A GRADIENT BASED ADAPTIVE CONTROL ALGORITHM FOR DUAL-RATE SYSTEMS

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ABSTRACT

In this paper, using a polynomial transformation technique, we derive a mathematical model for dual-rate systems. Based on this model, we use a stochastic gradient algorithm to estimate unknown parameters directly from the dual-rate input-output data, and then establish an adaptive control algorithm for dual-rate systems. We prove that the parameter estimation error converges to zero under persistent excitation, and the parameter estimation based control algorithm can achieve virtually asymptotically optimal control and ensure the closed-loop systems to be stable and globally convergent. The simulation results are included.

KeyWords: Multirate systems, adaptive control, parameter estimation, convergence analysis, stochastic gradient.

I. INTRODUCTION

For decades, the study of multirate systems has been quite active. For example, Chen and Qiu [1], Qiu and Chen [2,3], and Sagförs et al. [4] studied $H_2$ and $H_\infty$ optimal control and design problems of multirate systems, considering the causality constraint. Lee et al. [5], Scattolini and Schiavoni [6], and Sheng et al. [7] presented model based predictive control of multirate systems. In the inferential control area, Lee and Morari developed a generalized inferential control scheme and discussed various optimal control problems for multirate/dual-rate systems [8]; Li et al. applied dual-rate modeling to Octane quality inferential control [9]. Finally, Ding and Chen presented a series of identification methods of multirate systems [10-14], e.g., the auxiliary model least squares [10], the auxiliary model stochastic gradient [11], the finite impulse response auxiliary model identification [12], hierarchical identification algorithms [13].

However, most control algorithms reported in the area of multirate systems assume that the model parameters are known, which is usually not the case. Also, most theoretical results on parameter estimation based adaptive control assume that both the estimator and the controller are updated at the same rate, e.g., the well-known Åström and Wittenmark self-tuning regulator [15]. These results are not suitable for the multirate setting.

In the area of multirate adaptive control, Albertos et al. [16] discussed adaptive control schemes for dual-rate systems; Zhang et al. studied an indirect model reference multirate adaptive control [17]; Ishitobi et al. presented a least squares based self-tuning control algorithm [18]. But these algorithms handle only noise-free systems. Also, Kanniah et al. proposed a control algorithm based on a parameterized model with its AR coefficients corresponding to the fast sampling rate and the MA coefficients to the slow sampling rate [19]. Since the prediction and control are all based on the slow sampling rate, the desired fast-rate system performance may not be achieved. For multirate sampled-data control systems, we expect that the control law is updated at a fast rate even if the output is sampled at a relatively slow rate.

Scattolini [20] presented an adaptive control algorithm for multirate systems based on CARIMA models from lifted state-space models obtained by using the lifting technique [9,21,22]. The main drawback is that it requires more
controllers than the number of actual inputs because lifting results in a multi-input system even if the original system is a single-input one [9,21,22]. In this paper, we study the adaptive control problem of dual-rate sampled-data systems from a new viewpoint. The basic idea is to use a polynomial transformation technique to derive a mathematical model for dual-rate systems, and to establish a stochastic gradient based adaptive control algorithm, and then to prove the global convergence of the closed-loop systems. The control algorithm proposed here differs from the one in [23], which is based on the least squares parameter estimation.

The paper is organized as follows. In Section 2, we simply introduce the adaptive control scheme of dual-rate systems. In Section 3, we use a polynomial transformation technique to derive the mathematical model for dual-rate systems and present a stochastic gradient based adaptive control scheme. In Sections 4, we analyze the convergence properties of the parameter estimation algorithm, and prove the global convergence of the control algorithm in Section 5. In Section 6, we give an illustrative example demonstrating the effectiveness of the algorithm proposed. Finally, we offer some concluding remarks in Section 7.

II. PROBLEM FORMULATION

This paper focuses on a class of dual-rate systems, shown in Fig. 1, where $P_c$ is a continuous-time process; the input $u(t)$ to $P_c$ is produced by a zero-order hold $H_a$ with period $h$, processing a discrete-time signal $u(k)$; the output $y_c(t)$ of $P_c$ is sampled by a sampler $S_{ab}$ with period $g h(q$ being a positive integer), yielding a discrete-time signal $y(k)$ with period $g h$. The input-output data available are $\{u(k) : k = 0, 1, 2, \ldots\}$ at the fast rate, and $\{y(k) : k = 0, 1, 2, \ldots\}$ at the slow rate. Suppose that due to physical constraints, the intersample outputs, $y(kq + j)$, $j = 1, 2, \ldots, q – 1$ are not available, and thus we have missing output samples. Here, we refer to $\{u(k), y(k)\}$ as the dual-rate measurement data.

The parameter estimation based adaptive control scheme we propose is also shown in Fig. 1, where $y_r(k)$ denotes a deterministic reference input or desired output signal. For such a scheme to work, we can exploit an identification algorithm to produce the estimates $\hat{\theta}$ of the unknown system parameters based on the dual-rate data $\{u(k), y(kq)\}$, and compute the intersample (missing) outputs by using the estimated model and input $u(k)$. In order to feed back to the controller a fast rate signal $y_r(k)$, representing the output $y(k)$, we use the slow sampled output $y(q)$ every $q$ period, giving $y(0), y(q), y(2q), \text{etc.}$, and use the estimated output $\hat{y}(q + j)$ to fill in the missing samples in $y(k)$. In Fig. 1, $y(k)$ connects to $y(q)$ at times $k = iq$, and connects to $\hat{y}(q + j)$ at $k = iq + j, j = 1, 2, \ldots, (q – 1)$. Thus the output of the switch is a fast rate signal given by $y_r(k)$. Due

$$y_r(k) = \begin{cases} y(iq), & k = iq, \\ \hat{y}(iq + j), & k = iq + j, j = 1, 2, \ldots, (q – 1). \end{cases}$$

(1)

To summarize, the dual-rate adaptive control scheme uses a fast single-rate controller and a periodic switch. It is conceptually simple, easy to implement in digital computers, and practical for industry.

Our objective is to propose an algorithm to estimate the system parameter and intersample outputs $\{y(kq + j) : j = 1, 2, \ldots, (q – 1)\}$ from the given dual-rate data, to design an adaptive controller so as to make the output $y(k)$ track a given desired output $y_r(k)$ by minimizing the tracking error function:

$$J[u(k)] = E[\|y_r(k + d) – y_r(k + d)\|^2 | \mathcal{F}_{k-1}],$$

and to study the properties of the closed-loop system. Here, $\{\mathcal{F}_k\}$ is the $\sigma$ algebra sequence generated by the observations up to and including time $k$.

III. THE PARAMETER ESTIMATION AND CONTROL ALGORITHM

We assume that the open-loop transfer function from $u(k)$ to $y(k)$ takes the following real-rational form:

$$P(z) = \frac{z^{-d}b(z)}{a(z)}, \quad y(k) = \frac{z^{-d}b(z)}{a(z)} u(k)$$

(2)

with

$$a(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n},$$

$$b(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}.$$  

Here, $d$ denotes the system delay and $z^{-1}$ represents a unit backward shift operator at the fast rate: $z^{-1} u(k) = u(k – 1)$.

This model in (2) is not suitable for dual-rate adaptive control because it would involve the unavailable outputs $\{y(kq + j) : j = 1, 2, \ldots, (q – 1)\}$. To obtain a model that we can use directly on the dual-rate data, by a polynomial transformation technique [12,14], we can convert $P(z)$ into

![Fig. 1. The adaptive control scheme (j = 1, 2, …, (q – 1)).](image-url)
a form so that the denominator is a polynomial in $z^{-q}$ instead of $z^{-1}$.

For a general discussion, let the roots of $a(z)$ be $z_i$ to get

$$a(z) = \prod_{i=1}^{n} (1 - z_i \cdot z^{-1})$$

Define

$$\phi_q(z) := \prod_{i=1}^{n} \left(1 + z_i \cdot z^{-1} + z_i^2 \cdot z^{-2} + \ldots + z_i^{q-1} \cdot z^{-(q-1)}\right) = \prod_{i=1}^{n} \frac{1 - z_i^q \cdot z^{-q}}{1 - z_i \cdot z^{-1}}.$$ 

Multiplying the numerator and denominator of $P(z)$ by $\phi_q(z)$ and using the formula,

$$1 - x^n = (1 - x)(1 + x + x^2 + \ldots + x^{n-1}),$$ 

transforms the denominator into the desired form:

$$P(z) = \frac{z^{-d}b(z) \phi_q(z)}{a(z) \phi_q(z)} = \frac{z^{-d} \beta(z)}{\alpha(z)},$$

or

$$\alpha(z) y(k) = z^{-d} \beta(z) u(k)$$

with

$$\alpha(z) = a(z) \phi_q(z) = 1 + \alpha_1 z^{-q} + \alpha_2 z^{-2q} + \ldots + \alpha_n z^{-nq},$$

$$\beta(z) = b(z) \phi_q(z) = \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \ldots + \beta_n z^{-nq}.$$ 

Equation (3) is the mathematical model for the dual-rate system and it has the advantage that the denominator is a polynomial of $z^{-q}$; arising from here is a recursive equation using only slowly sampled outputs. The parameter estimation and control algorithm to be discussed later for dual-rate systems will be based on this model which does not involve the unavailable intersample outputs. Notice that the number of model parameters will be increased after the polynomial transform; this will lead to increased computational complexity, which may be reduced by using the hierarchical identification scheme [13,24-27].

Next, we derive an adaptive control algorithm based on the model discussed in (4). Define the parameter vector $\theta$ and information vector $\varphi(k)$ as

$$\theta = [\alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_n \quad \beta_0 \quad \beta_1 \quad \ldots \quad \beta_n]^T \in \mathbb{R}^{2n},$$

$$n_0 := qn + n + 1,$$ 

$$\varphi(k-d) = [-y(k-q) - y(k-2q) \ldots - y(k-qn)]^T \in \mathbb{R}^{n_0},$$

$$u(k-d) u(k-d-1) \ldots u(k-d-qn)]^T \in \mathbb{R}^{n_0}.$$ 

Notice that $\theta$ contains all parameters in the model in (3) to be estimated, and $\varphi(k-d)$ uses only available dual-rate data — if $k$ is an integer multiple of $q$, then $\varphi(k-d)$ contains only the past measurement outputs (slow rate) and inputs (fast rate). Substituting the polynomials $\alpha(z)$ and $\beta(z)$ into (4) leads to the following regression equation:

$$y(k) = \varphi^T (k-d) \theta,$$

where the superscript $T$ denotes the matrix transpose. For stochastic systems, based on the model in (5), introducing a disturbance term $v(k)$, we have

$$y(k) = \varphi^T (k-d) \theta + v(k),$$

where $\{v(k)\}$ is assumed to be a zero-mean random white noise sequence. Replacing $k$ in (6) by $kq$ gives

$$y(kq) = \varphi^T (kq-d) \theta + v(kq)$$

with

$$\varphi(kq-d) = [-y(kq-q) - y(kq-2q) \ldots - y(kq-nq)]^T u(kq-d) u(kq-d-1) \ldots u(kq-d-qn)]^T.$$ 

Let $\hat{\theta}(kq)$ be the estimate of $\theta$ at current time $kq$. We adopt the following stochastic gradient algorithm for estimating $\theta$ of the dual-rate system in (7):

$$\hat{\theta}(kq) = \hat{\theta}(kq-q) + \frac{\varphi(kq-d)}{r(kq)} \left[ y(kq) - \varphi^T (kq-d) \hat{\theta}(kq-q) \right];$$

(8)

$$r(kq) = r(kq-q) + \varphi^T (kq-d) \varphi(kq-d), \quad r(0) = 1;$$

(9)

$$\hat{\theta}(kq) = [\hat{\alpha}_0(kq) \quad \hat{\alpha}_1(kq) \ldots \hat{\alpha}_n(kq) \quad \hat{\beta}_0(kq) \quad \hat{\beta}_1(kq) \ldots \hat{\beta}_n(kq)]^T.$$ 

(11)

According to the certainty equivalence principle [28], minimizing $J[u(k)]$ yields the control law:

$$\varphi^T (kq+j) \hat{\theta}(kq) = y_r(kq+d+j), \quad j = 0, 1, \ldots, q-1.$$ 

(12)

The control signal $u(kq+j)$ in (12) may be obtained from the following recursive equation

$$u(kq+j) = \frac{1}{\hat{\beta}_0(kq)} \left[ y_r(kq+d+j) + \sum_{i=1}^{n_0} \hat{\alpha}_i(kq) y(kq+d+j-iq) - \sum_{i=1}^{n_0} \hat{\beta}_i(kq) u(kq+j-i) \right], \quad j = 0, 1, \ldots, q-1.$$ 

(13)
Here, a difficulty arises in that on the interval \([kq, kq + q]\), except for \(j = q - d\), the expression on the right-hand side of (13) contains the future missing outputs \(y(kq + j)\) if \(j_1 := d + j - iq > 0\), and the past missing outputs \(y(kq - j)\) if \(j_2 := -d - j + iq > 0\) and \(j_2\) is not an integer multiple of \(q\). In fact, only when \(j = q - d\), the control term \(u(kq + j)\) does not involve the missing outputs, and can be generated by

\[
u(kq + q - d) = \frac{1}{\hat{\beta}_i(kq)} \left[ y_i(kq + q) + \sum_{i=1}^{n} \hat{\alpha}_i(kq) y_i(kq + q - iq) - \sum_{i=1}^{n} \hat{\beta}_i(kq) u(kq + q - d - i) \right]. \tag{14}
\]

So it is impossible to compute the control law by (13) and to realize the algorithm in (8) ~ (13). Our solution is based on the adaptive control scheme stated in Section 2: These unknown outputs \(y(kq + j)\) in (13) are replaced by their estimates \(\hat{y}(kq + j)\). Hence,

\[
u(kq + j) = \frac{1}{\hat{\beta}_i(kq)} \left[ y_i(kq + d + j) + \sum_{i=1}^{n} \hat{\alpha}_i(kq) \hat{y}_i(kq + d + j - iq) - \sum_{i=1}^{n} \hat{\beta}_i(kq) u(kq + d - i) \right], \tag{15}
\]

Equations (8) ~ (11), (14), and (15) form the gradient based adaptive control algorithm.

When \(q = 1\), the control algorithm in (8) ~ (12) reduces to the well-known Åström-Wittenmark gradient-based self-tuning regulator. Thus, our algorithm includes the Åström-Wittenmark self-tuning regulator as a special case.

**IV. CONVERGENCE OF THE PARAMETER ESTIMATION**

Let us introduce some notation first. The symbol \(\lambda_{\max}(X)\) represents the maximum (minimum) eigenvalue of a square matrix \(X\). \(|X|\) denotes the norm of the matrix \(X\). \(f(\hat{\theta}) = o(g(\hat{\theta}))\) represents \(f(\hat{\theta})/g(\hat{\theta}) \to 0\) as \(k \to \infty\); for \(g(\hat{\theta}) \geq 0\), we write \(f(\hat{\theta}) = O(g(\hat{\theta}))\) if there exists a constant \(\delta_3 > 0\) such that \(|f(\hat{\theta})| \leq \delta_3 g(\hat{\theta})\).

We assume that \(\{v(\hat{\theta}), \mathcal{F}_k\}\) is a martingale difference sequence defined on a probability space \((\Omega, \mathcal{F}, P)\), where \(\mathcal{F}_k\) is the \(\sigma\) algebra sequence generated by \(\{v(\hat{\theta})\}, i.e., \mathcal{F}_k = \sigma(v(\hat{\theta}), v(k - 1), v(k - 2), \ldots)\), and that the noise sequence \(\{v(\hat{\theta})\}\) satisfies the following conditions [28]:

(A1) \(E[v(\hat{\theta})|\mathcal{F}_{k-1}] = 0\), a.s.,

(A2) \(E[v^2(\hat{\theta})|\mathcal{F}_{k-1}] = \sigma_v^2(\hat{\theta}) \leq \sigma_v^2 < \infty\), a.s.,

(A3) \(\lim\sup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} v^2(i) \leq \sigma_v^2 < \infty\), a.s.

In other words, \(\{v(\hat{\theta})\}\) may be regarded as an uncorrelated random noise sequence with zero mean and bounded time-varying variance.

In order to study the performances of the parameter estimation algorithm in (8) ~ (11) and the adaptive control algorithm in (8) ~ (11), (14), and (15), the following lemma is required.

**Lemma 1.** The following inequality holds:

1. \(\sum_{i=1}^{k} \frac{\|v(iq - d)\|^2}{r^2(iq)} < \infty\), a.s. \tag{16}

2. \(\sum_{i=1}^{k} \frac{\|v(iq - d)\|^2}{r(iq)} \leq \ln r(kq), \text{ a.s.}) \tag{17}

**Proof.** From the definition of \(r(kq)\), noting that \(\|v(kq)\|^2/r(kq) \to 0\), we have

\[
\sum_{i=1}^{k} \frac{\|v(iq - d)\|^2}{r(iq)} \leq \sum_{i=1}^{k} \frac{\|v(iq - d)\|^2}{r(iq) - r(0)} \leq \sum_{i=1}^{k} \frac{1}{r(iq) - r(0)} \leq 1 \frac{1}{r(0) - r(\infty)} < \infty, \text{ a.s.} \tag{18}
\]

2. \(\sum_{i=1}^{k} \frac{\|v(iq - d)\|^2}{r(iq)} \leq \sum_{i=1}^{k} \left[ \frac{\|v(iq - d)\|^2}{r(iq)} \right] \frac{dx}{x} \leq \ln r(kq) - \ln r(0), \text{ a.s.} \tag{19}
\]

Next, we shall prove the main results of this paper by formulating a martingale process and by using stochastic process theory and the martingale convergence theorem (Lemma D.5.3 in [28]).

**Theorem 1.** For the dual-rate system in (7), assume that (A1)-(A3) hold and let

\[
R(kq) = \sum_{i=1}^{k} q(iq - d) v^2(iq - d).
\]

If a persistently excited signal \(u(k)\) is chosen such that \(r(kq) \to \infty\) and \((kq)/\lambda_{\max}(R(kq)) \to 0\), then the parameter estimation error given by the algorithm in (8)-(11) consistently converges to zero, i.e., as \(k \to \infty\),
\[ \| \hat{\theta}(kq) - \theta \|^2 = 2 \left( \frac{r(kq)}{\lambda_{\min}[R(kq)]} \right) \to 0 , \text{ a.s.} \]

**Proof.** Define the parameter estimation error vector
\[ \hat{\theta}(kq) = \hat{\theta}(kq) - \theta . \]

Subtracting \( \theta \) from (8) and using (7) yield
\[ \hat{\theta}(kq) = \hat{\theta}(kq-q) + \frac{q(kq-d)}{r(kq)} [\phi^T(kq-d)\theta + v(kq)] \]
\[ - \phi^T(kq-d) \hat{\theta}(kq-q) ] \]
\[ = \hat{\theta}(kq-q) + \frac{q(kq-d)}{r(kq)} [-\hat{\gamma}(kq) + v(kq)] \]
\[ = \hat{\theta}(kq-q) + \Delta \hat{\theta}(kq) , \]
where
\[ \Delta \hat{\theta}(kq) \equal \frac{q(kq-d)}{r(kq)} [-\hat{\gamma}(kq) + v(kq)] , \]
\[ \hat{\gamma}(kq) \equal \phi^T(kq-d) \hat{\theta}(kq-q) - \phi^T(kq-d)\theta \]
\[ = \phi^T(kq-d) \hat{\theta}(kq-q) . \]

Define a non-negative definite function
\[ V(kq) = \| \hat{\theta}(kq) \|^2 . \]

Using (19) and (20), we have
\[ V(kq) = V(kq-q) + 2 \frac{\hat{\gamma}(kq)}{r(kq)} [-\hat{\gamma}(kq) + v(kq)] \]
\[ + \frac{\| q(kq-d) \|^2}{r^2(kq)} [-\hat{\gamma}(kq) + v(kq)]^2 \]
\[ = V(kq-q) - \frac{r(kq-q)}{r^2(kq)} \hat{\gamma}^2(kq) \]
\[ + \frac{\| q(kq-d) \|^2}{r(kq)} v^2(kq) + \frac{2r(kq-q)}{r(kq)} \hat{\gamma}(kq) v(kq) \]
\[ \leq V(kq-q) - \frac{1}{r(kq)} \hat{\gamma}^2(kq) \]
\[ + \frac{\| q(kq-d) \|^2}{r(kq)} v^2(kq) + \frac{2r(kq-q)}{r(kq)} \hat{\gamma}(kq) v(kq) . \]

Since \( \hat{\gamma}(kq) \), \( \phi^T(kq-d) \) and \( r(kq) \) are uncorrelated with \( v(kq) \) and are \( F_{kq-1} \)-measurable, taking the conditional expectation on both sides of (21) with respect to \( F_{kq-1} \) and using (A1)-(A2) give
\[ E[V(kq) \mid F_{kq-1}] \leq V(kq-q) - \frac{\hat{\gamma}^2(kq)}{r(kq)} + \frac{\| q(kq-d) \|^2}{r^2(kq)} \sigma^2(kq) \]
\[ \leq V(kq-q) - \frac{\hat{\gamma}^2(kq)}{r(kq)} + \frac{\| q(kq-d) \|^2}{r^2(kq)} \sigma^2(kq) . \]

(22)

By using Lemma 1, it is clear that the sum of the last right-hand term of (22) for \( k \) from 1 to \( \infty \) is finite. Applying the martingale convergence theorem (Lemma D.5.3 in [28]) to (22), we conclude that \( V(kq) \) converges a.s. to a finite random variable, say, \( V_0 \); i.e.,
\[ V(kq) = \| \hat{\theta}(kq) \|^2 \to V_0 < \infty , \text{ a.s.} , \]
and
\[ \sum_{i=1}^{\infty} \hat{\gamma}^2(kq) < \infty , \text{ a.s.} \]

Since \( r(kq) \to \infty \), applying the Kronecker lemma (Lemma D.5.5 in [28]) yields
\[ \sum_{i=1}^{\infty} \frac{\hat{\gamma}^2(kq)}{r(kq)} = 0 , \text{ a.s.} \]

(23)

From (19), we have
\[ \hat{\theta}(kq-iq) = \hat{\theta}(kq) - \sum_{j=0}^{i-1} \Delta \hat{\theta}(kq-jq) . \]

(24)

Replacing \( k \) in (20) with \( k-i \) gives
\[ \phi^T(kq-iq-d) \hat{\theta}(kq-iq) = -\hat{\gamma}(kq-iq) . \]

Using (24), it follows that
\[ \phi^T(kq-iq-d) \left[ \hat{\theta}(kq) - \sum_{j=0}^{i-1} \Delta \hat{\theta}(kq-jq) \right] = -\hat{\gamma}(kq-iq) , \]

or
\[ \phi^T(kq-iq-d) \hat{\theta}(kq) \]
\[ = -\hat{\gamma}(kq-iq) - \phi^T(kq-iq-d) \sum_{j=0}^{i-1} \Delta \hat{\theta}(kq-jq) . \]

Squaring and using the relation, \((a+b)^2 \leq 2(a^2+b^2)\), give
\[ \| \hat{\theta}(kq-iq) \|^2 \leq 2 \hat{\gamma}^2(kq-iq) + 2 \left[ \phi^T(kq-iq-d) \sum_{j=0}^{i-1} \Delta \hat{\theta}(kq-jq) \right]^2 \]
\[ = 2 \hat{\gamma}^2(kq-iq) + 2 \| v(kq-iq-d) \|^2 \left[ \sum_{j=0}^{i-1} \frac{q(kq-jq-d)}{r(kq-jq)} [-\hat{\gamma}(kq-jq) + v(kq-jq)] \right] . \]
Noting that \( \hat{y}(kq - iq) \), \( \varphi(kq - iq - d) \) and \( r(kq - iq) \) are uncorrelated with \( v(kq - iq) \) and are \( \mathcal{F}_{kq-1} \)-measurable, taking the conditional expectation with respect to \( \mathcal{F}_{kq-1} \) and using (A1) and (A2) yield

\[
E\{\hat{y}^2(kq - iq) | \mathcal{F}_{kq-1}\} \leq 2\hat{y}^2(kq - iq)
\]

Summing for \( i \) from \( i = 0 \) to \( i = k - 1 \) and dividing \( r(kq) \) give

\[
\frac{E\{\hat{y}^2(kq) R(kq) \hat{y}(kq) | \mathcal{F}_{kq-1}\}}{r(kq)} \leq \frac{2}{r(kq)} \sum_{i=0}^{k-1} \hat{y}^2(kq - iq)
\]

\[
+ 2 \sum_{i=0}^{k-1} \frac{|| \varphi(kq - iq - d) ||^2}{r(kq)} \sum_{j=0}^{i-1} \frac{|| \varphi(kq - jq - d) ||^2}{r^2(kq - jq)} \hat{y}^2(kq - jq)
\]

\[
\leq \frac{2}{r(kq)} \sum_{i=0}^{k} \hat{y}^2(iq) + \frac{2}{r(kq)} \sum_{i=0}^{k} \frac{|| \varphi(iq - d) ||^2}{r(iq)} \hat{y}^2(\hat{y}^2(iq))
\]

\[
+ 2 \sum_{i=0}^{k} \frac{|| \varphi(iq - d) ||^2}{r(iq)} \hat{y}^2(\hat{y}^2(iq))
\]

\[
\leq \frac{2}{r(kq)} \sum_{i=0}^{k} \hat{y}^2(iq) + \frac{2}{r(kq)} \sum_{i=0}^{k} \hat{y}^2(\hat{y}^2(iq))
\]

\[
+ 2 \ln \frac{r(kq)}{r(kq)} \frac{r(kq)}{r(kq)} \rightarrow 0, \text{ a.s., as } t \rightarrow \infty.
\]

Here, we have used Lemma 1 in the last inequality and (23) in the last step. Also

\[
\frac{\lambda_{\text{min}}[R(kq)] \|| \hat{y}(kq) \||^2}{r(kq)} \leq \frac{\hat{y}^2(kq) R(kq) \hat{y}(kq)}{r(kq)}.
\]

Thus, from assumption of \( r(kq) / \lambda_{\text{min}}[R(kq)] < \infty \), we have

\[
|| \hat{y}(kq) ||^2 = o \left( \frac{r(kq)}{\lambda_{\text{min}}[R(kq)]} \right) \rightarrow 0, \text{ a.s.}
\]

This proves Theorem 1.

Here, we prove that the parameter estimation is consistently convergent, i.e., \( \hat{y}(kq) \rightarrow \theta \), which improves the parameter norm and parameter difference convergence in [28].

V. THE GLOBAL STABILITY OF THE CLOSED-LOOP SYSTEM

We state and prove some global closed-loop properties in this section.

Theorem 2. For the dual-rate system in (7), assume that (A1)-(A3) hold, \( b(z) \) is stable, the system delay \( d \leq q \) is known, and the reference input \( y_i(k) \) is bounded in the sense

\[
(A4) \quad \| y_i(k) \| < \infty.
\]

Then the adaptive control algorithm in (8) ~ (11), (14), and (15) ensures that the output tracking error at the output sampling instants has the property of minimum variance, i.e.,

1. \( \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k} [y_i(iq) - \hat{y}(iq) + v(iq)]^2 = 0 \), a.s.;

2. \( \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k} E\{[y_i(iq) - y_i(iq)]^2 | \mathcal{F}_{kq-1}\} \leq \sigma_i^2 < \infty, \text{ a.s.}
\]

Proof. Since \( b(z) \) is stable, applying Lemma B.3.3 in [28] to (7) and using (A3) yield

\[
\frac{1}{k} \sum_{i=1}^{k} u^2(iq) \leq \frac{c_1}{k} \sum_{i=1}^{k} y^2(iq) + c_4, \text{ a.s.,}
\]

where \( c_i \) represent finite positive constants. According to the definitions of \( r(kq) \) and \( \varphi(kq) \), it is not difficult to get

\[
\frac{r(kq)}{k} \leq \frac{c_1}{k} \sum_{i=1}^{k} y^2(iq) + c_4
\]

\[
\leq \frac{c_1}{k} \sum_{i=1}^{k} (y_i(iq) - \hat{y}(iq) + v(iq))^2 + c_4
\]

\[
\leq \frac{c_1}{k} \sum_{i=1}^{k} y^2(iq) + c_6, \text{ a.s.}
\]
From (23), we have
\[
\lim_{k \to \infty} \frac{1}{r(kq)} \sum_{i=1}^{k} \hat{y}^2(iq) = 0, \text{ a.s.}
\]
Thus
\[
0 = \lim_{k \to \infty} \frac{1}{r(kq)} \sum_{i=1}^{k} \hat{y}^2(iq) \geq \lim_{k \to \infty} \frac{1}{r(kq)} \sum_{i=1}^{k} \hat{y}^2(iq) \geq 0, \text{ a.s.;}
\]
and hence
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \hat{y}^2(iq) = 0, \text{ a.s.}
\]
From (20), (7), and (12), for \( d \leq q \), we have
\[
\hat{y}(kq) = y_r(kq) - y(kq) + v(kq).
\]
So
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} [y_i(iq) - y(iq) + v(iq)]^2 = 0, \text{ a.s.} \quad (25)
\]
Since
\[
E\{y_r(kq) - y(kq) + v(kq)^2 \mid F_{kq-1}\}
\]
\[
= E\left[ (y_r(kq) - y(kq))^2 + 2y_r(kq) v(kq)
\right]
\]
\[
- 2y_r(kq) v(kq) + \nu^2(kq) \mid F_{kq-1}\]
\[
= E\left[ (y_r(kq) - y(kq))^2 \mid F_{kq-1}\right] + 0 - 2\sigma^2_r(kq) + \sigma^2_y(kq)
\]
\[
= E\left[ (y_r(kq) - y(kq))^2 \mid F_{kq-1}\right] - \sigma^2_r(kq), \text{ a.s.,}
\]
and \( y_r(kq) = y(kq) \) at the output sampling instants, we have
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} E\{[y_r(iq) - y(iq)]^2 \mid F_{kq-1}\}
\]
\[
= \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} E\{[y(iq) - y(iq)]^2 \mid F_{kq-1}\}
\]
\[
= \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \sigma^2_i(iq) \leq \sigma^2_{\epsilon}, \text{ a.s.}
\]
This proves Theorem 2. \[\blacksquare\]

**Theorem 3.** Assume that the conditions of Theorem 2 hold, the open-loop system \( h(z)/a(z) \) is minimum phase. Then the adaptive control algorithm in (8)-(11), (14), and (15) ensures the closed-loop system to be stable and globally convergent with probability 1, i.e., the input and output variables are uniformly bounded:
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} [u^2(i) + y^2(i) + y^2(i)] < \infty, \text{ a.s.}
\]
\[
\text{Proof.} \text{ From (1) and (7), we have}
\]
\[
y_r(kq) = y(kq) = \Phi^\top(kq - d) \Theta + v(kq), \quad (26)
\]
\[
y_r(kq + j) = \hat{y}(kq + j), \quad j = 1, 2, \ldots, q - 1. \quad (27)
\]
From Fig. 1 and (4), all missing output estimates \( y(kq + j) \) can be obtained by
\[
\hat{y}(kq + j) = \frac{z^{-d} \hat{\beta}(kq, z)}{\alpha(kq, z)} u(kq + j), \quad (28)
\]
where
\[
\hat{\alpha}(kq, z) = 1 + \hat{\alpha}_1(kq) z^{-q} + \hat{\alpha}_2(kq) z^{-2q} + \ldots + \hat{\alpha}_n(kq) z^{-mn},
\]
\[
\hat{\beta}(kq, z) = \hat{\beta}_0(kq) + \hat{\beta}_1(kq) z^{-1} + \hat{\beta}_2(kq) z^{-2} + \ldots + \hat{\beta}_{mn}(kq) z^{-mn},
\]
\[
\hat{\theta}(kq) = \left[\hat{\alpha}_1(kq) \hat{\alpha}_2(kq) \ldots \hat{\alpha}_n(kq) \hat{\beta}_0(kq) \hat{\beta}_1(kq) \ldots \hat{\beta}_{mn}(kq)\right].
\]
The output estimates \( \hat{y}(kq + j) \) also can be computed from the recursive equation:
\[
\hat{y}(kq + j) + \sum_{i=1}^{m_n} \hat{\alpha}_i(kq) \hat{y}(kq + j - iq)
\]
\[
= \sum_{i=0}^{q-1} \hat{\beta}_i(kq) u(kq - d + j - i), \quad j = 0, 1, \ldots, q - 1. \quad (29)
\]
This means
\[
\hat{y}(kq + j) = \Phi^\top (kq + j) \hat{\Theta}(kq), \quad j = 1, 2, \ldots, q - 1,
\]
with
\[
\Phi(kq + j) = [-\hat{y}(kq - q + j) - \hat{y}(kq - 2q + j) \ldots - \hat{y}(kq - qn + j)]^T
\]
\[
u(kq - d + j - 1) \quad u(kq - d + j - 1) \quad \ldots \quad u(kq - d + j - qn - 1)\]
Comparing (15) with (29), we find that the missing intersample output estimates \( \hat{y}(kq + j) \), \( j = 1, 2, \ldots, q - 1 \), equal the desired outputs \( y_r(kq + j) \), so we have
\[
y_r(kq + j) = \hat{y}(kq + j)
\]
\[
= \Phi^\top (kq + j) \hat{\Theta}(kq), \quad j = 1, 2, \ldots, q - 1. \quad (30)
\]
\[
\hat{\Phi}(kq + j) = [-y_r(kq - q + j) - y_r(kq - 2q + j) \ldots - y_r(kq - qn + j)]^T
\]
\[
u(kq - d + j - 1) \quad u(kq - d + j - 1) \quad \ldots \quad u(kq - d + j - qn - 1)\]

It is easy to understand that the unknown intersample outputs \(y(kq + j)\) are replaced by the desired outputs \(y_i(kq + j)\) in that our goal is to make \(y(k)\) track \(y_i(k)\). Hence, combining (14) with (30) also generates the control signal sequence \(\{u(kq + j), j = 0, 1, \ldots, q - 1\}\) based on the obtained estimates \(\hat{\theta}(kq)\).

Since \(y_i(k)\) is bounded, from Theorem 2 and Condition (A3), we can draw that outputs \(y(kq)\) at the output sampling instants are uniformly bounded, i.e.,

\[
\lim_{k \to \infty} \sup \frac{1}{k} \sum_{i=1}^{k - 1} y^2(iq) \leq \delta_j < \infty, \quad \text{a.s.}
\]

Also, the intersample output estimates \(\hat{y}(kq + j), j = 1, 2, \ldots, (q - 1)\), satisfy

\[
\hat{y}(kq + j) = y_i(kq + j), \quad j = 1, 2, \ldots, q - 1.
\]

So \(y_i(kq + j)\) is bounded. According to (26) and (27), \(y_i(k)\) is bounded. Since \(b(z) = a(z)\) is minimum phase, so are \(\beta(z) = a(z)\). Thus \(\hat{\theta}(k)\) is bounded by Lemma B.3.3 in [28]. Hence, we have

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k - 1} u^2(i) < \infty, \quad \text{a.s.,} \quad \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k - 1} y^2(i) < \infty, \quad \text{a.s.,}
\]

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k - 1} \hat{y}^2(i) < \infty, \quad \text{a.s.}
\]

We complete the proof of Theorem 3.

Theorems 2 and 3 show that for minimum phase systems, the proposed control algorithm in the dual-rate setting achieves the property of minimum variance at the output sampling instants. Between the output sampling instants, we have \(\hat{y}(kq + j) = y_i(kq + j) = y_i(kq + j), j = 1, 2, \ldots, q - 1\), which implies zero tracking error for intersamples. The control scheme here is easily extended to non-minimum phase cases by defining a new weighted criterion:

\[
J(u(k)) = E[\{P(z)y_i(k + d) - Q(z)y_i(k + d)\}^2] + \{R(z)u(k)\}^2 | F_{k-1} |.
\]

Here, \(P(z)\), \(Q(z)\), and \(R(z)\) are polynomials in \(z^{-1}\).

In order to avoid generating excessive \(u(k)\), for a given small positive \(\varepsilon\), if \(|\hat{\beta}_0(kq)| < \varepsilon\), we take \(\hat{\beta}_0(kq) = \text{sgn}(\hat{\beta}_0(kq))\varepsilon\), where the sign function is defined by

\[
\text{sgn}(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

The gradient based control algorithm has low computational effort, but its convergence is very slow. In order to improve the convergence performance, one may introduce a forgetting factor \(\lambda\) in (11) to get

\[
r(kq) = \lambda r(kq - q) + \varphi^T(kq - d) \varphi(kq - d), \\
0 \leq \lambda \leq 1, \quad r(0) = 1,
\]

and obtain the control algorithm with a forgetting factor in (8)-(9), (31), (14), and (15). When \(\lambda = 0\), we get the projection based adaptive control algorithm.

VI. EXAMPLE

Suppose that the discrete-time model with period \(h = 2\) s takes the following form

\[
R(z) = \frac{z^{-d}b(z)}{a(z)} = \frac{0.412z^{-1} + 0.309z^{-2}}{1 - 1.60z^{-1} + 0.80z^{-2}}, \quad d = 1.
\]

For \(q = 2\), only \(u(k)\) and \(y(2k)\) are available; thus the missing output \(y(2k + 1)\) is replaced by the estimated \(\hat{y}(2k + 1)\) as the feedback signal \(y_i(k)\). Use \(\{v(k)\}\) as a white noise sequence with zero mean and variance \(\sigma_v^2\). Take the desired output to be

\[
y_i(400i + j) = (-1)^j, \quad i = 0, 1, 2, \ldots, j = 1, 2, \ldots, 399,
\]

and \(\sigma_v^2 = 0.01^2\) and \(\lambda = 0.10\). We apply the adaptive control algorithm with a forgetting factor to this system, the system output \(y(k)\) and the desired output \(y_i(k)\) are shown in Fig. 2 with \(q = 2\). Figure 3 with \(q = 1\) is the simulated results of

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**Fig. 2.** \(y(k)\) and \(y_i(k)\) versus \(k(q = 2)\).

**Fig. 3.** \(y(k)\) and \(y_i(k)\) versus \(k(q = 1)\).
the Åström-Wittenmark gradient-based self-tuning regulator (A-W STR).

From Figs. 2 and 3, we can see that the control algorithm proposed in this paper can achieve less average tracking error than the A-W STR algorithm. This is due to the Åström-Wittenmark gradient-based self-tuning regulator (A-W STR).

Performance analysis in the stochastic framework indicates that the algorithm can achieve the desired control properties. The control algorithm for the case \( d > q \) requires further research. Of course, the control algorithm proposed can be extended to dual-rate nonlinear systems \([29]\).

VII. CONCLUSIONS

A gradient based adaptive control scheme for dual-rate systems is presented, and uses only slow-rate output measurement data to generate a relatively fast control signal. Performance analysis in the stochastic framework indicates that the algorithm can achieve the desired control properties. The control algorithm for the case \( d > q \) requires further research. Of course, the control algorithm proposed can be extended to dual-rate nonlinear systems \([29]\).

REFERENCES


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