ROBUST $H_\infty$ CONTROL FOR UNCERTAIN
STOCHASTIC SYSTEMS WITH MARKOVIAN SWITCHING
AND TIME-VARYING DELAYS

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ABSTRACT

This paper is concerned with the problem of robust $H_\infty$ control for uncertain stochastic systems with Markovian jump parameters and time-varying state delays. A linear matrix inequality approach is developed and state feedback controllers are designed, which guarantee mean square asymptotic stability of the closed-loop system and a prescribed $H_\infty$ performance level for all modes and admissible uncertainties. A numerical example is provided to demonstrate the application of the proposed method.

KeyWords: $H_\infty$ control, linear matrix inequality, Markovian jump, robust control, stochastic systems.

1. INTRODUCTION

Much attention has focused on Markovian jump systems during recent decades. The essence of Markovian jump systems is that their structures vary in response to random changes [1]. This class of systems is used to model many practical dynamic systems that may experience abrupt changes, caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances [2]. A great number of results for Markovian jump systems have been reported in the literature. For example, concepts of stochastic controllability and stochastic stabilizability were developed for continuous-time Markovian jump linear systems in [2], while in [3], the stochastic stability properties and the relationship among various moment and sample path stability properties were investigated. Also, the robust linear quadratic problem for uncertain Markovian jump linear systems was considered in [4].

On the other hand, stochastic modeling has also come to play an important role in many branches of science and industry [5]. Recently, stochastic systems with Markovian jumping parameters were studied in [6], where some stabilization results were obtained under the assumption that the nominal closed-loop system without the Brownian motion is stable. It should be noted that no time delays were considered in [6]. This paper investigates the problem of robust $H_\infty$ control for uncertain stochastic systems with Markovian jumping parameters and time delays. Our aim is to design state feedback controllers such that the closed-loop system not only is mean square asymptotically stable but also achieves a prescribed $H_\infty$ performance level for all modes and all admissible uncertainties. A linear matrix inequality (LMI) approach is developed to solve this problem, and desired state feedback controllers are constructed after the given LMIs are shown to be feasible.

II. PROBLEM FORMULATION

Considering the following stochastic system with Markovian jump parameters ($\Sigma$):

$$
dx(t) = [\bar{A}(t, r(t)) x(t) + \bar{A}_1(t, r(t)) x(t - \tau(t)) + \bar{B}(t, r(t)) u(t) + B_1(r(t)) v(t)] dt
$$

where $x(t)$ is the state vector, $u(t)$ and $v(t)$ are the input and Brownian motion, respectively, $\tau(t)$ and $\bar{\tau}(t)$ are the state dependent delay and leakage delay, and $\Sigma$ is a Markov chain with transition probability matrix $P(t)$.

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where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the control input vector; $v(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L_2[0, \infty)$; $z(t) \in \mathbb{R}^q$ is the controlled output, and $w(t)$ is a standard one-dimensional Brownian motion satisfying $E\{dw(t)\} = 0$ and $E\{dw(t)^2\} = dt$. $\tau(t)$ is the time-varying state delay satisfying $0 < \tau(t) \leq \mu < \infty$ and $\tau(t) \leq h < 1$, and

$$
A(t, r(t)) = A(r(t)) + \Delta A(t, r(t)),
$$

$$
\bar{A}_q(t, r(t)) = A_q(r(t)) + \Delta A_q(t, r(t)),
$$

$$
B(t, r(t)) = B(r(t)) + \Delta B(t, r(t)),
$$

$$
\bar{G}(t, r(t)) = G(r(t)) + \Delta G(t, r(t)),
$$

$$
\bar{G}_q(t, r(t)) = G_q(r(t)) + \Delta G_q(t, r(t)).
$$

We assume that

$$
[\Delta A(t, i) A_q(t, i) \Delta B(t, i) G(t, i) G_q(t, i)]
= H_i F_i(t) \begin{bmatrix} E_{i1} & E_{i2} & E_{i3} & E_{i4} \\ \end{bmatrix}
$$

where $F_i(t)$ is an unknown real matrix satisfying $F_i(t) F_i(t)^T \leq I$, $i \in S$, $t > 0$. The uncertain matrices are said to be admissible if (4) holds. In system (5), the parameter $r(t)$ represents a continuous-time Markov process taking values in the set $S = \{1, 2, \ldots, N\}$ with transition probability given by

$$
P[r(t + \Delta) = j | r(t) = i] = \begin{cases} \lambda_{ij} + o(\Delta) & \text{if } i \neq j \\ \left[1 + \lambda_{ij} + o(\Delta) \right] \text{if } i = j \end{cases}
$$

where $\lambda_{ij}$ stands for the transition probability rate from mode $i$ to mode $j$ and satisfies

$$
\lambda_{ij} \geq 0, \quad \lambda_{ij} = - \sum_{j \in S, j \neq i} \lambda_{ij}.
$$

Then, the robust $H_\infty$ problem to be addressed can be formulated as follows: Given a scalar $\gamma > 0$, find a state feedback controller such that the closed-loop system is robustly stochastically stabilizable and under zero initial conditions,

$$
\|z(t)\|_{L_2} < \gamma \|v(t)\|_2
$$

holds for all nonzero $v(t) \in L_2[0, \infty)$.

### 3. MAIN RESULTS

The main result in this section is given in the following theorem.

**Theorem 1.** Given a scalar $\gamma > 0$ and the stochastic delay system (5), there exists a state feedback controller such that the closed-loop system is robustly stochastically and (7) is satisfied under zero initial conditions if there exist scalars $\rho_i > 0$, $\delta_i > 0$, matrices $P_i > 0$, $X_i > 0$ and $Y_i$ such that, for each $i \in S$,

$\begin{bmatrix}
\Omega_i & A_{id} & B_i & W_i^T & X_i E_{i1} & X_i G_i^T & \Delta_i \\
-[(1-h)I] R^T & 0 & \text{RE}_{i2} & \text{RE}_{i3} & \text{RG}_{i4} & 0 \\
B_i^T & 0 & -\gamma_i I & 0 & 0 & C_i^T \\
W_i & E_{i2} R & 0 & -\rho_i I & 0 & 0 \\
E_{i3} X_i & E_{i4} R & 0 & 0 & -\delta_i I & 0 \\
G_i X_i & G_i d_i & G_i e_i & 0 & 0 & \delta_i H_i H_i^T - X_i \\
\Delta_i^T & 0 & 0 & 0 & 0 & \Lambda_i 
\end{bmatrix} < 0,
$}

with

$$
\Omega_i = A_i X_i + X_i A_i^T + B_i Y_i + Y_i B_i^T + \lambda_{ij} X_i + \rho_i H_i H_i^T, \\
\Pi_i = \begin{bmatrix} \sqrt{\Lambda_{i1}} X_i & \sqrt{\Lambda_{i2}} X_i & \cdots & \sqrt{\Lambda_{iN}} X_i \end{bmatrix}, \\
\Theta_i = \text{diag}\{X_i, \Pi_i, J_i^T\}, \quad \Lambda_i = \text{diag}\{-R - \Theta_i - I\}, \\
W_i = E_{i2} X_{i2} + E_{i3} Y_i, \quad J_i = C_i X_i + D_i Y_i.
$$

In this case, the desired state feedback controllers can be chosen as follows:

$$
u_i(t) = K_i x(t) = Y_i X_i^{-1} x(t).
$$

**Proof.** Let $x_i = x(t + s), \quad t - \tau(t) \leq s \leq t_i$ and choose the following stochastic Lyapunov functional candidate:

$$
V(x_i, t) = x^T(t) P_i x(t) + \int_{t - \tau(t)}^{t} x^T(s) Q(x(s)) ds.
$$

Based on the definition of the weak infinitesimal operator $\bar{A}$ of the joint process, and denote $x(t - \tau(t))$ as $x_i(t)$, we have

$$
\bar{A} V(x, t) = 2 x^T(t) P_i \left(x(t) + \bar{A}_d x_i(t) + B_i v(t)\right) + \sum_{j \in S} \lambda_{ij} V(x_j, t)
$$

\begin{align*}
V(x, t) &= x^T(t) P_i x(t) + \int_{t - \tau(t)}^{t} x^T(s) Q(x(s)) ds \\
&= x^T(t) P_i x(t) + \int_{t - \tau(t)}^{t} x^T(s) Q(x(s)) ds + \int_{t - \tau(t)}^{t} x^T(s) \left[\bar{A}_d x_i(s) + B_i v(s)\right] ds \\
&\leq x^T(t) P_i x(t) + \int_{t - \tau(t)}^{t} x^T(s) \left[\bar{A}_d x_i(s) + B_i v(s)\right] ds \\
&\leq x^T(t) P_i x(t) + \int_{t - \tau(t)}^{t} x^T(s) \left[\bar{A}_d x_i(s) + B_i v(s)\right] ds.
\end{align*}


+ (\mathbf{G}_i x(t) + \mathbf{G}_{ii} x_i(t) + \mathbf{G}_v x(v(t)))^T \\
\cdot P_i (\mathbf{G}_i x(t) + \mathbf{G}_{ii} x_i(t) + \mathbf{G}_v x(v(t))) + x(t)^T Q x(t) \\
- (1-h) x^T(t) Q x(t), \quad (11)
\]

and when \( v(t) = 0 \),

\[ \mathbb{A} V(x_i,t) \leq [x^T(t) \quad x^T_i(t)] \Xi_i [x^T(t) \quad x^T_i(t)]^T, \]

where

\[ \Xi_i = \left[ P_i A_{ii} + (P_i A_{ii})^T + \sum_{j \in S} \lambda_j P_j + \rho_i P_i H_i H_i^T P_j P_i A_{ii} \right] \]

\[ + \left[ \begin{array}{c} E_{ic}^T \\ E_{si}^T \end{array} \right] \left[ \begin{array}{c} E_{ic} \\ E_{si} \end{array} \right] \]

\[ + \mathbf{G}_i^T \left( P_i - \delta_i H_i H_i^T \right)^{-1} \left[ \begin{array}{cc} \mathbf{G}_i & \mathbf{G}_{ii} \end{array} \right] \]

\[ + \delta_i^{-1} \left[ \begin{array}{c} E_{ic}^T \\ E_{si}^T \end{array} \right] \left[ \begin{array}{c} E_{ic} \\ E_{si} \end{array} \right], \quad (12) \]

\[ A_{ii} = \mathbf{A}_i + \mathbf{B}_i K_i, \quad E_{ic} = E_{ii} + E_{si} K_i. \quad (13) \]

By the Schur Complement, and letting \( X_i = P_i^{-1}, \ R = Q^{-1}, \) (8) implies that \( \Xi_i < 0 \). Therefore, \( \mathbb{A} V(x_i,t) < 0 \) for each \( i \in S \) and \( [x^T(t) \quad x^T_i(t)]^T \neq 0 \). Then, by [7], we have that the closed loop system is robustly stochastically stable. Next, we shall show that the closed-loop system with controllers given in (9) satisfies (7) for all nonzero \( v(t) \in \mathcal{L}_2(0,\infty) \). To this end, we set

\[ \Psi(t) = E \left[ \int_0^t \left( x^T(s) z(s) - \gamma^2 v^T(s) v(s) \right) ds \right], \quad (14) \]

where \( t > 0 \). Then, for all nonzero \( v(t) \in \mathcal{L}_2(0,\infty) \),

\[ \Psi(t) \leq E \left[ \int_0^t [x^T(s) \quad x^T_i(s) \quad v^T(s) \quad v^T(s)] \Xi(s) [x^T(s) \quad x^T_i(s) \quad v^T(s)] ds \right], \quad (15) \]

where

\[ \Xi(s) = \left[ \begin{array}{cccc} \mathbf{G}_i & \mathbf{G}_{ii} & \mathbf{G}_v & \mathbf{G}_v \\ \mathbf{G}_i & \mathbf{G}_{ii} & \mathbf{G}_v & \mathbf{G}_v \\ \sum_{j \in S} \lambda_j P_j & P_i A_{ii} & P_i B_i & P_i \\ \mathbf{A}_i & -h Q & 0 & -\gamma^2 I \end{array} \right] \]

\[ \mathbf{V} = \mathbf{Q} + \mathbf{A}_i^T P_i + P_i \mathbf{A}_i + \rho_i P_i H_i H_i^T P_i, \]

\[ \Delta \mathbf{A}_i = \Delta \mathbf{A}_i + \Delta \mathbf{B}_i K_i, \quad \Delta C_i = \Delta C_i + \Delta D_i K_i. \]

Now, pre- and post-multiplying (8) by means of \( \mathbf{Q} \) and then applying the Schur complement, we have \( \Gamma(s) \leq 0 \). Thus, \( \Psi(t) < 0 \) for all \( t > 0 \) which implies (7). This completes the proof. \[ \blacksquare \]

**IV. A NUMERICAL EXAMPLE**

Consider an uncertain linear stochastic system with two modes: for mode 1,

\[ \mathbf{A}_i = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -5 & -5 \\ -7 & -2 & 0 \end{bmatrix}, \quad \mathbf{A}_{ii} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \]

\[ \mathbf{B}_i = \begin{bmatrix} 5 & 1 \\ -9 & 10 \\ 2 & 0 \end{bmatrix}, \quad \mathbf{B}_{ii} = \begin{bmatrix} 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \]

\[ \mathbf{C}_i = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{G}_i = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \]

\[ \mathbf{G}_{ii} = \begin{bmatrix} -1 & 0.5 & 0 \\ 0.1 & -1 & 0 \\ 0.8 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_{ii} = \begin{bmatrix} 0.4 \\ 0.5 \\ 0 \end{bmatrix}, \]

\[ \mathbf{D}_i = \begin{bmatrix} 0.5 & 1 \\ 0.1 & 0 \end{bmatrix}; \]

for mode 2,
$A_2 = \begin{bmatrix} -4 & 0 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix},$

$B_2 = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 8 \end{bmatrix}, \quad B_{d2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$

$C_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$

$G_{d2} = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad G_{d2} = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.6 \end{bmatrix},$

$D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}.$

In this example, we assume that $\gamma = 0.8$ and for $i = 1, 2,$

$\Pi = \begin{bmatrix} -0.6 & 0.6 \\ 0.5 & -0.5 \end{bmatrix}, \quad H_i = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix},$

$E_{ii} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{xi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$

$E_{2i} = E_{i1} = E_{5i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$

Then, using the Matlab LMI control toolbox to solve the LMIs in (8), we have that desired state feedback control laws can be obtained as in (9) with

$K_1 = \begin{bmatrix} -3.1899 & 3.9737 & 0.2350 \\ -1.0746 & -2.7144 & -0.2801 \end{bmatrix},$

$K_2 = \begin{bmatrix} -27.7262 & -4.8615 & -2.8860 \\ 0.9945 & -5.3727 & -0.0358 \end{bmatrix}.$

For this case, the disturbance attenuation level of the controlled system is shown in Fig.1, which shows the effectiveness of the method proposed in the paper.

REFERENCES


