\textbf{H}_2\text{-OPTIMAL SAMPLED-DATA CONTROL FOR PLANTS WITH MULTIPLE INPUT AND OUTPUT DELAYS}

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\section*{ABSTRACT}

The sampled-data \textbf{H}_2-optimization problem for plants with multiple input and output delays is considered. An equivalent discrete-time system is constructed and numerical algorithm for computing matrices of its state-space realization is presented. It is proved that stability of this system is equivalent to stability of original sampled-data system. The proposed method can be applied to a wide class of digital control problems for continuous-time plants with multiple input and output delays.

\textit{KeyWords:} Sampled-data systems, optimization, delay, state-space methods, stability.

\section*{I. INTRODUCTION}

Time delay phenomenon is often unavoidable in models of real-world plants in chemical industry, biological systems, real-time and networked systems, combustion control, and other fields. Existing works mostly investigate purely continuous-time or purely discrete-time systems with delay, and great progress was achieved in this field as regards stability analysis and robust control design [1-8]. Nevertheless, sampled-data time-delay systems, i.e., systems with continuous-time plant and digital controller, received much less attention. Most papers dealing with this topic investigate systems with a single input delay [9-13]. Discretization of continuous-time systems with various delays was studied in [14,15], state-feedback \textbf{H}_\infty-control for sampled-data systems with state delays was considered in [16].

The \textbf{H}_2-problem for sampled-data systems without delay is a well-studied topic [17-20]. A method to handle time-delayed plants was presented for a special class of systems in [21], but it was assumed that the plant itself is stable. A hybrid state-space solution to the \textbf{H}_2-problem for the standard sampled-data system with a single input delay was given in [22]. Now the most popular methods of direct sampled-data control design are known as “lifting” [18,20,23] and “FR-operator” approach [19], but so far they have not been extended to cover time-delayed systems. As was shown in [16,24,25], application of the lifting technique to such problems is at least problematic.

An alternative frequency-domain approach to sampled-data control design, which is based on the parametric transfer function concept, was proposed in [26]. As was shown in [26-28], it can easily be applied to SISO time-delay systems, and the presence of delays causes no difficulties. Moreover, this approach was successfully used for solving some sampled-data preview control problems that can be reduced to equivalent schemes with delay units [29,30]. Unfortunately, frequency-domain techniques for MIMO systems are not fairly reliable numerically, therefore, it is desirable to develop relevant state-space methods.

In the present paper we consider the so-called simple \textbf{H}_2-problem for sampled-data systems [21] with multiple input and output delays. It is well known that problems of this class can often be reduced, using either state-space or frequency-domain approach, to an equivalent discrete-time problem, which is solvable by means of standard algorithms. Using the classical approach, we propose a method to construct such an equivalent state-space model for a sampled-data system with multiple input and output delays. Our basic investigation technique is closely related to that of [22], but we specially consider the case of multiple delays and present computationally convenient numerical procedure.

The paper is organized as follows. The \textbf{H}_2-problem
for the standard sampled-data system with four scalar delays is formulated in Sec. 2. In Sec. 3 and 4 we perform an equivalent discretization of the plant and cost function, respectively. A complete equivalent discrete state-space model is given in Sec. 5, where we also prove the equivalence property with respect to stability and cost function. In Sec. 6 we generalize the main results onto sampled-data systems with multiple rather than scalar delays. Computational aspects of the equivalent discretization procedure are discussed in Sec 7. The results are illustrated by a numerical example in Sec. 8, where we solve a problem that was previously solvable only by the frequency-domain approach. The paper is finalized by conclusions in Sec. 9.

II. STATEMENT OF THE PROBLEM

Consider the standard sampled-data system shown in Fig. 1, where \( w, z, y, \) and \( u \) are vector signals called input, output, measurement and control, respectively. Discrete-time signals are denoted by dashed lines. Continuous-time plant is given by the following state equations:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B_1 u(t) + B_2 u(t - \tau) , \\
y(t) &= C_1 x(t - \tau_1) + D u(t - \tau_2) , \\
\end{align*}
\]

(1)

\[
\begin{align*}
\dot{y}(t) &= C_2 x(t - \tau) , \\
\end{align*}
\]

where \( A, B_1, B_2, C_1, C_2, \) and \( D \) are constant matrices of compatible dimensions; \( \tau_1, \tau_2, \tau_1, \) and \( \tau_2 \) are nonnegative delay intervals. This simplest model with four scalar delays makes it possible to demonstrate first the main ideas of the proposed approach. The results obtained below will then be extended onto the general case in Sec. 6.

Measured signal \( y(t) \) is sampled with period \( T \) so that only the values \( y_k = y(kT) \) (for integer \( k \)) are accessible for control. The block \( K \) denotes a linear digital control law described by the following difference equations:

\[
\begin{align*}
\xi_{k+1} &= A_k \xi_k + B_k y_k , \\
v_k &= C_k \xi_k + D_k y_k , \\
\end{align*}
\]

(2)

where \( \xi_k \) denotes controller state vector, and \( A_k, B_k, C_k, \) and \( D_k \) are constant matrices of compatible dimensions. For simplicity, we assume that the hold circuit \( H \) uses only the latest value of discrete control signal \( v_k \) so that

\[
u(t) = h(t - kT) v_k , \quad kT \leq t < (k+1) T ,
\]

(3)

where \( h(t) \) is a function defined over \([0,T]\). Throughout the paper we suppose that the following assumptions hold:

A1: The plant is stabilizable, i.e., there exists a discrete controller \( K \) such that the system in Fig. 1 is internally stable.

A2. Each element of the matrix function \( h(t) \) is bounded on \([0,T]\), has finitely many points of discontinuity and bounded derivative inside each continuity interval [31].

These two assumptions are fairly standard. Stabilizability of such a time-delayed system in terms of initial plant parameters is a special topic and will be considered elsewhere. Due to this reason, assumption A1 is formulated in an indirect manner. To check A1, one has to check stabilizability of the equivalent discrete system derived below.

In the present paper we investigate the so-called simple \( H_2 \)-optimization problem for the sampled-data system in Fig. 1 [20,21]. Let \( \chi \) be the dimension of \( w \) and \( \{ e_i \} (i = 1, \ldots, \chi) \) be the standard basis in \( \mathbb{R}^\chi \). Let \( \zeta(t) \) be the response of the sampled-data system to the input \( w(t) = \delta(t) e_i \), where \( \delta(t) \) is a unit impulse (Dirac delta-function). Then, in analogy with the continuous-time case, the cost function can be chosen as

\[
J = \left( \sum_{i=1}^{\chi} \int_0^T \zeta_i'(t) \zeta_i(t) \, dt \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{\chi} \| \zeta_i \|^2 \right)^{\frac{1}{2}} ,
\]

(4)

where prime denotes transposition and \( \| \zeta_i \| \) is the \( L_2 \)-norm of the output. The problem of minimizing \( J \) over the set of stabilizing controllers is called the simple \( H_2 \)-problem for sampled-data systems [20].

As was already mentioned, most of existing direct optimization methods for sampled-data systems reduce original hybrid problem to a similar problem for an equivalent discrete-time system (model) shown in Fig. 2. The generalized discrete-time plant \( \hat{G} \) is described by the state-space equations

\[
\begin{align*}
\hat{\xi}_{k+1} &= \hat{A} \hat{\xi}_k + \hat{B}_1 \hat{w}_k + \hat{B}_2 v_k , \\
\hat{z}_k &= \hat{C}_1 \hat{\xi}_k + \hat{D} v_k , \\
\end{align*}
\]

(5)

where \( \hat{\xi}_k, \hat{w}_k, \) and \( \hat{z}_k \) are new state, input and output vectors, while \( \hat{A}, \hat{B}_1, \hat{B}_2, \hat{C}_1, \hat{C}_2, \) and \( \hat{D} \) are constant matrices.
Let \( \delta_k(\gamma) \) denote a unit discrete impulse applied at time \( k = \gamma \) and \( \gamma \) be the dimension of \( \hat{y}_k \). By \( \hat{\zeta}_i \) we denote the response of the system in Fig. 2 to the input \( \hat{w}_k = \delta_k(0) e \). The equivalence property means that

1. Controller (2) stabilizes the sampled-data system in Fig. 1 if it stabilizes the discrete-time system in Fig. 2.
2. Controller (2) minimizes the cost function \( J(4) \) if it minimizes the \( H_2 \)-norm of the discrete-time system in Fig. 2:

\[
J_d = \left( \sum_{k=1}^{\infty} \| \hat{\zeta}_i \|_2^2 \right)^{\frac{1}{2}},
\]

where \( \| \hat{\zeta}_i \|_2 \) denotes the \( \ell_2 \)-norm of the output sequence \( \hat{\zeta}_i \) [20].

The primary aim of the present paper is to construct an equivalent discrete-time state-space model for the \( H_2 \)-optimization problem with the plant (1). Then, we will generalize the results onto the case of plants with multiple input and output delays.

### III. PLANT DISCRETIZATION

For any continuous-time signal \( f(t) \) we introduce the notation \( f_\varepsilon(t) \triangleq f(t + \varepsilon) \), where \( \varepsilon \) is local time and \( \varepsilon \) means "is defined as". Also, denote \( f_\varepsilon \triangleq f_\varepsilon(0) \).

Let \( \tau_a = d_a T + \tau_g \), where \( d_a \) is a nonnegative integer such that \( 0 \leq \tau_g < T \). Then, integrating the first equation in (1) with the input \( u(t) = f(t) e_1 \) after some routine manipulations (see [11,20]) we can easily obtain

\[
x_k(\varepsilon) = \Phi(\varepsilon) x_k + \Gamma(\varepsilon) y_{k-d_a} + \Gamma_0(\varepsilon) v_{k-d_a} , \quad x(0) = B_1 e_1 ,
\]

where \( \Phi(\varepsilon) \triangleq e^{A \varepsilon} \) and

\[
\Gamma_i(\varepsilon) = \int_{0}^{\varepsilon} e^{A (\varepsilon-\sigma)} B_2 h(\sigma - \tau_g + T) d\sigma , \quad 0 \leq \varepsilon < T ,
\]

\[
\Gamma_0(\varepsilon) = \begin{cases} 0, & 0 \leq \varepsilon < \tau_g, \\ \int_{0}^{\varepsilon} e^{A (\varepsilon-\sigma)} B_2 h(\sigma - \tau_g) d\sigma, & \tau_g \leq \varepsilon < T . \end{cases}
\]

Equation (6) can be written as \( x_k(\varepsilon) = Q(\varepsilon) p_k \), where

\[
Q(\varepsilon) \triangleq \begin{bmatrix} \Phi(\varepsilon) & \Gamma_i(\varepsilon) & \Gamma_0(\varepsilon) \end{bmatrix} , \quad p_k \triangleq \begin{bmatrix} x_k \\ v_{k-d_a} \\ x_{k-d_a} \end{bmatrix} .
\]

For \( \varepsilon = T \) we obtain

\[
x_{k+1} = \Phi x_k + \Gamma_1 v_{k-d_a} + \Gamma_0 v_{k-d_a} = Q p_k ,
\]

with the notation \( \Phi \triangleq \Phi(T) \), \( \Gamma_i \triangleq \Gamma_i(T) \), \( \Gamma_0 \triangleq \Gamma_0(T) \), and \( Q \triangleq Q(T) \).

Then, we consider the third equation in (1) for \( t = kT \). Let \( \tau_y = d_y T - \theta_y \), where \( \theta_y \) is a nonnegative integer such that \( 0 \leq \theta_y < T \). Then, we find

\[
y_k = C_2 x_{k-d_y}(\theta_y) = C_2 Q(\theta_y) p_{k-d_y} .
\]

Consider a new vector

\[
q_k = \begin{bmatrix} x_{k-m}^{T} & x_{k-m}^{T} & \cdots & x_k^{T} & v_{k-\ell}^{T} & v_{k-\ell+1}^{T} & \cdots & v_{k-1}^{T} \end{bmatrix}^{T},
\]

where \( m \) and \( \ell \) are integers to be defined later such that \( m \geq d, \) and \( \ell \geq d_y + d_a + 1 \). Then, for any integer \( g (0 \leq g \leq d) \) we can always find a constant matrix \( W_g \) such that \( p_{k-g} = W_g q_k \). Notice that \( q_k \) can be partitioned as

\[
q_k = \begin{bmatrix} \hat{n}_{k} \\ \hat{n}_{k} \\ \hat{n}_{k} \\ \hat{n}_{k} \\ \hat{n}_{k} \end{bmatrix}, \quad \hat{n}_{k} = \begin{bmatrix} \hat{n}_{k-m} \\ \hat{n}_{k} \\ \hat{n}_{k} \\ \hat{n}_{k} \\ \hat{n}_{k} \end{bmatrix}, \quad \hat{n}_{k} = \begin{bmatrix} x_{k-m} \\ v_{k-\ell} \\ v_{k-\ell+1} \end{bmatrix} .
\]

Then, separating the last column block in \( W_g = [W_g^{(r)} \ W_g^{(r)}] \), we find

\[
p_{k-g} = W_g^{(r)} \hat{n}_{k} + W_g^{(r)} v_k .
\]

Using this notation, we can write (9) and (10) in the form

\[
x_{k+1} = \Phi^{(r)}(\varepsilon_0) x_{k} + \Gamma^{(r)}(\varepsilon_0) v_{k+1} + \Gamma_0^{(r)}(\varepsilon_0) v_{k+1} ,
\]

\[
y_k = C_2^{(r)} Q(\theta_y) p_{k+1} .
\]

Here we used the fact that \( \hat{n}_{k} \) is independent of \( v_k \), therefore, \( Q(\theta_y) W_d^{(r)} \hat{n}_{k} = 0 \) for any \( \tau_y > 0 \).

Assuming that \( \hat{n}_{k} \) is a new state vector of an equivalent discrete model, we can write the state equation as

\[
\hat{n}_{k+1} = \hat{A} \hat{n}_{k} + \hat{B}_1 \hat{w}_k + \hat{B}_2 v_k .
\]

The matrices \( \hat{A} \), \( \hat{B}_1 \), and \( \hat{B}_2 \) are easily determined. For example, if \( m = 1 \) and \( \ell = 2 \), we obtain

\[
\hat{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad \hat{B}_1 = \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} , \quad \hat{B}_2 = \begin{bmatrix} 0 \\ QW^{(r)}_0 \\ 0 \end{bmatrix} .
\]

The matrix \( \hat{B}_1 \) results from equivalent input discretization explained in detail in [20] (Sec. 12.1).

Equations (14) and (13) form a discrete-time model of the plant. Therefore, it remains to perform discretization of system output in an appropriate way.
IV. OUTPUT DISCRETIZATION

For output discretization, we employ the following chain of equalities:

\[
\sum_{k=0}^{\infty} z_k' = \sum_{k=0}^{\infty} \|z_k\|, \tag{16}
\]

where \(\{z_k\}\) is any vector sequence satisfying

\[
\sum_{k=0}^{\infty} (z_k' + \epsilon) = \sum_{k=0}^{\infty} z_k' + \sum_{k=0}^{\infty} \epsilon. \tag{17}
\]

From the second equation in (1) it follows that

\[
z_k(\epsilon) = C_z(z_k) + D_z(u_k(\epsilon)),
\]

where

\[
\tilde{x}_k(\epsilon) = x(t - \tau_z), \quad \tilde{u}_k(\epsilon) = u(kT + \epsilon - \tau_u).
\]

Let \(\tau_z = d_z T + \tau_y\) and \(\tau_u = d_u T + \tau_y\), where \(d_z\) and \(d_u\) are nonnegative integers such that \(0 \leq \tau_z < T\) and \(0 \leq \tau_u < T\). Then,

\[
\tilde{x}_k(\epsilon) = \begin{cases} x_{k-d_z}(\epsilon - \tau_z + T), & 0 \leq \epsilon < \tau_z \\ x_{k-d_u}(\epsilon - \tau_u), & \tau_z \leq \epsilon < T \end{cases},
\]

\[
\tilde{u}_k(\epsilon) = \begin{cases} Q(\epsilon - \tau_y + T) W_{d_z} q_k, & 0 \leq \epsilon < \tau_y \\ Q(\epsilon - \tau_y) W_{d_u} q_k, & \tau_y \leq \epsilon < T \end{cases}. \tag{18}
\]

We also notice that for any integer \(\rho (0 \leq \rho \leq \ell)\) there exists a matrix \(V_\rho\) such that \(v_\rho(\epsilon) = V_\rho \tilde{q}_k\). Hence,

\[
\tilde{u}_k(\epsilon) = \begin{cases} h(\epsilon - \tau_v + T) V_{d_z} q_k, & 0 \leq \epsilon < \tau_v \\ h(\epsilon - \tau_v) V_{d_u} q_k, & \tau_v \leq \epsilon < T \end{cases}. \tag{19}
\]

Therefore,

\[
Z_k = q_k' \Pi q_k, \tag{21}
\]

where \(\Pi\) is a constant nonnegative definite matrix. Therefore, there exists a matrix \(A\) such that

\[
\Pi = A' A. \tag{22}
\]

Hence, comparing (17), (20), and (21), we conclude that equivalence condition (17) holds for a vector sequence \(z_k = \Lambda q_k\), where \(A\) is any matrix satisfying (22). Recall that \(q_k\) can be partitioned as (12) so that, with appropriate partition of \(\Lambda = [\Lambda^{(0)} \Lambda^{(1)}]\),

\[
z_k = \Lambda^{(0)} \tilde{q}_k + \Lambda^{(1)} v_k. \tag{23}
\]

This equation defines the output signal of an equivalent discrete-time system.

V. EQUIVALENT DISCRETE MODEL

Now we construct a complete equivalent discrete-time model for the problem at hand and prove the equivalence property. As follows from the preceding sections, the values \(m\) and \(\ell\), which were introduced in Sec. 3, must be chosen as

\[
m = \max(d_z, d_u + 1), \quad \ell = \max(d_v + d_u + 1, d_z + d_u + 2, d_v + 1).
\]

Then, it can be easily shown that the state equations of an equivalent discrete-time model can be written in the form (5), where the constant matrices \(A, B_1, B_2, C_1, C_2,\) and \(D\) are determined appropriately. Namely, \(A, B_1,\) and \(B_2\) have the form (15), and

\[
\hat{C}_1 = \Lambda^{(0)}, \quad \hat{D} = \Lambda^{(1)}, \quad \hat{C}_2 = C_2 Q(0) W_{d_u}^{(q)}.
\]

To complete the solution, we will prove the equivalence property defined in Sec. 2. First, stability of both systems will be analyzed. Hereinafter \(\|\cdot\|\) denotes a vector norm or the matrix norm consistent with the vector norm.

**Definition 1.** Sampled-data system in Fig. 1 will be called internally stable if for \(w(t) = 0\) and any initial conditions the following estimates hold:

\[
\|x(t)\| < L_x e^{-\alpha t}, \quad \|u(t)\| < L_u e^{-\alpha t}, \quad \|z_k\| < L_z e^{-\alpha kT}, \tag{24}
\]

where \(L_x, L_u, L_z,\) and \(\alpha\) are constants, and \(\alpha > 0\) is independent of initial conditions.

**Definition 2.** Discrete-time system in Fig. 2 will be called internally stable if for \(\tilde{w}_k = 0\) and any initial conditions the following estimates hold:
Therefore, 
\[ || x(k\varepsilon) || < L_0 M_0 e^{-\beta T} + L_2 M_2 e^{-\beta (k-\delta_2-1)T} + L_3 M_3 e^{-\beta (k-\delta_3)T} = \tilde{L}_\varepsilon e^{-\beta T}, \]
where \( \varepsilon \) is the same \( \varepsilon \) as in the statement of Theorem 1.

\[ || x(t) || < \tilde{L}_\varepsilon e^{\beta T} \]
so that all the estimates (24) hold.

\[ || x(k\varepsilon) || < M_k e^{-\beta \varepsilon T}, \]
\[ || v_k || < M_v e^{-\beta \varepsilon T}, \]
\[ || \xi_k || < M_\xi e^{-\beta \varepsilon T}. \]

\[ \tag{25} \]
where \( M_k, M_v, M_\xi, \) and \( \beta \) are constants, and \( \beta > 0 \) is independent of initial conditions.

**Theorem 1.** The following two statements are equivalent:
(i) The sampled-data system in Fig. 1 with controller \( K(2) \) is internally stable.
(ii) The discrete-time system in Fig. 2 with controller \( K(2) \) is internally stable.

**Proof.** (i) \( \rightarrow \) (ii). Assume that the sampled-data system in Fig. 1 is internally stable, i.e., (24) holds. Obviously, in this case the third condition in (25) holds with \( M_k = L_k \) and \( \beta = \alpha. \) Let us show that the sequence \( \{v_k\} \) decreases exponentially. Equation (2) yields
\[ || v_k || \leq || C_k || || \xi_k || + || D_k || || y_k ||. \]

Since \( || x(t) || < L_\varepsilon e^{-\alpha T} \) from the second equation in (1) it follows that \( || x(t) || < L_\varepsilon e^{-\alpha T} \) with appropriate finite \( L_\varepsilon. \)
Then, \( || v_k || < M_v e^{-\alpha T} \) with
\[ M_v = || C_k || L_\varepsilon + || D_k || L_\varepsilon. \]

Therefore, all components of the state vector \( \hat{\eta}_k \) are exponentially bounded, and it is easy to show that \( || \hat{\eta}_k || < M_n e^{-\alpha T} \) with appropriately chosen \( M_n. \)

(ii) \( \rightarrow \) (i). Let the discrete system be stable so that (25) holds. Hence, the second condition in (24) is valid with \( L_\varepsilon = M_k \) and \( \alpha = \beta. \) From the second condition in (25) it follows that
\[ || u(t) || = || h(t-kT) v_k || \leq \sup_{0 < t < T} || h(\varepsilon) || || M_v || e^{-\beta \varepsilon T}, \]
\[ kT \leq t < (k+1)T. \]

In this case \( || u(t) || < L_\varepsilon e^{-\alpha T} \) with \( L_\varepsilon = \sup_{0 < t < T} || h(\varepsilon) || || M_v || e^{\beta T} \) and \( \alpha = \beta. \)

Using (6), for \( 0 \leq \varepsilon < T \) we obtain
\[ || x(k \varepsilon) || \leq L_\varepsilon || x_k || + L_2 || v_{k,\delta_2-1} || + L_3 || v_{k,\delta_3} ||, \]
where
\[ L_1 = \max_{0 < \varepsilon < T} || \Phi(\varepsilon) ||, \]
\[ L_2 = \max_{0 < \varepsilon < T} || \Gamma_1 (\varepsilon) ||, \]
\[ L_3 = \max_{\tau \leq \varepsilon < T} || \Gamma_0 (\varepsilon) ||. \]

Finally, from Theorem 1 it follows that the sets of stabilizing controllers are the same for both the sampled-data and equivalent discrete systems. This completes the proof.

**VI. GENERAL CASE**

Here we generalize the results onto the case of a plant with multiple input and output delays:
\[ \dot{x}(t) = A x(t) + B_1 w(t) + \sum_{i=1}^{n_1} B_{2i} u(t - \tau_{w_i}), \]
\[ z(t) = \sum_{j=1}^{n_2} C_{1j} x(t - \tau_{x_j}) + \sum_{j=1}^{n_3} D_{ij} u(t - \tau_{y_j}), \]
\[ y(t) = \sum_{j=1}^{n_4} C_{2j} x(t - \tau_{y_j}), \]
where \( A, B_1, B_{2i}, C_{1j}, C_{2j}, \) and \( D_j \) are constant matrices of compatible dimensions; \( \tau_{w_i}, \tau_{x_j}, \tau_{y_j} \) are nonnegative values (some of them can be zero); \( n_1, n_2, n_3, \) and \( n_4 \) are positive integers. Partition the delay intervals as
\[ \tau_{w_i} = d_w T + \tau_{w_i}, \quad i = 1, \ldots, n_w. \]
\[ \tau_{ij} = d_{ij} T + \tau_{ij}, \quad j = 1, \cdots, n_z, \]
\[ \tau_{ij} = d_{ij} T + \tau_{ij}, \quad j = 1, \cdots, n_z, \]
\[ \tau_{ij} = d_{ij} T - \theta_{ij}, \quad j = 1, \cdots, n_y, \]

where \( d_{i1}, d_{i2}, d_{i3} \) and \( d_{i4} \) are nonnegative integers such that
\( 0 \leq \tau_{i1} < T, \quad 0 \leq \tau_{i2} < T, \quad 0 \leq \tau_{i3} < T, \) and \( 0 \leq \theta_{i4} < T \) for all relevant indices. Obviously, the integers \( m \) and \( \ell \) in (11) must be chosen as
\[ m = \max(f_i + 1), \]
\[ \ell = \max(f_j + 1), \]
\[ f_i = \max(d_{i1}, f_i), \quad f_j = \max(d_{i2}, f_j), \]
\[ f_i = \max(d_{i3}, f_i), \quad f_j = \max(d_{i4}, f_j). \]

The plant can be discretized, similarly to Sec. 3, as
\[ x_i(\varepsilon) = \Phi(\varepsilon) x_i + \sum_{j=1}^{n_z} \left[ \Gamma_{i1}(\varepsilon) v_{k-d_{i1}} + \Gamma_{i2}(\varepsilon) v_{k-d_{i2}} \right], \]
\[ y_k = \sum_{j=1}^{n_y} C_{ij} x_{k-d_{i3}}, \]

where
\[ \Gamma_{i1}(\varepsilon) = \int_0^{\min(\varepsilon, \tau_{i1})} e^{(\varepsilon - \sigma) B_{i1}} h(\sigma - \tau_{i1} + T) d\sigma, \]
\[ \Gamma_{i2}(\varepsilon) = \int_{\tau_{i1}}^\varepsilon e^{(\varepsilon - \sigma) B_{i1}} h(\sigma - \tau_{i1} + T) d\sigma, \]
\[ \tau_{i1} \leq \varepsilon < T, \]
\[ \tau_{i1} \leq \varepsilon \leq \tau_{i2}, \]
\[ \tau_{i1} \leq \varepsilon \leq \tau_{i3}, \]
\[ \tau_{i1} \leq \varepsilon \leq \tau_{i4}. \]

For any \( i = 1, \cdots, n_z \) and integer \( g (0 \leq g \leq f_i) \) there exists a matrix \( W_{ig} \) such that
\[ p_{i,k-g} = \begin{bmatrix} x_{k-g} \\ v_{k-g-d_{i1}-1} \\ v_{k-g-d_{i2}} \end{bmatrix} = W_{ig} q_k, \]
where \( q_k \) is defined by (11). Therefore, (27) can be written in the form
\[ x_i(\varepsilon) = \sum_{j=1}^{n_y} Q_i(\varepsilon) \tilde{W}_{i,j} q_k, \]
where \( Q_i(\varepsilon) \) is \( \Phi(\varepsilon) \Gamma_{i1}(\varepsilon) \Gamma_{i2}(\varepsilon) \), while \( \tilde{W}_{i,j} = W_{ig} \) for \( i = 1 \) and
\[ \tilde{W}_{i,g} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{for} \quad i = 2, \cdots, n_z. \]

The representation (31) makes it possible to write the final results in a compact form. Equation (28) can be written as
\[ y_k = \sum_{j=1}^{n_y} C_{ij} \bar{W}_{i,j} k-d_{i3} q_k, \]
and the second equation in (26) takes the form
\[ z_k(\varepsilon) = \sum_{j=1}^{n_y} C_{ij} \bar{w}_{i,j}(\varepsilon) + \sum_{j=1}^{n_y} D_{ij} \bar{w}_{i,j}(\varepsilon), \]

where
\[ \bar{w}_{i,j}(\varepsilon) = \begin{bmatrix} x_{k-d_{i1}}(\varepsilon) \tau_{i2} + T, \quad 0 \leq \varepsilon \leq \tau_{i1} \\ x_{k-d_{i2}}(\varepsilon) \tau_{i3} + T, \quad \tau_{i2} \leq \varepsilon < \tau_{i3} \\ \frac{h(\varepsilon - \tau_{i3} + T) v_{k-d_{i3}+1}}{\tau_{i3} \leq \varepsilon < \tau_{i4}} \end{bmatrix} \]
\[ \bar{w}_{i,j}(\varepsilon) = \begin{bmatrix} \frac{h(\varepsilon - \tau_{i3} + T) v_{k-d_{i3}+1}}{\tau_{i3} \leq \varepsilon < \tau_{i4}} \end{bmatrix} \]
\[ \bar{w}_{i,j}(\varepsilon) = \begin{bmatrix} \frac{h(\varepsilon - \tau_{i4} + T) v_{k-d_{i4}+1}}{\tau_{i4} \leq \varepsilon < \tau_{i5}} \end{bmatrix} \]

Similarly to (20), we have
\[ z_k = \int_0^T z_k(\varepsilon) d\varepsilon = \sum_{i \in \mu} \sum_{j \in \nu} X_{i,j}(\varepsilon) \mu + X_{i,j}(\varepsilon) \mu + \sum_{\mu \in \nu} U_{\mu,k}, \]

where
\[ X_{i,j}(\varepsilon) = \int_0^T \bar{w}_{i,j}(\varepsilon) C_{ij} \bar{w}_{i,j}(\varepsilon) d\varepsilon \]
\[ N_{i,j}(\varepsilon) = \int_0^T \bar{w}_{i,j}(\varepsilon) D_{ij} \bar{w}_{i,j}(\varepsilon) d\varepsilon \]
\[ U_{\mu,k} = \int_0^T \bar{w}_{i,j}(\varepsilon) D_{ij} \bar{w}_{i,j}(\varepsilon) d\varepsilon. \]

Further discretization is performed similarly to that of Sec. 4 and 5. The matrix \( Z_k \) can be presented in the form
\[ Z_k = q_k \Pi q_k \] with a constant nonnegative definite matrix \( \Pi \), which is factorized as \( \Pi = \Lambda^T \Lambda \) to obtain the output discretization equation (23).

**VII. COMPUTATIONAL ASPECTS**

Obviously, computation of the matrix \( \Pi \) is the most involved step in the discretization procedure. First, we show how to compute \( X_{i,j,k} \) on the basis of (31). Assume that
\[ 0 \leq \tau_{i3} \leq \tau_{i1} < T, \]
without loss of generality. Indeed, if this is not the case, i.e., \( \tau_{i3} > \tau_{i1} \), we can exploit the property \( X_{i,j,k} = X_{i,j,k}^T \) to satisfy (34). Using (32) and (31), we easily find
\[ X_{i,j,k} = q_k \sum_{i \in \mu} \sum_{j \in \nu} \left( W_{i,j} x_{i,j} q_k \right) \]
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where

\[ P_{v_i} \triangleq \int_0^\tau \dot{Q}_i'(e - \tau_m + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dc , \]

\[ P_{v_{i+2}} \triangleq \int_0^\tau \dot{Q}_i'(e - \tau_m + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dc , \]

\[ P_{v_{i+3}} \triangleq \int_0^\tau \dot{Q}_i'(e - \tau_m + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dc , \]

\[ = \int_0^\tau \dot{\tau}_m \dot{Q}_i'(e - \tau_m + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dt , \]

\[ P_{v_{i+1}} \triangleq \int_0^\tau \dot{Q}_i'(e + \tau_m - \tau_g + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dc , \]

\[ = \int_0^\tau \dot{\tau}_m \dot{Q}_i'(e + \tau_m - \tau_g + t)C_{iu} \dot{Q}_i(e - \tau_g + t) \, dt . \]

Therefore, to compute \( X_{\mu,j} \), we need to find integrals of the form

\[ \int_0^T Q_i'(t + \alpha \tau - \beta \tau) \, \Omega \, Q_j(t + \beta \tau) \, dt , \quad (35) \]

where \( \Omega \) is a constant matrix, and \( \alpha \) and \( \beta \) are real constants. Further it will be shown that this problem can be reduced to computing matrix exponentials (see [32,33]).

For simplicity we consider the case of a zero-order hold, i.e., \( h(t) = 1 \) for \( 0 \leq t < T \). Introduce the following square matrices:

\[ \widetilde{A}_i \triangleq \begin{bmatrix} A & B_{2i} \\ 0 & 0 \end{bmatrix} , \quad i = 1, \ldots, n_u . \]

Then, as follows from [32],

\[ e^{\tilde{A}_i \sigma} \triangleq \begin{bmatrix} \Phi(\sigma) & Y_i(\sigma) \\ 0 & I \end{bmatrix} , \quad Y_i(\sigma) \triangleq \int_0^\sigma e^{\tilde{A}_i(-\sigma)} B_{2i} \, d\sigma . \]

For \( h(t) = 1 \), Eqs. (29) and (30) can be transformed as

\[ \Gamma_{ii}(\varepsilon) = \begin{cases} Y_i(\varepsilon) , & 0 \leq \varepsilon < \tau_m \\ \Phi(\varepsilon - \tau_m)Y_i(\tau_m) , & \tau_m \leq \varepsilon < T , \end{cases} \]

\[ \Gamma_{ij}(\varepsilon) = \begin{cases} 0 , & 0 \leq \varepsilon < \tau_m \\ Y_i(\varepsilon - \tau_m) , & \tau_m \leq \varepsilon < T . \end{cases} \]

Hence,

\[ Q_i(\varepsilon) \triangleq [\Phi(\varepsilon) \quad \Gamma_{ii}(\varepsilon) \quad \Gamma_{ij}(\varepsilon)] \]

\[ = \begin{cases} Le^{\tilde{A}_i \varepsilon} R_i , & 0 \leq \varepsilon < \tau_m \\ Le^{\tilde{A}_i(\varepsilon - \tau_m)} R_{2i} , & \tau_m \leq \varepsilon < T , \end{cases} \quad (36) \]

where

\[ L \triangleq [I \ 0] , \quad R_i \triangleq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \quad R_{2i} \triangleq \begin{bmatrix} \Phi(\tau_m) & Y_i(\tau_m) \\ 0 & I \end{bmatrix} . \]

Equation (36) will now be substituted into (35). Assume, without loss of generality,

\[ \tau_m - \alpha \leq \tau_g - \beta . \]

Then, allowing for any real \( \alpha \) and \( \beta \), we obtain

\[ P_j(\alpha, \beta) = M_1 + M_2 + M_3 , \]

where

\[ M_1 \triangleq R_i^r e^{\tilde{A}_j(\eta) T_j} e^{\tilde{A}_j \beta \tau_j} R_i , \]

\[ M_2 \triangleq R_{2i}^r e^{\tilde{A}_j(\eta - \tau_m + \tau_i) T_j} (T_j - \eta) e^{\tilde{A}_j \beta \tau_j} R_{2i} , \]

\[ M_3 \triangleq R_{2i}^r e^{\tilde{A}_j(\eta - \tau_m + \tau_i) T_j} (T_j - \eta) e^{\tilde{A}_j \beta \tau_j} R_{2j} , \]

with the notation

\[ T_j(0) \triangleq \int_0^\tau e^{\tilde{A}_j \eta} L' \Omega \, L e^{\tilde{A}_j \eta} \, d\eta , \quad (37) \]

\[ t_1 = \max (0, \min (\tau_m - \alpha, T)) , \]

\[ t_2 = \max (t_1, \min (\tau_g - \beta, T)) . \]

Finally, we notice that the integrals (37) appearing in these equations can be easily computed using the matrix exponential formulas given in [20] (Sec. 10.6). As regards computation of \( N_{\mu,j} \) and \( U_{\mu,j} \), they can be found in a similar way.

VIII. EXAMPLE

Consider the sampled-data tracking system shown in Fig. 3. The continuous-time system contains a reference generator \( R \), plant \( F \) and ideal operator \( Q \). Introduce the usual notation

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \triangleq C (sI - A)^{-1} B + D . \]

and suppose that the continuous-time elements are described by rational transfer functions

\[ R(s) = \begin{bmatrix} A_R & B_R \\ C_R & 0 \end{bmatrix} , \quad F(s) = \begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} , \quad Q(s) = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix} , \]

with constant matrices of compatible dimensions. Moreover, the blocks \( F \) and \( Q \) are attached with pure delay elements with delay times \( \tau_F \) and \( \tau_Q \), respectively.

\[ \begin{pmatrix} w \ \\
R \ y \ g \ K \ H \ m \ F e^{-\tau_F} \ \varphi \ z \ \varphi_0 \end{pmatrix} \]

Fig. 3. Sampled-data tracking system.
The input signal is \( w(t) = \delta(t) \) and the cost function (4) has the form

\[
J = \left( \int_0^{\infty} z'(t) z(t) \, dt \right)^{1/2},
\]

where \( z = \phi - \phi_0 \) is the difference between actual and desired output signals.

The presence of delays makes the problem hardly solvable rigorously using either “lifting” or “FR-operator” method. The only working approach is the parametric transfer function method [26,28], which uses the frequency domain concept. Up to the author’s knowledge, no exact state-space technique for solving this type of problem have been reported before.

Using the results of Sec. 6, we can easily construct the equivalent discrete-time model. After routine state-space transformations [34] the plant equations can be written in the form (26) as

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B_1 w(t) + B_2 u(t), \\
z(t) &= C_{11} x(t - \tau_p) + C_{12} x(t - \tau_d), \\
y(t) &= C_{21} x(t) + C_{22} x(t - \tau_F),
\end{align*}
\]

where

\[
\begin{align*}
A &\triangleq \begin{bmatrix} A_R & 0 & 0 \\ B_R C_R & A_Q & 0 \\ 0 & 0 & A_F \end{bmatrix}, & B_1 &\triangleq \begin{bmatrix} B_R \\ 0 \\ 0 \end{bmatrix}, & B_2 &\triangleq \begin{bmatrix} 0 \end{bmatrix}, \\
C_{11} &\triangleq \begin{bmatrix} 0 & 0 & C_F \end{bmatrix}, & C_{12} &\triangleq \begin{bmatrix} -D_B C_R & -C_Q & 0 \end{bmatrix}, \\
C_{21} &\triangleq \begin{bmatrix} C_R & 0 & 0 \end{bmatrix}, & C_{22} &\triangleq \begin{bmatrix} 0 & 0 & -C_F \end{bmatrix}.
\end{align*}
\]

For numerical calculations we take

\[
R(s) = \frac{1}{s}, \quad F(s) = \frac{1}{3s + 1}, \quad Q(s) = \frac{1}{s + 1},
\]

\[
T = 0.3, \quad \tau_p = 0.1, \quad \tau_d = 0.5.
\]

Notice that time delays are not integer multiples of \( T \). We assume that the hold \( H \) is a zero-order hold so that \( h(t) = 1 \) for \( 0 \leq t < T \).

Using the proposed algorithms, we find the following equivalent discrete model with 6 states, 1 input and 6 outputs (in minimal realization):

\[
\hat{A} =
\begin{bmatrix}
0.3258 & 0.4911 & 0.2256 & -0.2962 & -0.3078 & 0.1893 \\
-0.3024 & 1.0254 & 0.1240 & 0 & 0 & 0 \\
0 & 0.3643 & 0.9413 & 0 & 0 & 0 \\
0.4020 & -0.2439 & -0.0909 & 0.3195 & -0.6510 & 0.3047 \\
0.4177 & -0.2534 & -0.0945 & -0.6083 & 0.0543 & 0.0334 \\
-0.2569 & 0.1559 & 0.0581 & 0.3740 & -0.0333 & -0.0206
\end{bmatrix}
\]

\[
\hat{B}_1 =
\begin{bmatrix}
0.2958 & -0.8463 & -0.4339 & -0.0565 & -0.0587 & 0.0361
\end{bmatrix},
\]

\[
\hat{B}_2 =
\begin{bmatrix}
0 & 0 & -0.0740 & 0.5784 & 0.8248
\end{bmatrix},
\]

\[
\hat{C}_1 =
\begin{bmatrix}
-0.0199 & -0.1516 & 0.2730 & -0.1555 & 0.1712 & -0.0764 \\
-0.0272 & 0.0026 & -0.0205 & 0.0099 & -0.0267 & 0.0096 \\
-0.0099 & -0.0208 & 0.0352 & -0.0261 & 0.0111 & -0.0608 \\
-0.0093 & -0.0028 & 0.0008 & -0.0035 & -0.0074 & 0.0018 \\
-0.0046 & -0.0021 & -0.0006 & 0.0058 & 0.0044 & -0.0024 \\
0.0004 & 0.0002 & 0 & -0.0005 & 0.0008 & 0.0019
\end{bmatrix},
\]

\[
\hat{C}_2 =
\begin{bmatrix}
0.2958 & -0.8463 & -0.4339 & 0.3012 & -0.3190 & 0.1725
\end{bmatrix}.
\]

Then, the standard \( H_2 \)-optimization algorithm for discrete-time systems [20] yields the optimal controller:

\[
K_{op}(z) = \frac{-0.22512(z + 9.991)(z - 0.9048)(z + 0.6787)(z + 0.01622)}{(z - 1)(z + 0.3299)(z + 0.0162)(z^2 + 0.1954z + 0.1421)},
\]

which coincides with the optimal controller obtained by the polynomial method of [35] realized in the DIRECSD tool- box for MATLAB [36]. The cost function (4) equals \( J_{opt} = 0.0145 \), i.e., we achieved almost perfect tracking (Fig. 4).

**IX. CONCLUSIONS**

In this paper we presented a novel method for constructing an equivalent discrete-time state-space model for the so-called simple sampled-data \( H_2 \)-optimization problem with multiple input and output delays. As distinct from [21], a complete state-space model is given, the most restrictive assumption on plant open-loop stability is lifted, and multiple delays are allowed. Unlike all other known
results in the field of direct sampled-data system design, our method is not restricted to the case of a single delay at the plant input.

It was demonstrated that this technique makes it possible to construct equivalent discrete-time models for many problems that were previously solvable only by the frequency-domain parametric transfer function method of [26]. This result opens a perspective of using numerically reliable state-space machinery for optimal design of a wide class of multivariable sampled-data systems with delay elements. The proposed algorithms will be realized in a future version of DIRECTSD toolbox [36].

We note that the next challenging problems in this direction are the generalized $\mathcal{H}_\infty$-problem and the $\mathcal{H}_2$-problem. State-space solutions for the delay-free case are well known [17-20,37], but similar results for systems with time-delayed plants (26) are still unavailable. Also, the systems with state delays are an important topic [16]. We hope that the proposed ideas will advance these investigations.

REFERENCES


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