CONSTRaining HEAT INPUT By Trajectory Optimization For MINIMUM-FUEL HYPERSONIC CRUISE

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ABSTRACT

Unsteady heat input effects are considered for the range cruise of a future hypersonic vehicle equipped with a turbo/ram jet engines combination. A realistic mathematical model for describing the unsteady heat effects has been developed. It is coupled to the model for the dynamics of the vehicle. To compute the heat load in hypersonic flight, several points on the vehicle surface are treated simultaneously. A two-step technique consisting of an efficient optimization algorithm and an ordinary differential equations (ODE) solver is applied to generate a solution. The results show that the heat load can be significantly reduced, with only a small increase in fuel consumption.

KeyWords: Hypersonic vehicle, optimal control, trajectory optimization, unsteady heat effects, thermal protection system.

I. INTRODUCTION

In recent years, there have been significant research efforts in the field of new space transportation systems. A promising concept is concerned with a two-stage aerospace vehicle, consisting of a winged carrier stage with turbo/ram jet engines and a rocket propelled winged orbital stage. The development of such future space transportation systems poses challenging problems.

One problem concerns the high temperatures to which the vehicle is exposed, yielding significant heat loads not only for the engines but also for the structure (Fig. 1). Heat load reduction by optimal trajectory control is a subject of recent research, e.g. [1-3].

For coping with the high temperatures at hypersonic speed, sophisticated thermal protection systems are necessary to reduce the heat input into the vehicle [4,5]. These thermal protection systems usually consist of different insulating layers of adequate material and thickness. To realistically simulate the heat transfer through the layers a suitable mathematical model is required which accounts for unsteady effects and material properties. In this paper results are presented which show a significant improvement concerning the overall heat load, using trajectory optimization with multipoint heat input constraints.

II. MODELLING

2.1 Flight system dynamics modelling

For describing the dynamics of the vehicle, a point mass model is applied. The equations of motion for a flight in the equatorial plane read ([6], Fig. 2):

\[ \dot{V} = \frac{1}{m} \left[ T(V, h; \alpha, \delta_f) \cos \alpha - D(V, h; \alpha) \right] - g(h) \sin \gamma + \omega^2 \sin h \sin \gamma, \]

\[ \dot{\gamma} = \frac{1}{mV} \left[ T(V, h; \alpha, \delta_f) \sin \alpha + L(V, h; \alpha) \right] \]

Fig. 1. Areas with significant heat loading.
\begin{equation}
\dot{h} = V \sin \gamma , \\
\dot{x} = V \cos \gamma , \\
\dot{m} = -m_f (V, h; \alpha, \delta) , \\
\end{equation}

with
\[
\begin{align*}
 r &= r_E + h \\
 g &= g_0 \left( \frac{r_E}{r} \right)^2 .
\end{align*}
\]

In the above relations, \( V \) denotes the speed, \( \gamma \) the flight path angle, \( h \) the altitude, \( x \) the horizontal coordinate and \( m \) the mass. There are two control variables, the angle of attack \( \alpha \) and the throttle setting \( \delta_t \).

In the mathematical models describing the aerodynamics and powerplant characteristics, piecewise defined polynomials of sufficient smoothness are used. There are multifunctional dependencies for the lift \( L \), the drag \( D \), the thrust \( T \) and the fuel consumption \( f \).

The lift and drag models read (Fig. 3)
\begin{equation}
L = C_L (M; \alpha) \frac{\rho (h)}{2} V^2 S , \\
D = C_D (M; \alpha) \frac{\rho (h)}{2} V^2 S .
\end{equation}

The powerplant consists of a turbo/ram jet engines combination. The mathematical model, which describes rather complex thrust and fuel consumption characteristics (Fig. 4), can be expressed as
\begin{equation}
T = \delta_t T^* (M, h) f_t (M; \alpha) , \\
\dot{m}_f = \phi_f (M; \delta_t) \sigma^* (M, h) T^* (M, h) f_s (M; \alpha) .
\end{equation}

The quantities \( T^* \) and \( \sigma^* \) denote reference values at stoichiometric combustion. The model includes the possibility of overfueled combustion in the ramjet mode (\( \phi_f > 1 \)) and accounts for nonzero fuel consumption at idling (Fig. 5). As an unique effect of hypersonic flight, the angle of attack also exerts an influence. This property is modeled using the factor \( f_\alpha \) as shown in Fig. 5.

For modelling the air density, reference is made to the Standard Atmosphere given in [7]. An approximation is applied based on a polynomial approach, yielding
\begin{equation}
\rho (h) = \exp \left( \sum_{j=0}^{5} b_j h^j \right) .
\end{equation}

The coefficients \( a_i \) and \( b_i \) are given in Table 1.

\subsection{Heat flux modelling}
A complex model for realistically simulating the unsteady heat flux into the vehicle was developed [8-10]. Three different regions are considered: regions at the lower side of the vehicle, at the upper side and at the nose. The wall structure of the thermal protection system for the lower and upper sides of the vehicle is schematically shown

![Fig. 2. Forces acting on the vehicle.](image2)

![Fig. 3. Lift and drag coefficients.](image3)

![Fig. 4. Reference thrust \( T^* \) and fuel consumption \( \sigma^* \).](image4)

![Fig. 5. Equivalence ratio \( \phi_f \) and angle of attack \( f_\alpha \) effect on the thrust.](image5)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\( a_i \) & \( b_i \) \\
\hline
1.02906 \times 10^{-2} & 5.20452 \times 10^{-2} \\
-6.06038 \times 10^{-3} \text{ km}^{-1} & -8.01210 \times 10^{-3} \text{ km}^{-1} \\
5.35980 \times 10^{-3} \text{ km}^{-1} & 2.64429 \times 10^{-3} \text{ km}^{-1} \\
1.99264 \times 10^{-3} \text{ km}^{-1} & 1.23161 \times 10^{-3} \text{ km}^{-1} \\
-3.42227 \times 10^{-3} \text{ km}^{-1} & 1.58725 \times 10^{-3} \text{ km}^{-1} \\
\hline
\end{tabular}
\end{table}
in Fig. 6. The thermal protection system consists of several layers of different material and thickness. The nose cap consists of C/SiC material.

To describe the heat transfer characteristics, the wall is considered to consist of \( n \) layers as shown in Fig. 7. The heat flux from one layer to the other is described with the use of a one-dimensional knot model.

For the heat flux into the first layer, the following equation holds

\[
q_i = q_{\text{air}} = q_{\text{rad}} = q_{\text{air}}(M, h, T_i; \alpha) - \varepsilon \sigma \left[ T_i^4 - T_\infty^4 \right].
\]  
(5)

The quantity \( T_i \) is the temperature on the vehicle surface, depending on the actual flight condition,

\[
T_i = T_i(M, h; \alpha),
\]  
(6)

and \( T_\infty \) denotes the temperature for free stream conditions. The quantity \( q_{\text{air}} \) describes the convective heat flux into the vehicle in the form

\[
q_{\text{air}} = q_{\text{air}}(M, h, T_i; \alpha; x_n, P).
\]  
(7)

The heat fluxes for the remaining \( n - 1 \) layers can be described by

\[
q_i(T_{i-1}, T_i) = C_i(T_i)[T_{i-1} - T_i] + \varepsilon_i(T_{i-1})[T_{i-1}^4 - T_i^4]
\]  
(8)

where the quantities \( \varepsilon_{i-1} \) denote the emissivities of the applied materials and \( \sigma \) is the Boltzmann constant.

The expressions in Eqs. (7) and (8) are based on semi-empirical methods for describing the heating in hypersonic flow [8,9]. A detailed description is given in [9].

Since different points of the vehicle are considered, a set of differential equations is obtained for the temperatures of each point denoted by an additional subscript \( P \):

\[
\dot{T}_{i,p} = \frac{q_{i,p}(T_{i-1,p}, T_{i,p}) - \varepsilon_{i-1}T_i^4}{C_{p,i}(T_i,p)},
\]  
\( i = 1, \ldots, n-1, \)

\[
\dot{T}_{n,p} = \frac{q_{n,p}(T_{n-1,p}, T_{n,p}) - \alpha(T_{n,p} - T_\infty) - \varepsilon_{n-1}T_i^4}{C_{n,p}(T_{n-1,p}, T_{n,p})},
\]  
(10)

with \( P = \{ \}

1 \colon \text{lower surface }

2 \colon \text{upper surface }

3 \colon \text{nose cap }

\( n \) for \( P = 1, 2 \)

2 for \( P = 3 \).

where \( T_{i,p} \) is the temperature related to the corresponding layer and \( T_\infty \) the temperature in the interior of the vehicle. \( C_i \) and \( C_{p,i} \) are polynomial model functions describing the properties of the different materials subject to the respective temperatures.

For the point on the lower surface (\( P = 1 \)) the transition between laminar and turbulent flow is also taken into account. This is done via an approximation of the Reynolds-Number at the transition point [8]. The transition process is of great interest for hypersonic vehicles since the heating rate for a turbulent flow along the vehicle surface is significantly higher when compared to a laminar one [11].

With Eqs. (9) and (10), a system of \( 3 \cdot n \) differential equations for the temperature \( T_i \) of each layer is obtained. This holds for the three regions addressed above.

To describe the overall heat flux inside the vehicle, the variable \( \Gamma \) is introduced as

\[
\Gamma_{i,j} = \begin{cases} 1 & \text{if } P = 1, 2, 3, \\ \text{for } i = 1, 2, 3, \end{cases}
\]  
(11)

\[
\Gamma_{n,j} = \begin{cases} q_{n,j}(T_{n-1,j}, T_{n,j}), & j = 1, 2. \end{cases}
\]  
(11)

### III. OPTIMAL CONTROL PROBLEM

The optimal control problem is to minimize the fuel consumption for a range cruise over a given distance for which a value of 9000 km was selected. This can be formulated as to find a state function

\[
x := (V, \gamma, h, x, m, T_{i,1}, T_{i,2}, T_{i,3}, T_0, T_f) : [t_0, t_f] \rightarrow \mathbb{R}^9
\]  
(12)

and a control function

\[
u := (\alpha, \delta)^T : [t_0, t_f] \rightarrow \mathbb{R}^2,
\]  
(13)

which minimize the cost functional
subject to a set of nonlinear differential equations $\dot{x}(t) = f(x(t), u(t))$, given by Eqs. (1), (9) – (11), as well as to state constraints $(N(x(t)) \geq 0)$

$$\begin{align*}
\bar{q}_{\text{min}} & \leq q(V, h) \leq \bar{q}_{\text{max}}, \\
n_{\text{min}} & \leq n(V, h, m, \alpha) \leq n_{\text{max}}, \\
0 & \leq \Gamma_{n,i}(T_{n-1,i}, T_{n,i}) \leq \Gamma_{n,i,\text{max}}, \\
0 & \leq T_{h,i}(V, h, \alpha) \leq T_{h,i,\text{max}},
\end{align*}$$

and control constraints $C(x(t), u(t)) \geq 0$.

By applying the state constraints (Eq. (15)), the differential equations for the dynamics of the vehicle (Eq. (1)) and for the heat input (Eqs. (9) – (11)) are coupled.

Boundary conditions $(B(x(0), x(t_f)) = 0)$ for the state variables at the initial and final times are given in Table 2.

Table 2. Boundary Conditions.

<table>
<thead>
<tr>
<th>$V$ [m/s]</th>
<th>$\gamma$ [$^\circ$]</th>
<th>$h$ [m]</th>
<th>$x$ [km]</th>
<th>$m$ [kg]</th>
<th>$T_i$ [K]</th>
<th>$\Gamma_{x,i}$ [kJ/m$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>150</td>
<td>0</td>
<td>500</td>
<td>0</td>
<td>244000</td>
<td>300</td>
</tr>
<tr>
<td>$t = t_f$</td>
<td>150</td>
<td>0</td>
<td>500</td>
<td>9000</td>
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</tbody>
</table>

To solve the optimal control problem, suitable numerical methods are required. These can be classified as indirect or direct methods [12]. The indirect approach is based on Pontryagin’s Minimum Principle [13,14], leading to a set of necessary conditions to solve the arising multi-point boundary value problem. An alternate approach is the direct method which is used in this paper.

The optimal control problem is transformed in a problem of nonlinear programming. The efficient optimization technique DIRCOL [15] is applied which is based on a direct collocation approach. Both state and control variables are discretized by piecewise polynomial approximations. The control vector $u$ is approximated by piecewise linear and continuous functions. For the state vector $x$ cubic polynomials are used. The sets of differential equations (Eqs. (1), (9) – (11)) are satisfied at $m$ discrete points of the time interval. This leads to the following nonlinear optimization problem

Minimize $\Phi(Y)$

subject to:

$$f((x_{\text{app}}, u_{\text{app}}), t) - \dot{x}_{\text{app}}(t) = 0,$$

$$B(x(t_0), x(t_f)) = 0,$$

$N(x(t)) \geq 0, C(x(t), u(t)) \geq 0.$

The quantity

$$Y = (x_1, u_1, \ldots, x_m, u_m, t_f), Y \in \mathbb{R}^{2^{1+m+1}}$$

is the vector of parameters to be determined. The quantities $u_{\text{app}}$ and $x_{\text{app}}$ denote the piecewise polynomial approximations. The coefficients for $u_{\text{app}}$ and $x_{\text{app}}$ have to be chosen in such a way, that the cost functional Eq. (14) is minimized and the constraints Eqs. (15), (16) and the boundary conditions (Table 1) are accounted for. Thus, the infinite-dimensional optimal control problem is transformed into a finite-dimensional nonlinear constrained optimization problem which is solved with a sequential quadratic programming (SQP) method [13,16]. With the SQP Method the nonlinear programming problem is replaced by a quadratic subproblem (QP) in each iteration step (e.g. [17]). A sequence of iterates $(x_k, \pi_k)$ is generated that is converging to a point $(x^*, \pi^*)$, fulfilling the first-order Karush-Kuhn-Tucker optimality conditions. Each single iterate is the result of an iteration involving computing the solution of a QP subproblem and updating the QP Hessian.

To restrain the dimensions of the optimal control problem, the thermal protection systems for the upper and lower sides are modeled with five layers while for the nose cap area two layers are applied. This leads to a system of 19 differential equations, with two control variables and seven state constraints. The resulting NLP Problem consists of about 3300 variables and more than 4000 equality and inequality constraints.

The obtained results are then recalculated with the ODE-Solver LSODA. It is capable of dealing with stiff and non-stiff problems, using an implicit Adams formula for non-stiff problems and a BDF formula for stiff ones [18,19]. Furthermore this solver contains an automatic selection between stiff and non-stiff solver, based on a stepsize approximation. Thus, it is possible to deal with the different time scales of vehicle dynamics and unsteady heat transfer. Comparing this method with more conventional ODE-Solvers (i.e. explicit Runge-Kutta methods, extrapolation methods) there is an advantage of LSODA. The conventional solvers may need very large computing times, in some cases the methods can fail. This is due to the partial stiffness of both the dynamics and unsteady heating systems, which can be seen by the different order of magnitude of the eigenvalues.

For the recalculation the number of layers is increased to ten for the lower and upper sides of the vehicle and to five for the nose cap area. It may be of interest to note that the results obtained with the simpler layer model used in the optimization compare well with the more elaborated one. Experience shows that a further refinement of the layer models yields practically no improvement.

IV. RESULTS

The case with no heat input constraint is used as a reference. Figure 8 shows the time histories of the Mach num-
ber and the altitude for the unconstrained and the constrained cases. Figure 9 depicts the time histories of the two control variables. The three-dimensional progression of the temperatures at the lower side of the vehicle, at a distance of 15 meters behind the nose of the vehicle (10-layer model), is shown in Fig. 10, and the integrated heat flow in the most inner layer for the same reference point in Fig. 11. In Fig. 12 the progression of the heat flux into the first layer for the point on the lower surface is depicted. Table 3 presents the most relevant quantities for both cases.

The results for the temperatures of the different layers show a significant reduction of the temperature level during the flight. This holds especially for the first layers at the reference points. The maximum temperature in the first layers decreases by almost 42 Kelvin (with 784 Kelvin for the unconstrained case) for the point on the lower surface of the vehicle. In the nose cap area the decrease of the maximum temperature is about 10%.

The reduction of the heat flux (Fig. 11) into the vehicle is due to the lower level of the Mach number in the constrained case. The overall flight time increases by 6.5%. There is a small fuel penalty which amounts to 2.1%.

An interesting effect as regards heat flux reduction concerns altitude oscillations in the optimized case. The reference case without a heat input constraint does not show such oscillations. In the optimized case the temperature of the first layer has an oscillatory behavior in consonance with the altitude. Thus, more heat can radiate into the outside environment. Furthermore, in parts of the flight trajectory more heat is radiated away from the vehicle than going inside.

<table>
<thead>
<tr>
<th>Table 3. Optimization Results.</th>
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<tr>
<td></td>
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<tr>
<td>$t_f$ [s]</td>
</tr>
<tr>
<td>$m_f$ [kg]</td>
</tr>
<tr>
<td>$\Gamma_{1,1}$ [kJ/m²]</td>
</tr>
<tr>
<td>$\Gamma_{1,2}$ [kJ/m²]</td>
</tr>
<tr>
<td>$T_{\text{max,Nose Cap}}$ [K]</td>
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</table>

### V. CONCLUSIONS

Heat load reduction is considered for a two-stage hypersonic vehicle equipped with a turbo/ramjet engines combination, using optimal trajectory control. A complex mathematical model is applied to simulate the unsteady heat flux through the thermal protection system for several points of the vehicle. The differential equations for vehicle dynamics and unsteady heat transfer are coupled. Solutions are constructed, using an efficient optimization technique.
and an advanced ODE-Solver. The results show that the heat input into the vehicle as well as the maximum temperatures in the different layers can be significantly reduced, with a small fuel penalty.

**NOMENCLATURE**

- $a$: speed of sound
- $C$: heat conductivity
- $C_D$: drag coefficient
- $C_L$: lift coefficient
- $C_P$: heat capacity
- $D$: drag
- $f_\alpha$: thrust factor for angle of attack dependency
- $g$: acceleration due to gravity
- $h$: altitude
- $L$: lift
- $M$: Mach number, $M = V/a$
- $m$: mass
- $m_f$: fuel mass
- $n$: load factor, $n = L/(mg_0)$
- $P$: reference point at vehicle
- $q$: heat flux
- $q$: dynamic pressure, $q = (\rho/2) V^2$
- $r$: distance to centre of the Earth
- $r_E$: radius of the Earth
- $S$: reference area
- $T$: thrust
- $T_i$: temperature (at location $i$)
- $t$: time
- $u$: control vector
- $V$: speed
- $x$: range
- $x$: state vector
- $x_{le}$: length of flow along vehicle
- $\alpha$: angle of attack
- $\alpha_q$: heat transfer coefficient
- $\Gamma$: integrated heat flux
- $\gamma$: flight path angle
- $\delta$: throttle setting
- $\varepsilon$: emissivity
- $\rho$: atmospheric density
- $\sigma$: Boltzmann constant
- $\sigma^*$: specific fuel consumption at stoichiometric combustion
- $\phi_f$: equivalence ratio
- $\omega_E$: angular velocity of the Earth

**REFERENCES**


