i = 0, 1, 2)\) are defined in (33) and are well-defined for the case where \(\Gamma\) is nonsingular.

Furthermore, (40) is equivalent to saying that there exists a sufficiently small \(\epsilon > 0\) such that \(\Psi_0 + \epsilon I\) is nonsingular and

\[
K^T (\Psi_0 + \epsilon I) K + K^T \Psi_1^T + \Psi_1 K + \Psi_2 < 0.
\]

For symmetric matrix \(\Psi_0 + \epsilon I\), there exists an orthogonal matrix \(D\) such that

\[
D^T (\Psi_0 + \epsilon I) D = \begin{pmatrix}
\Lambda_1 & 0 \\
0 & -\Lambda_2
\end{pmatrix},
\]

where \(\Lambda_1\) and \(\Lambda_2\) are positive and diagonal. Let

\[
K = D \begin{pmatrix}
K_1 \\
K_2
\end{pmatrix},
\]

Then (41) is equivalent to the following matrix inequality:

\[
\Pi := \begin{bmatrix}
\Lambda_1 K_1 + (I - 0) D^T \Psi_1^T \\
\Lambda_2 K_2 - (0 I) D^T \Psi_1^T
\end{bmatrix} \begin{bmatrix}
\Lambda_1 K_1 + (I - 0) D^T \Psi_1^T \\
\Lambda_2 K_2 - (0 I) D^T \Psi_1^T
\end{bmatrix}^T < 0.
\]

Let

\[
\Phi_1 := \Lambda_1^+ K_1 + \Lambda_1^+ (I - 0) D^T \Psi_1^T, \\
\Phi_2 := \Lambda_2^+ K_2 - \Lambda_2^+ (0 I) D^T \Psi_1^T,
\]

then (44) is equivalent to (35). In this case, the controller gain can be obtained from (43) and (45), and is identical to
Remark 3.2. Using Theorem 2.1 and reasoning similar to that in the proof of Theorem 3.1, Theorem 3.1 can be extended to the case in which control law \( u(k) = Kx(t) + K_d x_d(k - d) \) instead of (30). In order to derive both \( K \) and \( K_d \), we need to obtain an inequality similar to (40) by replacing \( K \) with \((K K_d)\).

Remark 3.3. Robust D-stability for unforced linear time-invariant uncertain discrete singular system with time-delay was discussed in [19], Theorem 3.1 presents a control design for a more general system (27). It is worth pointing out that the approach developed here based on the S-procedure is quite different from the robust pole location approach was discussed in [19]. Theorem 3.1 presents a control design procedure in this paper offers some freedom, which can be used to improve other aspects of the system performance, such as tracking and \( H_\infty \) performance, which will be one of our future research topics.

Theorem 3.2. Suppose \( A_d = 0 \) and \( M_d = 0 \) in singular system (27) with constraint (29). Then (27) is generalized quadratically stabilizable via (30) if and only if there exists a symmetric matrix \( P \) satisfying

\[
E^T P E \succeq 0, \quad G^T P G - I < 0, \quad (47)
\]

and at the same time, there exist matrix parameters \( \Phi_1, \Phi_2 \) and a scalar \( \epsilon > 0 \) such that \( S_e \) is nonsingular and

\[
\Phi_1^T \Phi_2 \geq \Phi_1^T \Phi_1 - (B^T S A + N^T M) Y S_e^{-1} (B^T S A + N^T M) + A^T S A - E^T P E + M^T M. \quad (48)
\]

In this case, controller gain \( K \) in (30) can be parameterized as follows:

\[
K = D \begin{pmatrix} \Lambda_1^{-1} \Phi_1 \\ \Lambda_2^{-1} \Phi_2 \end{pmatrix} - S_e^{-1} (B^T S A + N^T M), \quad (49)
\]

where \( \Lambda_1 \) and \( \Lambda_2 \) are defined based on the decomposition of \( S_e \) as follows:

\[
D^T S_e D = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix}, \quad (50)
\]

where \( D \) is orthogonal and \( \Lambda_1 \) and \( \Lambda_2 \) are positive diagonal.

The proof of Theorem 3.2 is similar to that of Theorem 3.1 and, hence, omitted here.

Remark 3.4. Theorem 3.2 presents a necessary and sufficient condition for generalized quadratic stabilization without any constraint on the system matrices. This is in contrast to the results presented in [14,17], where an additional assumption about the system matrices was necessary (see assumptions \( \text{rank}(E, M) = \text{rank}(E) \) and \( \text{rank}(E, B) = \text{rank}(E) \) in [14,17], respectively). In addition, only linear time-invariant uncertainty was considered in the above works. Therefore, Theorem 3.2 is less conservative than the main results given in [14,17].

Remark 3.5. From the choices of parameter matrices \( \Phi_1 \) and \( \Phi_2 \) in the control gain design of Theorems 3.1 and 3.2, we can see that the control design procedure in this paper presents a necessary and sufficient conditions; however, they are usually very hard to use to search for a solution of the control gain efficiently if the system order is relatively high. Fortunately, we can transform the problem into an LMI, which can be solved efficiently by means of interior-point optimization algorithms [23].

Before we present the main results in this section, we give the following key lemma.

Lemma 4.1. Closed-loop system (31) with (32) is generalized quadratically stable if and only if there exist a symmetric matrix \( P \in \mathbb{R}^{n \times n} \) and a positive definite matrix \( Q \in \mathbb{R}^{m \times m} \), and matrices \( X \in \mathbb{R}^{m \times n} \) and \( Y \in \mathbb{R}^{n \times n} \) such that the following LMI is solvable:

\[
E^T P E \succeq 0, \quad \begin{pmatrix} A^T X^T + X A + \Lambda & \tilde{A}^T Y^T - X \\ Y \tilde{A} - X^T & P - Y - Y^T \end{pmatrix} < 0, \quad (51)
\]

where \( \tilde{A} = (A_e A_d G) \), and
\[
\Lambda = \begin{pmatrix}
-E^T P E + Q + M_d^T M_d & M_d^T M_d & 0 \\
-M_d^T M_d & -Q + M_d^T M_d & 0 \\
0 & 0 & -I
\end{pmatrix}.
\] (52)

**Proof.** First, we have the following matrix equality:

\[
\begin{pmatrix}
I & 0 \\
A & I
\end{pmatrix}
\begin{pmatrix}
\tilde{A}^T X^T + X \tilde{A} + \Lambda \\
Y \tilde{A} - X^T \\
P - Y - Y^T
\end{pmatrix}
\begin{pmatrix}
I \\
A
\end{pmatrix} = \begin{pmatrix}
\tilde{A}^T \tilde{P} \tilde{A} + \Lambda \\
\tilde{P} \tilde{A} - \tilde{A}^T Y - X \\
P \tilde{A} - Y^T \tilde{A} - X^T \\
P - Y - Y^T
\end{pmatrix} < 0,
\] (53)

If (51) holds, then it follows from (53) that \( \tilde{A}^T \tilde{P} \tilde{A} + \Lambda < 0 \). Furthermore, we have that \( \tilde{A}^T \tilde{P} \tilde{A} + \Lambda < 0 \) is equivalent to (20), where \( A \) is replaced with \( A_a \), which completes the proof of sufficiency.

In addition, if there exist a symmetric matrix \( P \) and a positive definite matrix \( Q \) such that \( E^T P E \geq 0 \) and \( \tilde{A}^T \tilde{P} \tilde{A} + \Lambda < 0 \), and if we choose \( Y = P + I \) and \( X = \tilde{A}^T P - \tilde{A} Y \), then we have

\[
\begin{pmatrix}
\tilde{A}^T \tilde{P} \tilde{A} + \Lambda \\
\tilde{P} \tilde{A} - \tilde{A}^T Y - X \\
P \tilde{A} - Y^T \tilde{A} - X^T \\
P - Y - Y^T
\end{pmatrix} < 0,
\] (54)

which completes the proof of necessity. ■

We will now present a control design based on LMIs and linear matrix equalities (LMEs). Without loss of generality, suppose that \( B \) has full column rank in this section; that is, rank(\( B \)) = \( m \). In fact, if \( B \) is not full column rank, then there exists a matrix \( T_0 \) such that \( \tilde{B} = B T_0 \) has full column rank. Let \( u(k) = T_0 x(k) \); then \( Bu(k) = \tilde{B} v(k) \), which implies that we can convert the problem into that of finding the auxiliary control input \( v(k) \) with full column rank matrix \( \tilde{B} \).

**Theorem 4.1.** Suppose \( N = 0 \); then singular system (27) with constraint (29) is generalized quadratically stabilizable if there exist a symmetric matrix \( P \in \mathbb{R}^{m \times m} \), a positive definite matrix \( Q \in \mathbb{R}^{m \times m} \), matrices \( X_i \in \mathbb{R}^{m \times m} \) (\( i = 1, 2, 3 \)), \( W \in \mathbb{R}^{m \times n} \) and \( Z \in \mathbb{R}^{m \times n} \), and nonsingular matrix \( Y \in \mathbb{R}^{m \times m} \) such that the following LMIs and LMEs are solvable:

\[
E^T P E \geq 0
\]

\[
\begin{pmatrix}
\Sigma_1 + \Sigma_1^T + \Lambda & \Sigma_1^T - X \\
\Sigma_1 - X^T & P - Y - Y^T
\end{pmatrix} < 0,
\]

\[
X_i B = Y B = B W, \quad i = 1, 2, 3,
\] (55)

where

\[
X = \begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix}
X_1 + \begin{pmatrix} B & B \end{pmatrix} Z X A_d & X G \\
B & B
\end{pmatrix},
\]

\[
\Sigma_2 = (Y A + B Z & Y A_d & Y G),
\]

\[
\Lambda_0 = \begin{pmatrix}
-E^T P E + Q + M_d^T M & M_d^T M_d & 0 \\
-M_d^T M_d & -Q + M_d^T M_d & 0 \\
0 & 0 & -I
\end{pmatrix}.
\] (56)

In this case, the state feedback controller can be \( u(k) = K x(h) = W^T Z x(k) \).

**Proof.** If (55) is solvable, then the nonsingularity of \( Y \) and full column rank of \( B \) imply that

\[
m \geq \text{rank}(W) \geq \text{rank}(B W)
\]

\[
= \text{rank}(Y B) \geq \text{rank}(Y^{-1} Y B) = \text{rank}(B) = m.
\] (57)

That is, \( W \) is nonsingular. Let \( K = W^{-1} Z \); then from (51), we have that

\[
Y \tilde{A} = Y (A + B K A_d G) = (Y A + Y B K Y A_d) G = (Y A + B W K Y A_d) G = \Sigma_2.
\] (58)

Similarly, we have \( \tilde{X} A = \Sigma_1 \). Thus, (55) implies (51) if \( \Lambda \) is replaced with \( \Lambda_0 \), which completes the proof. ■

In order to use the Matlab LMI Toolbox [23], we have to transform the nonstrict LMI (55) with the matrix equation constraint into a strict LMI. To this end, we present the singular value decomposition of \( B \) as

\[
B = U \begin{pmatrix} B_0 \\
0
\end{pmatrix} V^T,
\] (59)

where \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{m \times m} \) are unitary matrices, and \( B_0 \in \mathbb{R}^{m \times m} \) is a diagonal matrix with positive diagonal elements arranged in decreasing order.

The following lemma presents an equivalent condition for the matrix equation \( Y B = X_i B = B W \) (\( i = 1, 2, 3 \)).

**Lemma 4.2.** For a given \( B \in \mathbb{R}^{m \times m} \) with rank (\( B \)) = \( m \), there exist matrices \( W \in \mathbb{R}^{m \times n} \) and \( Y, X_i \in \mathbb{R}^{m \times m} \) (\( i = 1, 2, 3 \)) such that \( Y B = X_i B = B W \) (\( i = 1, 2, 3 \)) if and only if there exist matrices \( Y_1 \in \mathbb{R}^{m \times m} \) and \( X_i \), \( Y_i \in \mathbb{R}^{m \times n} \) (\( i = 1, 2, 3 \)) such that

\[
X_i = U \begin{pmatrix} Y_1 \\
0
\end{pmatrix} X_i U^T, \quad Y = U \begin{pmatrix} Y_1 \\
0
\end{pmatrix} Y_i U^T.
\] (60)
\textbf{Proof.} If \( m = n \), then \( B \) is non-singular, and it is clear that the result is true. Without loss of generality, suppose \( m < n \).

From \( YB = XB = BW \) and the singular value decomposition of \( B \), that is, \( B = U_{0}B_{0}V^{T} \), we have that matrix equation \( YB = XB = BW \) is equivalent to

\[ YU_{0}B_{0}V^{T} = X_{i}U_{0}B_{0}V^{T} = U_{0}B_{0}V^{T}W. \quad (61) \]

Let \( Y = U(Y_{0}Y_{2})^{T} \) and \( X_{i} = U(X_{i0}X_{i})U^{T} \); then (61) implies that

\[ U(Y_{0}Y_{2})^{T}B_{0}V^{T} = U(X_{i0}X_{i})B_{0}V^{T}. \quad (62) \]

That is,

\[ Y_{0} = X_{i0} = (B_{0}V^{T}WV_{0}^{-1}). \quad (63) \]

Matrix equation (63) is solvable if and only if there exists a nonsingular matrix \( Y_{1} \in \mathbb{R}^{m \times n} \) such that \( Y_{0} = X_{i0} = \begin{pmatrix} Y_{1} \\ 0 \end{pmatrix} \), which completes the proof. \( \blacksquare \)

From Theorem 4.1 and Lemma 4.2, we have the following result, which includes a sufficient condition based on the LMI under which the state feedback controller (30) for system (27) can be obtained.

\textbf{Theorem 4.2.} Suppose \( N = 0 \); then singular system (27) with constraint (29) is generalized quadratically stabilizable if there exist two positive definite matrices, \( P, Q \in \mathbb{R}^{m \times m} \), a symmetric matrix, \( P_{0} \in \mathbb{R}^{(m-r) \times (m-r)} \), matrices \( Z \in \mathbb{R}^{m \times n} \) and \( X_{i}, Y_{i} \in \mathbb{R}^{(m-r) \times n} \) \((i = 1, 2, 3)\), and a nonsingular matrix \( Y_{1} \in \mathbb{R}^{m \times n} \) such that the following LMI problem is solvable:

\[ \begin{bmatrix} \Sigma_{1} + \Sigma_{1}^{T} + \Lambda_{1} & \Sigma_{2} - X^{T} \\ \Sigma_{2} - X^{T} & P_{0} + E_{1}^{T}P_{0}E_{1} - Y - Y^{T} \end{bmatrix} < 0, \quad (64) \]

where \( E_{1} \in \mathbb{R}^{(m-r) \times n} \) is a full column rank matrix with \( E_{1}E_{1}^{T} = 0 \), and

\[ X_{i} = U \begin{pmatrix} Y_{1} \\ 0 \end{pmatrix} X_{i0}U^{T}, \]

\[ Y = U \begin{pmatrix} Y_{1} \\ 0 \end{pmatrix} Y_{2}U^{T}, \quad X = \begin{pmatrix} X_{1} \\ X_{2} \\ X_{3} \end{pmatrix}. \]

\[ \Sigma_{1} = \begin{pmatrix} XA + \begin{pmatrix} B \\ B \end{pmatrix} Z & XA_{d} & XG \\ \begin{pmatrix} B \\ B \end{pmatrix} & M^{T}M_{d} & 0 \\ M_{d}^{T} & 0 & 0 \end{pmatrix}. \]

\[ \Sigma_{2} = \begin{pmatrix} (YA + BZ) & YA_{d} & YG \\ BZ & -Q & 0 \\ 0 & 0 & -I \end{pmatrix}. \quad (65) \]

In this case, the state feedback controller can be \( u(k) = Kx(k) = VB_{0}^{-1}Y_{1}^{-1}B_{0}V^{T}Zx_{k} \).

\textbf{Proof.} From (59) and noting that \( K = VB_{0}^{-1}Y_{1}^{-1}B_{0}V^{T}Z \), we have that

\[ YBK = U \begin{pmatrix} Y_{1} \\ 0 \end{pmatrix} U^{T}U \begin{pmatrix} B_{0} \\ 0 \end{pmatrix} V^{T}VB_{0}^{-1}Y_{1}^{-1}B_{0}V^{T}Z = BZ. \quad (66) \]

Similarly, we have that \( XB_{K} = BZ \). Choose \( P = P_{1} + E_{1}^{T}P_{0}E_{1} \) in (55); then by means of Theorem 4.1, the proof is completed. \( \blacksquare \)

\textbf{Remark 4.1.} In Theorem 4.1, without loss of generality, we assume that matrix \( Y \) is nonsingular. If \( Y \) is singular, let \( Y_{2} = Y + \varepsilon I_{1} \), \( X_{k} = X_{i} + \varepsilon I_{1} \) and \( W_{2} = W + \varepsilon I_{1} \) with scalar \( \varepsilon > 0 \); then \( Y_{2}, X_{k}, W_{2} \) \((i = 1, 2, 3)\) and \( W_{2} \) can simultaneously guarantee that LMI and LME (55) hold, and that \( Y_{2} \) is nonsingular for sufficiently small \( \varepsilon > 0 \). The case is similar for matrix \( Y_{1} \) in Theorem 4.2. In addition, for a similar reason, symmetric matrix \( P \) can be replaced with nonsingular symmetric matrix \( P \) in Theorems 2.1, 3.1, 3.2, 4.1, and 4.2.

\textbf{Remark 4.2.} Theorem 4.2 presents an LMI approach to constructing, using the Matlab LMI Toolbox, a state feedback controller which is efficient even for a high-order system (27) [23]. Although [14,17] presented necessary and sufficient conditions for the existence of a state feedback controller for linear (uncertain) DSS, no efficient algorithms were presented. The approach proposed in this paper can be used to efficiently design state feedback controllers for the linear (uncertain) DSS discussed in [14,17].

\textbf{Remark 4.3.} The main result in [3] is a special case of Theorem 4.2, in which linear perturbed system with no time-delay is addressed.

\textbf{V. NUMERICAL EXAMPLES}

\textbf{Example 5.1.} Consider system (1) with \( G = 0.1 I_{1} \) and
In order to determine $M, M_d$, consider

\[
\|g[k, x(k), x(k-d)]\| \leq \|\sin^2[x_1(k) + x_2(k-d)] + \sin^2[x_3(k) + x_4(k-d)] + \sin^2[x_5(k) + x_6(k-d)] + \sin^2[x_7(k) + x_8(k-d)] + \sin^2[x_9(k) + x_10(k-d)]\|.
\]

That is, $\|g[k, x(k), x(k-d)]\| \leq \|x(k) + x(k-d)\|$, which implies that $M = M_d = I_3$.

By means of Theorem 2.2, we have that the following optimal solution:

\[
X = \begin{bmatrix} 0.3424 & -0.2389 & -0.0762 \\ -0.2389 & 6.7038 \times 10^3 & 0.4410 \\ -0.0762 & 0.4410 & 0.1237 \end{bmatrix},
\]

\[
Q = \begin{bmatrix} 0.2100 & -0.0482 & -0.0005 \\ -0.0482 & 0.0734 & -0.0135 \\ -0.0005 & -0.0135 & 0.0613 \end{bmatrix},
\]

\[
Y = -1.2078 \times 10^5.
\]

In the numerical example presented in [19], the modulus matrix constraint for $d = 1$ and the special structure of linear uncertainties were provided to guarantee stability. Here, our results obtained in this example are more general for any $d > 0$ and for any perturbation with $\|g\| \leq \|x(k) + x(k-d)\|$, in which $g$ can be nonlinear functions or uncertainties. Even though $g$ is chosen to be a linear uncertainty, it may not satisfy the modulus matrix constraint discussed in [19], which implies that Theorem 2.2 offers a new approach to DSSDP.

**Example 5.2.** Consider the uncertain DSS discussed in [17], that is, systems (27) and (29) with

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0.5 \\ 0 & 0 & -0.4 \end{bmatrix},
A_d = \begin{bmatrix} 0.12 & 0.2 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.1 & -0.1 \end{bmatrix},
B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},
G = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},
M = (0.5 & 1),
M_d = N = (0 & 0).
\]

(70)

From (47), we can obtain a solution of $P = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$; then

\[
S_0 = \begin{bmatrix} 2.0408 & -1.0204 \\ -1.0204 & -0.4898 \end{bmatrix},
\]

which is non-singular. In this case, from (48) and (49) in Theorem 3.2, the control gain can be parameterized as follows:

\[
K = \begin{bmatrix} -1.4612 & 0.3069 \\ 0.5158 & 0.8694 \end{bmatrix} \big(\Phi_1 \big) + \begin{bmatrix} -1.0000 & -0.5000 \\ 2.0000 & 1.0000 \end{bmatrix},
\]

(71)

where $\Phi_i = (\Phi_1, \Phi_2)$ $(i = 1, 2)$ satisfies

\[
\Phi_1^T \Phi_2 > \Phi_1^T \Phi_1 + \begin{bmatrix} -0.7500 & 0.5000 \\ 0.5000 & 1.0000 \end{bmatrix}.
\]

(72)

For a special case, if $\Phi_1 = (0, 0)$ and $\Phi_2 = \mu(1, 2)$ with $\mu > 0.5$, then (72) holds. Thus, $K$ can be chosen as follows:

\[
K = \mu \begin{bmatrix} 0.3069 & 0.6138 \\ 0.8694 & 1.7388 \end{bmatrix} + \begin{bmatrix} -1.0000 & -0.5000 \\ 2.0000 & 1.0000 \end{bmatrix},
\]

(73)

Example 5.2 shows that the approach developed in this paper is also effective for solving the control problem for linear uncertain discrete singular systems without time-delay discussed in [17].

**Example 5.3.** Consider the DSSDP in (27) and (29) with $G = 0.2I_3, N = 0$ and

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},
A = \begin{bmatrix} 1.5 & 1 & 1 \\ 0 & 0.5 & -1.6 \\ 0 & 0.3 & 1.5 \end{bmatrix},
A_d = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0 & 0.1 & -0.1 \\ 0.1 & 0.1 & 0 \end{bmatrix},
B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
M = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0 & 0 & -0.2 \end{bmatrix},
M_d = \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}.
\]

(74)
It is easy to compute the eigenvalues of the nominal system as follows:

$$\sigma(E, A, A) = \{1.6232, -0.1232, 0.6791, -0.1097, 0.0639\}, \quad (75)$$

which implies that the nominal system is unstable [19].

We will now construct a stabilizing controller for the system. Choosing $E = (0 \ 1 \ 0)$ and by Theorem 4.2, we can obtain the following solutions [5]:

$$X_{11} = 10^5 \begin{pmatrix} -1.8871 & -0.7177 \\ -1.5078 & -1.6655 \\ -0.4588 & 3.3199 \end{pmatrix},$$

$$X_{21} = 10^5 \begin{pmatrix} -2.5161 & -0.9569 \\ -2.5161 & -0.9570 \\ -0.4996 & -2.3737 \end{pmatrix},$$

$$X_{31} = \begin{pmatrix} -1.2580 \times 10^4 & -0.4784 \times 10^4 \\ -0.4995 \times 10^4 & -2.3738 \times 10^4 \\ -0.0870 & 0.1990 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 1.2580 \times 10^5 & 0.4785 \times 10^5 \\ -0.4785 \times 10^5 & -2.3742 \times 10^5 \\ -0.0215 & 0.2185 \end{pmatrix},$$

$$Y_1 = 0.3560,$$

$$Z = (-0.5415 \quad -0.5257 \quad 0.9215),$$

$$P_1 = \begin{pmatrix} 0.0177 & 1.5794 & -0.0630 \\ 1.5794 & 1.2362 \times 10^6 & 0.5899 \\ -0.0630 & 0.5899 & 0.5058 \end{pmatrix},$$

$$P_6 = -2.9471 \times 10^6,$$

$$Q = \begin{pmatrix} 1.1190 & 1.3002 & -2.9070 \\ 1.3002 & 7.5982 & -20.3324 \\ -2.9070 & -20.3324 & 59.0584 \end{pmatrix}. \quad (76)$$

Therefore, the stabilizing controller is as follows:

$$u(k) = Kx(k) = (-1.5210 \quad -1.4766 \quad 2.5884) x(k). \quad (77)$$

The control problem in Example 5.3 was not solved in [19] or else where in the literature. This example shows that the approach developed in this paper presents a new way to solve the control problem for high-order discrete singular systems with perturbation or uncertainty.

VI. CONCLUSION

Generalized quadratic stability and generalized quadratic stabilization have been addressed in this paper for a type of DSSDP in which the perturbation is in the form of a quadratic constraint. Necessary and sufficient conditions for generalized quadratic stability and generalized quadratic stabilization have been obtained by using the S-procedure approach and matrix inequality technique. An explicit expression of the linear static state feedback laws has been obtained along with some free parameters. The proposed feedback laws guarantee that the resulting closed-loop systems are regular, causal and globally exponentially stable for all tolerable perturbations. In addition, a sufficient condition for the existence of state feedback laws has been obtained by means of an LMI approach, and the resulting stabilization feedback has also been obtained. This paper has also presented an efficient algorithm for computing the feedback control gain for DSS or DSSDP. The new approach presented in this paper improves and generalizes the design techniques presented previously in the literature.

REFERENCES


