ROBUST STABILITY AND STABILIZATION OF A CLASS OF
SINGULAR SYSTEMS WITH MULTIPLE TIME-VARYING DELAYS

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ABSTRACT

This paper deals with the problem of robust stability and robust stabilization for uncertain continuous singular systems with multiple time-varying delays. The parametric uncertainty is assumed to be norm bounded. The purpose of the robust stability problem is to give conditions such that the uncertain singular system is regular, impulse free, and stable for all admissible uncertainties. The purpose of the robust stabilization problem is to design a feedback control law such that the resulting closed-loop system is robustly stable. This problem is solved via generalized quadratic stability approach. A strict linear matrix inequality (LMI) design approach is developed. Finally, a numerical example is provided to demonstrate the application of the proposed method.

KeyWords: Singular systems, multiple time-varying delays, robust stability, robust stabilization, linear matrix inequality.

I. INTRODUCTION

In recent years, considerable efforts have been devoted to the analysis and synthesis of singular systems (known also as descriptor systems, semi-state systems, differential algebraic systems, generalized state-space systems, [1-3]). These systems arise naturally in various fields including electrical networks, robotics, social, biological, and automatic control. Alike the case of uncertain systems without delay, methods based on the concepts of quadratic stability and quadratic stabilizability have been shown to be effective in dealing with these problems in both continuous and discrete contexts [4,5].

On the other hand, control of singular systems has been extensively studied in the past years due to the fact that singular systems describe physical systems better than regular ones. Recently, robust stability and robust stabilization for uncertain singular systems have been considered.

The notions of quadratic stability and quadratic stabilization of regular systems have been extended [5,6]. It should be pointed out that the robust stability problem for singular systems is much more complicated than that for regular systems because it requires to consider not only stability robustness but also regularity and absence of impulses for continuous singular systems or causality for discrete singular systems [7-9]. Very recently, much attention has been paid to singular systems with time delay. For the continuous case, numerical methods for such systems were discussed in [10-12] for instance. To the best of our knowledge, there is few contributions dealing with the problems of robust stability or robust stabilization for singular systems, or neutral systems as a particular case, with multiple time-varying delays [13,14].

In this note, we address the problems of stability/stabilization and robust stability/stabilization for uncertain singular systems with multiple time-varying delays. The parameter uncertainties are time invariant and unknown, but norm bounded.

The paper is organized as follows. In section 2, the problem is stated and the required assumptions are formulated. Section 3 deals with the stability problem. In section 4 we address the robust stability problem and in section 5 we address the stabilization problem. Section 6 deals with the robust stabilization. In section 7 we present a numerical example to show the usefulness of the proposed results.

Notation: In the sequel Sym{ } is defined as
The singular delay system (1) is said to be stable if for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\dot{\phi}(t)$ satisfying $\sup_{t \in [0, \infty)} \|\dot{\phi}(t)\| \leq \delta(\varepsilon)$, the solution $x(t)$ of system (1) satisfies $\|x(t)\| \leq \varepsilon$ for $t \geq 0$. Furthermore
\[
\lim_{t \to \infty} x(t) = 0.
\]
The following three lemmas are very useful for our development in this paper.

**Lemma 2.2.** [16] Let $Z$, $E$, $F$, $R$, and $\Delta$ be matrices of appropriate dimensions. Assume that $Z$ is symmetric, $R$ is symmetric and positive definite and $\Delta' R \Delta \leq R$, then
\[
Z + E \Delta F + F^T \Delta^T E^T < 0
\]
if and only if there exists a scalar $\lambda > 0$ satisfying
\[
Z + E(\lambda R) E^T + F^T (\lambda R)^{-1} F < 0.
\]

**Lemma 2.3.** [17] Let $\Phi$, $a$, and $b$ be given matrices of appropriate dimension, then the two statement are equivalent
(a) the LMI
\[
\begin{bmatrix}
\Phi & a \\
0 & 0
\end{bmatrix} + \text{Sym} \begin{bmatrix} f \\ b^T - I \end{bmatrix} < 0
\]
is feasible in the variable $f$ and $g$.
(b) $\Phi$, $a$ and $b$ satisfy:
\[
\Phi + ab^T + ba^T < 0.
\]

**Lemma 2.4.** [17] Let $\Phi$, $a$, and $b$ be given matrices of appropriate dimension, then the two statements are equivalent
(a) the following LMI
\[
\begin{bmatrix}
\Phi & a + bG^T \\
0 & 0
\end{bmatrix} + \text{Sym} \begin{bmatrix} G \\ b^T - I \end{bmatrix} < 0
\]
is feasible in the variable $G$.
(b) $\Phi$, $a$, and $b$ satisfies
\[
\Phi < 0 \text{ and } \Phi + ab^T + ba^T < 0.
\]

In our subsequent developments we need the following lemma:

**Lemma 2.5.** [15] Consider the function $\varphi: \mathbb{R}^+ \to \mathbb{R}$, if $\varphi$ is bounded on $[0, \infty)$, that is, there exists a scalar $\alpha > 0$
such that $|\psi(t)| \leq \alpha$ for all $t \in [0, \infty)$, then $\psi$ is uniformly continuous on $[0, \infty)$.

**Lemma 2.6. (Barbalat’s Lemma)** Consider a function $\psi: R^+ \to R$. If $\psi$ is uniformly continuous and $\int_0^\infty \psi(s)ds < \infty$, then

$$\lim_{t \to \infty} \psi(t) = 0$$

In the rest of the paper the notation is standard unless it is specified otherwise. $L > 0$ ($L < 0$) means that the matrix $L$ is symmetric and positive-definite (symmetric and negative-definite).

**Assumption 2.2.** The delays $h_j(t)$, $j = 1, 2, ..., p$ are assumed to satisfy the following constraint:

$$0 \leq h_j(t) \leq \bar{h}_j \quad \text{and} \quad 0 \leq h_j(t) \leq \bar{T}_j < 1,$$

where $\bar{h}_j$ and $\bar{T}_j$, $j = 1, 2, ..., p$, are given positive constants. In addition we define the scalar $\bar{h} = \max(\bar{h}_1, ..., \bar{h}_p)$.

### III. STABILITY ANALYSIS

The goal of this section consists of establishing what will be the sufficient conditions that can be used to check whether or not the class of systems under study is stable. We consider the nominal system given by the following dynamics:

$$E \dot{x}_i = A_i x(t) + \sum_{j=1}^p A_j x(t-h_j(t))$$

or in a compact form as

$$E \dot{x}_i = A_i x(t) + A_d x_d(t)$$

where

$$A_d = [A_1 \quad A_2 \quad \ldots \quad A_p]$$

$$x_d(t) = [x(t-h_1)^T \quad x(t-h_2)^T \quad \ldots \quad x(t-h_p)^T]^T$$

The goal of this section consists of developing some conditions that can be used to check whether the class of systems under study is stable or not. The conditions we are looking for should depend on the upper bound of the delay as given in Assumption 2.2. The following theorem states such a result.

**Theorem 3.1.** Assume that the assumption 2.2 is satisfied. If there exist $F_i$, $i = 1, ..., 4$, $P$, $Q_j > 0$, $W_j > 0$, $Y_j$ and $Z_j$ for $j = 1, 2, ..., p$ such that the following hold:

$$E^T P = P^T E \geq 0$$

and

$$\begin{bmatrix} Z_j & Y_j \\ Y_j^T & E^T W_j E \end{bmatrix} \geq 0 \quad (9)$$

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P^T \\ -\Psi_3 & \Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W} \\ P & 0 & \mathcal{W} & 0 \end{bmatrix} + \text{Sym} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A_0 & A_d & 0 & -I \end{bmatrix} < 0 \quad (10)$$

are feasible with $A_d$ defined as in (6) and

$$W = \sum_{j=1}^p \bar{h}_j W_j$$

$$\Psi_1 = \sum_{j=1}^p (Q_j + (1-\bar{T}_j)(\bar{h}_j Z_j + Y_j + Y_j^T))$$

$$\Psi_2 = \text{diag}((1-\bar{T}_j) Q_1, \ldots, (1-\bar{T}_p) Q_p)$$

$$\Psi_3 = [(1-\bar{h}_1) Y_1 \quad (1-\bar{h}_2) Y_2 \quad \ldots \quad (1-\bar{h}_p) Y_p]$$

then system (5) is asymptotically stable.

**Proof.** Note that the regularity and the absence of impulses of the pair $(E, A_d)$ implies that there exist two invertible matrices $G$ and $H \in \mathbb{R}^{n \times n}$ such that [1]

$$\bar{E} = GEH = \begin{bmatrix} L_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_d = GA_d H = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_n \end{bmatrix}$$

(11)

$$\bar{A}_d = GA_d H = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

(12)

$$\bar{A}_d = \text{diag}(H, H, ..., H)$$

(13)

where $L_r \in \mathbb{R}^{nr \times nr}$ and $I_n \in \mathbb{R}^{n \times n}$ are identity matrices.

Using the same transformation as in (15), we get

$$\bar{P} = G^{-T} P H, \quad \bar{W}_j = G^{-T} W_j G^{-1}, \quad \bar{Q}_j = H^{-T} Q_j H, \quad \bar{Z}_j = H^T Z_j H, \quad \bar{Y}_j = H^T Y_j H$$

Taking account of (15) and using (8), we deduce that $\bar{P}_{11} = \bar{P}_{44} > 0$ and $\bar{P}_{12} = 0$.

Therefore $\bar{P}$ reduces to
\[
P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix} \quad (18)
\]

Now, let
\[
\zeta(t) = [\zeta_1^T (t) \quad \zeta_2^T (t)]^T = H_1^T x(t) \quad (19)
\]
where \( \zeta_1 \in \mathbb{R}^r \) and \( \zeta_2 \in \mathbb{R}^{n-r} \). Using (5), the singular delay system (1) can be decomposed as
\[
\begin{bmatrix}
\tilde{F}_1 \\
\tilde{F}_2
\end{bmatrix}
= \sum_{j=1}^{p} \begin{bmatrix} A_{1j} \zeta_1(t) - h_j(t) + A_{2j} \zeta_2(t - h_j(t)) \\
A_{1j} \zeta_1(t) - h_j(t) + A_{2j} \zeta_2(t - h_j(t)) \end{bmatrix}
\]

\[
0 = \zeta_2(t) + \sum_{j=1}^{p} \left[ A_{1j} \zeta_1(t - h_j(t)) + A_{2j} \zeta_2(t - h_j(t)) \right] \quad (20)
\]

It is easy to see that the stability of the singular delay system (1) is equivalent to that of the system (20). By virtue of this, we will prove that system (20) is stable.

Consider the Lyapunov function candidate:
\[
V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \quad (21)
\]
with
\[
\begin{align*}
V_1(\zeta_1) &= \zeta_1^T P_1^T E \zeta_1 \\
V_2(\zeta_2) &= \sum_{j=1}^{p} \int_{t-h_j(t)}^{t} \left( \zeta_2^T \tilde{W}_j^T \tilde{E} \zeta_2 \right) dz ds \\
V_3(\zeta_1) &= \sum_{j=1}^{p} \int_{t-h_j(t)}^{t} \left( \zeta_1^T \tilde{Q}_j \zeta_1 \right) dz \\
V_4(\zeta_1) &= \sum_{j=1}^{p} \int_{t-h_j(t)}^{t} \left( 1 - \tilde{T}_j \right) \left( \zeta_1^T \zeta_1 - \zeta_2^T \tilde{Q}_j \zeta_2 \right) dz
\end{align*}
\]

In the following we will compute and bound the first derivatives of the various functions \( V_i(\zeta) \) for \( i = 1, 2, 3, 4 \).

The first derivative of \( V_1 \) is given by:
\[
\dot{V}_1(\zeta_1) = \zeta_1^T \tilde{P}_1^T \tilde{E} \zeta_1 + \zeta_1^T \tilde{P}_1^T \tilde{E} \zeta_1(t)
= \zeta_1^T A_{1j} \zeta_1(t) + \zeta_1^T \tilde{A}_{ij} \tilde{P}_1^T \zeta_1(t) \\
+ \zeta_2^T \tilde{A}_{ij}^T \tilde{E} \zeta_2(t) + \zeta_2^T \tilde{A}_{ij} \tilde{P}_1^T \zeta_2(t) \\
= \zeta_1^T (\tilde{A}_{ij} \zeta_1 + \tilde{A}_{ij} \zeta_1(t)) + 2\zeta_2^T \tilde{A}_{ij} \zeta_2(t)
\]

The first derivative of \( V_2 \) is given by:
\[
\dot{V}_2(\zeta_2) = \sum_{j=1}^{p} \left( 1 - \tilde{h}_j(t) \right) \left( \zeta_2^T \tilde{W}_j^T \tilde{E} \zeta_2(t) - \zeta_2^T \tilde{W}_j^T \tilde{E} \zeta_2(t) ds \\
- \sum_{j=1}^{p} \left( \tilde{T}_j \right) \left( \zeta_2^T \tilde{Q}_j \zeta_2(t) - \zeta_2^T \tilde{Q}_j \zeta_2(t) \right) dz
\]

and we obtain after bounding the delays and their derivatives:
\[
\dot{V}_2(\zeta_2) \leq \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \zeta_2^T \tilde{W}_j^T \tilde{E} \zeta_2(t) \right) ds \\
- \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \zeta_2^T \tilde{Q}_j \zeta_2(t) - \zeta_2^T \tilde{Q}_j \zeta_2(t) \right) dz
\]

The derivative of \( V_3 \) is given by
\[
\dot{V}_3(\zeta_1) = \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \tilde{Q}_j \zeta_2(t) - \tilde{Q}_j \zeta_2(t) \right) \\
- \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \tilde{Q}_j \zeta_2(t) - \tilde{Q}_j \zeta_2(t) \right) dz
\]

which can be bounded as
\[
\dot{V}_3(\zeta_1) \leq \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \zeta_1(t) - \zeta_1(t) \right) - \sum_{j=1}^{p} \left( 1 - \tilde{T}_j \right) \left( \zeta_1(t) - \zeta_1(t) \right) dz
\]
The first derivatives of $V_4$ is given by

$$
\dot{V}_4(\zeta) = \sum_{j=1}^{n} (1 - \tau_j) \left[ \xi_j^\top(t) Z_j \xi(t) + \xi_j^\top(t) Y_j \xi(s) \right]
+ \xi_j^\top(s) F_j \xi(t) + \xi_j^\top(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

and using the fact that

$$
\int_{-h(t)}^{t} \zeta(s) ds = \zeta(t) - \zeta(t - h(t))
$$

we get

$$
\dot{V}_4(\zeta) = \sum_{j=1}^{n} (1 - \tau_j) \left[ \xi_j^\top(t) Z_j \xi(t) + 2 \xi_j^\top(t) Y_j \xi(t) \right]
- 2 \xi_j^\top(t) \left[ \int_{-h(t)}^{t} \zeta(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

$$
= \sum_{j=1}^{n} (1 - \tau_j) \left[ \xi_j^\top(t) Z_j \xi(t) + \xi_j^\top(t) Y_j + \xi_j^\top(t) \bigg( 0 \right) \xi(t) \right]
+ \sum_{j=1}^{n} (1 - \tau_j) \int_{-h(t)}^{t} \zeta(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

and using $\Psi_3$ we get

$$
\dot{V}_4(\zeta) \leq \zeta^\top(t) \left( [\overline{W}_j Z_j \xi(t) + \xi_j^\top(t) Y_j + \xi_j^\top(t) \bigg( 0 \right) \xi(t) \right)
- 2 \zeta^\top(t) \Psi_3 \xi(t)
$$

Finally, the first derivative of the Lyapunov functional candidate is given by the following expression

$$
\dot{V} \leq \zeta^\top(t) \left( \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \overline{A}_0 \overline{W} \overline{A}_0 \right) \zeta(t) + \zeta^\top(t) \overline{P} \zeta(t) + \zeta^\top(t) \overline{W} \overline{A}_0 \zeta(t)
$$

$$
+ \zeta^\top(t) \overline{W} \overline{A}_0 \zeta(t) + \zeta^\top(t) \overline{A}_0 \overline{W} \overline{A}_0 \zeta(t)
$$

$$
+ \zeta^\top(t) \overline{A}_0 \overline{W} \overline{A}_0 \zeta(t)
$$

$$
+ \sum_{j=1}^{n} \zeta_j^\top(t) \bigg( \int_{-h(j)}^{t} \zeta(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

$$
+ \sum_{j=1}^{n} \zeta_j^\top(t) \left( 1 - \tau_j \right) \xi_j^\top(t) Z_j \xi(t) + \xi_j^\top(t) Y_j + \xi_j^\top(t) \bigg( 0 \right) \xi(t) \right)
$$

$$
- 2 \zeta^\top(t) \Psi_3 \xi(t)
$$

$$
- 2 \zeta^\top(t) \Psi_3 \xi(t)
$$

$$
+ \sum_{j=1}^{n} (1 - \tau_j) \int_{-h(t)}^{t} \zeta(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

$$
- \sum_{j=1}^{n} (1 - \tau_j) \int_{-h(t)}^{t} \zeta(s) E^\top \overline{W}_j \overline{E} \xi(s) \right] ds
$$

which yields

$$
\dot{V} \leq \zeta(t)^\top \left( \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \overline{A}_0 \overline{W} \overline{A}_0 \right) \zeta(t)
$$

$$
+ \sum_{j=1}^{n} (1 - \tau_j) \left( \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \overline{A}_0 \overline{W} \overline{A}_0 \right) \zeta(t)
$$

$$
+ \sum_{j=1}^{n} (1 - \tau_j) \left( \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \overline{A}_0 \overline{W} \overline{A}_0 \right) \zeta(t)
$$

and using the expression of $\Psi_1$ or precisely $\Psi_1 = H^\top \Psi_1 H$ we get

$$
\dot{V} \leq \left[ \begin{array}{c} \zeta(t) \\ \zeta(t) \end{array} \right] M \left[ \begin{array}{c} \zeta(t) \\ \zeta(t) \end{array} \right]
$$

with

$$
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix}
$$

and

$$
M_{11} = \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \overline{A}_0 \overline{W} \overline{A}_0 + \Psi_1
$$

$$
M_{12} = \overline{A}_0 \overline{P} - \Psi_1 + \overline{A}_0 \overline{W} \overline{A}_0
$$

$$
M_{22} = \overline{A}_0 \overline{P} - \Psi_1 + \overline{A}_0 \overline{W} \overline{A}_0
$$

It comes then that satisfying the condition $M < 0$ is a sufficient condition to insure that $\dot{V}(\zeta) \leq 0$ is negative.

Notice that matrix $M$ can be expressed as follows:

$$
M = \begin{bmatrix} \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \Psi_1 & \overline{P}^\top \overline{A}_0 - \Psi_2 \\ (\overline{P}^\top \overline{A}_0 - \Psi_3)^\top & -\Psi_2 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \Psi_1 & \overline{P}^\top \overline{A}_0 - \Psi_2 \\ (\overline{P}^\top \overline{A}_0 - \Psi_3)^\top & -\Psi_2 \end{bmatrix}
$$

$$
< 0
$$

Using the Schur complement, we get

$$
\begin{bmatrix} \overline{A}_0 \overline{P} + \overline{P}^\top \overline{A}_0 + \Psi_1 & \overline{P}^\top \overline{A}_0 - \Psi_2 \\ (\overline{P}^\top \overline{A}_0 - \Psi_3)^\top & -\Psi_2 \end{bmatrix}
$$

$$
< 0
$$
which can be written as:

\[
\begin{bmatrix}
\tilde{\Psi}_1 & -\tilde{\Psi}_3 & 0 \\
-\tilde{\Psi}_3 & -\tilde{\Psi}_2 & 0 \\
0 & 0 & -\tilde{\Psi}
\end{bmatrix} + \text{Sym}\left[
\begin{bmatrix}
F^T \\
0 \\
[\mathcal{A}_0 \ -\mathcal{A}_d]
\end{bmatrix}\right] < 0
\]

According to lemma 2.3, the condition above holds if there exist \(\mathcal{F}_n, \, m = 1, \ldots, 4\), such that (10) is satisfied.

Note that we have

\[
\int_0^t \dot{V}(\zeta(s)) \, ds = V(\zeta(t)) - V(\zeta_0) \geq \lambda_1 \| \zeta(t) \|^2 - V(\zeta_0)
\]

with \(\lambda_1 = \lambda_{\max}\{\mathcal{P}_{11}\}\).

Further, taking account of (22) we get

\[
\int_0^t \dot{V}(\zeta(s)) \, ds \leq \lambda_{\max}\{|\mathcal{M}|\} \left[\int_0^t \| \zeta(t) \|^2 + \| \zeta(s) \|^2 \right] \, ds \\
\leq \lambda_{\max}\{|\mathcal{M}|\} \int_0^t \| \zeta(t) \|^2 \, ds
\]

bearing in mind that \(\mathcal{M} < 0\).

It comes then that (26) with (27), we get

\[
\lambda_1 \| \zeta(t) \|^2 + \lambda_2 \int_0^t \| \zeta(s) \|^2 \, ds \leq V(\zeta_0)
\]

with \(\lambda_2 = -\lambda_{\max}\{\mathcal{M}\}\).

Therefore

\[
\| \zeta(t) \|^2 \leq c_1\quad \text{and} \quad \int_0^t \| \zeta(s) \|^2 \, ds \leq c_2
\]

where

\[
c_1 = \frac{1}{\lambda_1} V(\zeta_0) \quad c_2 = \frac{1}{\lambda_2} V(\zeta_0)
\]

Thus, \(\| \zeta(t) \|^2\) is bounded and from system (20) we note that \(\frac{d}{dt} \| \zeta(t) \|^2\) is bounded too. By Lemma 2.5, we have that \(\| \zeta(t) \|^2\) is uniformly continuous. Therefore, with (29) in mind and using Lemma 2.5, we obtain

\[
\lim_{t \to \infty} \| \zeta(t) \| = 0
\]

Now, note that for any \(t > 0\), there exists a positive integer \(k\) such that \(\frac{h}{k} - \frac{h}{k} \leq t < k \frac{h}{k}\), we have

\[
\zeta_k(t) = -\sum_{j=1}^{k} \sum_{j=1}^{p} \left( -\mathcal{A}_{11} \right)^{j-1} \zeta_1 \left( t - j h_j(t) \right) \\
+ \sum_{j=1}^{p} \left( -\mathcal{A}_{22} \right)^j \zeta_2 \left( t - k h_j(t) \right)
\]

with \(\mathcal{h} = \max(h_1, \ldots, h_p)\).

Since \(\| \zeta(t) \|^2\) is bounded and if

\[
p(A_{22}) < 1 \quad \text{for} \quad j = 1, \ldots, p
\]

This, together with (32) and (31), implies that

\[
\lim_{t \to \infty} \| \zeta(t) \| = 0
\]

Thus, the system (20) is stable.

**Remark 3.1.** The results of Theorem 3.1 are only sufficient and therefore if these conditions are not verified we can’t claim that the system under study is not stable.

**IV. ROBUST STABILITY**

We assume that the system has uncertainties on all the matrices, i.e.: 

\[
E_x = [\mathcal{A}_0 + \mathcal{D}(t)N_0] x(t) + \sum_{j=1}^{p} [\mathcal{A}_j + \mathcal{D}(t)N_j] x \left( t - h_j(t) \right)
\]

with

\[
\tilde{\mathcal{A}}_0(t) = \mathcal{A}_0 + \mathcal{D}(t) N_0 \\
\tilde{\mathcal{A}}_j(t) = \mathcal{A}_j + \mathcal{D}(t) N_j
\]

and \(N_j = [N_1, \ldots, N_p]\) (\(\mathcal{A}_j\) and \(x_i\) are defined respectively by (6) and (7)).

Note that conditions (8) and (9) do not depend on the system matrices so they do not need to be adapted to the uncertain case. Besides, we have to replace \(\mathcal{A}_0\) and \(\mathcal{A}_d\) respectively by \(\tilde{\mathcal{A}}_0(t)\) and \(\tilde{\mathcal{A}}_d(t)\) in condition (10). Separating the nominal and the uncertain part and applying Lemma 2.2 and using the Schur complement we get a condition for the robust case which is stated by Theorem 4.1.

**Theorem 4.1.** Assume that assumptions 2.1 and 2.2 are satisfied. If there exist \(\mathcal{F}_i\) for \(i = 1, \ldots, 4\) and \(\mathcal{P}, \mathcal{Q}_j > 0, \mathcal{W}_j > 0, \mathcal{Y}_j, \mathcal{Z}_j\), for \(j = 1, 2, \ldots, p\), and a scalar \(\lambda > 0\) such that conditions (8), (9) and

\[
\begin{bmatrix}
\Psi_1 + f_{11} & -\Psi_1 + f_{12} & 0 & P^T & 0 \\
-\Psi_1 + f_{11} & -\Psi_2 + f_{22} & 0 & 0 & 0 \\
0 & 0 & -\mathcal{W} & \mathcal{W} & 0 \\
P & 0 & \mathcal{W} & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda \mathcal{R}
\end{bmatrix}< 0
\]

are feasible, with

\[
\mathcal{M}_n + \text{Sym} \begin{bmatrix}
\mathcal{F}_1 \\
\mathcal{F}_2 \\
\mathcal{F}_3 \\
\mathcal{F}_4 \\
0
\end{bmatrix} \begin{bmatrix}
\mathcal{A}_0 & \mathcal{A}_d & 0 & -I & D
\end{bmatrix} < 0
\]
and
\[ f_{11} = \lambda \, N_0^T \, R \, N_0 \]  
(35)
\[ f_{12} = \lambda \, N_0^T \, R_d \, N_d \]  
(36)
\[ f_{22} = \lambda \, N_d^T \, R_d \, N_d \]  
(37)
then the uncertain system under study is asymptotically stable for all admissible uncertainties.

**Proof.** We have to replace \( A_0 \) and \( A_d \) respectively by \( \tilde{A}_0(t) \) and \( \tilde{A}_d(t) \) in condition (10)

\[
\begin{bmatrix}
\Psi_1 & -\Psi_3 & 0 & P^T \\
-\Psi_3^T & -\Psi_2 & 0 & 0 \\
0 & 0 & -\mathcal{W} & \mathcal{W} \\
P & 0 & \mathcal{W} & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
\end{bmatrix}
+ \text{Sym}
\begin{bmatrix}
A_0 & A_d & 0 & -I \\
\end{bmatrix}
< 0
\]  
(38)
which gives after separating the uncertain terms

\[
\begin{bmatrix}
\Psi_1 & -\Psi_3 & 0 & P^T \\
-\Psi_3^T & -\Psi_2 & 0 & 0 \\
0 & 0 & -\mathcal{W} & \mathcal{W} \\
P & 0 & \mathcal{W} & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
\end{bmatrix}
+ \text{Sym}
\begin{bmatrix}
D \, \Delta(t) \, [N_0 & N_d & 0 & 0] \\
\end{bmatrix}
< 0
\]  

Applying Lemma 2.2 for expression (3) above, then there exists \( \lambda > 0 \) such as:

\[
\begin{bmatrix}
\Psi_1 & -\Psi_3 & 0 & P^T \\
-\Psi_3^T & -\Psi_2 & 0 & 0 \\
0 & 0 & -\mathcal{W} & \mathcal{W} \\
P & 0 & \mathcal{W} & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
\end{bmatrix}
+ \text{Sym}
\begin{bmatrix}
D \, (\lambda \, R)^{-1} \, D^{-1} & 0 & N_0^T & N_d^T \\
0 & (\lambda \, R)^{-1} & N_0^T & N_d^T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
< 0
\]  

The use of the Schur complement allows one to state that condition (34) holds with \( f_{11}, f_{12}, \) and \( f_{22} \) given, respectively, by (35), (36), and (37). Hence, the uncertain system under study is asymptotically stable for all admissible uncertainties.

**V. STABILIZABILITY ANALYSIS**

We consider the nominal system (1) given as

\[
E \, \dot{x}(t) = A_0 \, x(t) + A_d \, x_d(t) + B \, u(t)
\]
\[
y(t) = C \, x(t)
\]
\[
x(t) = \phi(t), \quad -\bar{h} \leq t \leq 0
\]  
(39)

We propose to synthesize a stabilizing output feedback controller. Notice that a dynamical output feedback could be obtained as a statical output feedback for an augmented system. In addition state feedback could be obtained by adopting \( C = I \), where \( I \) indicates the unit matrix. The controller is thus given by

\[
u_t = Ky_t
\]  
(40)

Substituting (40) in the plant model (39) and with \( A^f = A_0 + BK_C \) we get the closed-loop dynamic

\[
E \, \dot{x}(t) = (A_0 + BK_C) \, x(t) + A_d \, x_d(t) + B \, u(t)
\]
\[
y(t) = C \, x(t)
\]
\[
x(t) = \phi(t), \quad -\bar{h} \leq t \leq 0
\]  
with

\[
A^f = A_0 + BK_C = A_0 + BK_s + B(KC - K_d)
\]
\[
A^f_j = A_j + BK_d - BK_d_j
\]  
(41)

Matrices \( K_s \) and \( K_d \) are introduced here as additional initialization parameters. The introduction of these matrices allows the linearisation of the problem with respect to the controller gain \( K \). These matrices, in principle, must lead to large values of the upper value of the delay. Matrices \( K_s \) and \( K_d \) behave as if the delays are known. The objective of this study is to develop a new delay-dependent stabilization method that provides an output feedback controller \( u(t) = Ky(t) \) for a class of dynamical singular systems. The following theorem states such a result:

**Theorem 5.1.** Assume that assumption 2.2 is satisfied and there exist the matrices \( F_m \) for \( m = 1, 2, \ldots, 4, P, W_j > 0, Q_j > 0, Y_j, Z_j \) for \( j = 1, 2, \ldots, p, L_j, G_j, \) and a scalar \( \lambda > 0 \) such that the conditions (8), (9) and
are feasible, then system (39) is asymptotically stable and the output feedback control law is given by

$$K = G^{-1} L$$

Proof. We have to replace $A_0$ and $A_d$ respectively by $A'$ and $A_d'$ in condition (10)

$$
\begin{bmatrix}
\Psi_1 & -\Psi_3 & 0 & 0 & P^T & 0 \\
-\Psi_3 & \Psi_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma \mathcal{W} & \mathcal{W} & 0 & 0 \\
P & 0 & \mathcal{W} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

are feasible, then system (39) is asymptotically stable and the output feedback control law is given by

$$E \dot{x}(t) = \tilde{A}'(t) x(t) + \tilde{A}_d(t) x_b(t)$$

$$x(t) = \phi(t), \quad -\bar{h} \leq t \leq 0$$

with

$$\tilde{A}'(t) = A(t) + B(t) KC,$$

and

$$\tilde{A}_d(t) = A_d + D \Delta(t) N_d$$

For $K$ and $K_{d_{k}}$ of appropriate dimension, we have

$$\tilde{A}'(t) = A(t) + B(t) KC + B(t) K_{d_{k}} - B(t) K_s$$

$$\tilde{A}_d(t) = A_d(t) + B(t) K_{d_{k}} - B(t) K_{d_{k}}$$

Note that conditions (8) and (9) do not depend on the system matrices so they do not need to be adapted to the uncertain case. Besides, we have to replace $A'$ and $A_d'$ respectively by $\tilde{A}'(t)$ and $\tilde{A}_d(t)$ in condition (10) to get a condition for the robust case which is stated by Theorem 6.1.

Theorem 6.1. Assume that assumptions 2.1 and 2.2 are satisfied. If there exist the matrices $F_i$ for $i = 1, \ldots, 4$ and $P$, $Q_i > 0$, $W_i > 0$, $Y$, $Z$ for $i = 1, 2, \ldots, p$, $L$, $G$, and a scalar $\lambda > 0$ such that conditions (8), (9) and

$$
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & 0 & P^T & \Phi_{15} & 0 \\
\Phi_{12} & \Phi_{22} & 0 & 0 & \Phi_{25} & 0 \\
0 & 0 & -\gamma \mathcal{W} & \mathcal{W} & 0 & 0 \\
P & 0 & \mathcal{W} & 0 & 0 & 0 \\
\Phi_{15} & \Phi_{25} & 0 & 0 & \Phi_{55} & 0 \\
0 & 0 & 0 & 0 & -\lambda R & 0
\end{bmatrix}
$$

are feasible, then system (39) is asymptotically stable and the output feedback control law is given by

$$E \dot{x}(t) = \tilde{A}'(t) x(t) + \tilde{A}_d(t) x_b(t)$$

$$x(t) = \phi(t), \quad -\bar{h} \leq t \leq 0$$

VI. ROBUST STABILIZATION

In this section, we are concerned by robust stabilization of the uncertain system under the control law (40). Introducing the uncertainty terms in (4), the closed loop system equation becomes
\[ \Phi_{11} = \Psi_1 + (N_0 + N_b K_s) \lambda R (N_0 + N_b K_s) \]
\[ \Phi_{12} = -\Psi_2 + (N_d + N_b K_{d, s}) \lambda R (N_d + N_b K_{d, s}) \]
\[ \Phi_{13} = (LC - G K_s) + N_b^T \lambda R (N_0 + N_b K_s) \]
\[ \Phi_{23} = (-G K_{d, s}) + N_b^T \lambda R (N_d + N_b K_{d, s}) \]
\[ \Phi_{25} = -(G + G) \lambda R (N_d + N_b K_{d, s}) \]

Then system (43) is robustly asymptotically stabilizable by the output feedback controller

\[ K = G^{-1} L \]

**Proof.** We have to replace \( A \) and \( A_d \) respectively by \( A(t) \) and \( A_d(t) \) in condition (10) to get

\[
\begin{bmatrix}
\Psi_1 & -\Psi_2 & 0 & p^T & 0 \\
-\Psi_3 & -\Psi_1 & 0 & 0 & 0 \\
0 & 0 & -\lambda W & \lambda W & 0 \\
p & 0 & \lambda W & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
0
\end{bmatrix}
\begin{bmatrix}
A & A_d & 0 & -I & B \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
(LC - G K_s) & -G K_{d, s} & 0 & 0 & -G
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
0
\end{bmatrix}
\begin{bmatrix}
D(\lambda R)^{-1} D^T & F_1 \\
F_2 & F_3 \\
F_4 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
(N_0 + N_b K_s)^T \\
(N_d + N_b K_{d, s})^T \\
(N_d + N_b K_{d, s})^T \\
N_b^T \\
N_b^T
\end{bmatrix}
\]

The use of lemma 2.2 for the last expression followed by some algebraic manipulation allows us to obtain expression (44).

Then system (43) is robustly asymptotically stabilizable by the output feedback controller

\[ K = G^{-1} L \]

**VII. NUMERICAL EXAMPLE**

In this section we present a numerical example where we consider the problem of state feedback robust stabilization for the system whose data are given by

\[ A_b = \begin{bmatrix} 1.5 & 0.5 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0.5 \\ 0.3 & 0.5 & -1 \end{bmatrix} \]

\[ E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \]

The state is assumed to be available and then matrix \( C \) satisfies \( C = I \).

The admissible uncertainties are given by

\[ D = \begin{bmatrix} 0.2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ N_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

We apply theorem 6.1 for the overall system and for

\[
K_s = \begin{bmatrix} 8.9291 & 15.2471 & 28.9799 \\ -27.3996 & -61.0604 & -78.6984 \end{bmatrix}
\]
\[ K_d = \begin{bmatrix} 1.8620 & 1.3457 & 1.0246 \\ 0.6899 & -1.2852 & -0.3757 \end{bmatrix} \]

we find out that this system is asymptotically stabilizable with the state feedback gain

\[ K = \begin{bmatrix} -0.1738 & 0.3713 & -1.6077 \\ 0.0396 & -0.1690 & 0.5175 \end{bmatrix} \]

for \( \bar{h} = 2.7 s \) and \( 0 \leq \dot{h}(t) \leq 0.8 \).

Figure 1 shows the behaviour of system (45) for a maximum delay.

\[ \text{Fig. 1. Evolution of states } x_1, x_2, \text{ and } x_3 \text{ of system (45).} \]

**VIII. CONCLUSION**

This paper deals with a class of dynamical uncertain singular systems with multiple time-varying states delays. Delay-dependent sufficient conditions have been developed to check whether a system of this class is stable or unstable, an output feedback controller with consequent parameters has been used to stabilize the system. The LMI technique is used in all the development. A numerical example is given to illustrate the obtained results.

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