SLIDING MODE PREDICTION TRACKING CONTROL DESIGN FOR UNCERTAIN SYSTEMS

Lingfei Xiao, Hongye Su, and Jian Chu

ABSTRACT

In this paper, a tracking control algorithm based on sliding mode prediction for a class of discrete-time uncertain systems is presented. By creating a special model to predict the future sliding mode function value and by combining feedback correction and receding horizon optimization approaches, which are extensively applied in predictive control strategy, a discrete-time sliding mode control law for tracking problem is constructed. With the designed control law, closed-loop systems have strong robustness to matched or unmatched uncertainties as they eliminate chattering. Besides, in the robustness analysis, the boundary condition for uncertainties, which is a universal presupposition in sliding mode control method, is not required. Numerical simulation and cart-pendulum experiment results illustrate the validity of the proposed algorithm.

KeyWords: Sliding mode prediction, sliding mode control, uncertain systems, discrete-time system.

I. INTRODUCTION

Sliding mode control (SMC) as an effective robust control method for uncertain systems is well established. The long history of its development and main results have been reported since the 1950s [1,2]. The method is mainly based on the design of a sliding mode function of state (or output) variable, which is used to compose a sliding surface (hyperplane). When this surface is attained, the sliding mode function keeps the trajectory on the manifold, thus yielding the desired system dynamic. For its easy implementation and good robustness, SMC is considerably attractive for many control system researchers. Due to the widespread use of digital controllers, SMC methods for discrete-time systems attract more and more attention. In [3], Dote and Holt first considered the discrete-time sliding mode control (DSMC), and obtained a discrete-time reaching condition (DRC) to ensure the existence of sliding mode by simply substituting the forward difference into the continuous-time reaching condition (CRC). In [4], the concept of a quasi-sliding mode was suggested and showed the DRC in [3] is necessary, but not sufficient in and of itself, for the existence of such a quasi-sliding mode. Later, a modified DRC, which is an inequality in the form of an absolute value, was presented in [5]; however, such a DRC is hard to solve. In [6], Furuta proposed a DRC using the equivalent form of Lyapunov-type of CRC, but it is difficult to extend this to multiple input systems. In [7], Gao et al. specified desired properties of DSMC systems and defined notions of reaching condition, quasi-sliding mode, and quasi-sliding mode band. Then, they used the so-called reaching law approach to design DSMC algorithms for the systems under nominal and perturbed conditions. In [8], Bartoszewicz proposed a different reaching law that was based on the concept of time-varying switching surfaces introduced in [9]. The reaching law in [8] has faster convergence speed of system error and chattering elimination; however, it possesses greater steady-state error as well, comparing with that of [7]. In [10], a proportional-constant-variable rate reaching law was formed to overcome the shortcoming that system states could only converge to a boundary of origin in [7]. When and how to ac-
tualize the switch between the two sub-reaching laws, however, were not explained, and chattering still exists. In [11], a time-vary reaching law was presented; although chattering was reduced, robustness was decreased as well. In [12], further improvement for [7] was made, however, chattering was not be eliminated and no robustness analysis was given.

As known to all, chattering is undesired in real applications because it will destroy actuators and excite system high frequency dynamics [1,2]. Besides, remarkable attention has been devoted by control system researchers to SMC systems, mainly because of their robustness characteristic, but the robustness is only for matched uncertainties [7-12]. In addition, although there are few works on DSMC for unmatched uncertainty systems, known boundaries of uncertainties are required, which are not feasible under many circumstances [13]. In order to overcome these problems, a discrete-time control algorithm based on sliding mode prediction, which has the merit of SMC method and predictive control strategy, is presented in this paper. On one hand, sliding mode function with the constructed control law enables the closed-loop systems to possess robustness and specified performance. On the other hand, due to the virtue of feedback correction, not only can the system states avoid piercing through the sliding mode surface, thus eliminating chattering, but also the control law has robustness to matched and unmatched uncertainties. Besides, the boundaries of uncertainties are not required in the algorithm, namely, the algorithm does not consider the worst case, therefore, the conservation is decreased. Furthermore, it is well known that the discrete-time control algorithm is capable of being implemented on a regular computer or single-chip computer, so the control law can be realized easily for real application.

The remainder of this paper is organized as follows. Section 2 describes the considered systems and preliminary of SMC. The main results, i.e., the construction of the sliding mode prediction model, the corresponding control law, and the robustness analysis of the closed-loop system are given in Section 3. In Section 4, advantages of the presented algorithm are verified by a numerical example and a cart-pendulum experiment. Finally, Section 5 draws the conclusions of the paper.

II. PROBLEM FORMULATION

Throughout this paper, only a single input system is considered, since an extension to systems with multiple inputs is possible without an increase of complexity.

Consider the following discrete-time system:

\[ x(k+1) = Ax(k) + \Delta Ax(k) + bu(k) + w(k) \]  

(1)

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( u(k) \in \mathbb{R} \) is the control input, \( A \) and \( b \) are of appropriate dimensions, \( \Delta A \in \mathbb{R}^{n \times n} \) represents parameter perturbations, \( w(k) \in \mathbb{R}^n \) denotes external disturbances. The control problem in this paper is to force the state vector \( x(k) \) to track a specific state vector \( x_d(k) \in C^- \) in the presence of uncertainties, where \( x_d(k) \in \mathbb{R}^n \) is known vector.

Let \( e(k) = x(k) - x_d(k) \) be the tracking error for state \( x(k) \), and define

\[ d(k) = \Delta Ax(k) + w(k) \]

\[ \tilde{x}_d(k) = Ax_d(k) - x_d(k+1) \]

then error system can be explained as:

\[ e(k+1) = x(k+1) - x_d(k+1) \]

\[ = Ax(k) + bu(k) + d(k) - x_d(k+1) \]

\[ = A[e(k) + x_d(k)] + bu(k) + d(k) - x_d(k+1) \]

\[ = Ae(k) + bu(k) + \tilde{x}_d(k) + d(k) \]  

(2)

The corresponding nominal system, namely \( d(k) = 0 \), is:

\[ e(k+1) = Ae(k) + bu(k) + \tilde{x}_d(k) \]  

(3)

The design objective is to construct a state feedback control law \( u(e) \), which guarantees tracking error toward zero in a stable manner. According to SMC theory, there are two steps to solve the problem. First, a sliding mode surface with desired performance is required; second, a suitable control law is needed to drive system error to the sliding mode surface and remain on it after that.

Here, we introduce the following definitions which will be used in this paper.

Definition 1. [8] We call the quasi-sliding mode in the \( \varepsilon \) vicinity of the sliding hyperplane \( s(k) = c^T x(k) = 0 \) such that a motion of the system \( |s(k)| \leq \varepsilon \) where the positive constant \( \varepsilon \) is called the quasi-sliding mode band (QSMB) width.

Definition 2. [8] We say that the system (1) satisfies the reaching condition of the quasi-sliding mode in the \( \varepsilon \) vicinity of the sliding surface \( s(k) = 0 \) if and only if, for any \( k > 0 \), the following conditions are satisfied:

\[ s(k) > \varepsilon \Rightarrow -\varepsilon \leq s(k+1) < s(k) \]
\[ s(k) < -\varepsilon \Rightarrow s(k) < s(k+1) \leq \varepsilon \]

\[ |s(k)| \leq \varepsilon \Rightarrow |s(k+1)| \leq \varepsilon \]  

(4)

Definition 3. [7] The quasi-sliding mode becomes an ideal quasi-sliding mode (IQSM) when the quasi-sliding mode band width is zero.

III. MAIN RESULTS

In predictive control strategy, prediction model, feedback correction and receding horizon optimization are em-
ployed during the design process [14,15]. Accordingly, in order to use predictive control approach to improve the performance of SMC, a suitable sliding mode prediction model (SMPM) should be created at first, then a satisfying sliding mode control law will be obtained for the sake of feedback correction and receding horizon optimization.

3.1 The design of sliding mode prediction model (SMPM)

Define the sliding mode function as follows:

\[ s(k) = \sigma^T e(k) \]  \hspace{1cm} (5)

where \( \sigma^T = [\sigma_1, \sigma_2, \ldots, \sigma_n] \). Therefore, the sliding surface is \( S = \{ e | s(e) = 0 \} \).

In Eq. (5), the choosing of \( \sigma_i \) (\( i = 1, 2, \ldots, n \)) should guarantee the stability and dynamic performance of the ideal quasi-sliding mode. According to classical control theory for linear systems, the suitable \( \sigma_i \) can be conveniently gained by pole assignment method.

Arising from the global sliding mode approach which is described in [16,17], and for the nominal system (3), the following sliding mode prediction surface is constructed:

\[ s_m(k) = \sigma^T e(k) - \alpha^k s(0) \]  \hspace{1cm} (6)

where \( s(0) = \sigma^T e(0) \) and \( 0 < \alpha < 1 \). The selection of \( \alpha \) has been introduced in [17]. The corresponding sliding mode prediction surface is \( S_m = \{ e | s_m(e) = 0 \} \).

From (6), it is obvious that:

1. When \( k = 0 \), (6) turns to \( s_m(0) = \sigma^T e(0) - \alpha^0 s(0) = s(0) \).

2. When \( k \rightarrow \infty \), (6) deduces to \( s_m(k) = \sigma^T e(k) \). From (5), yields \( s_m(k) = s(k) \). As the stability of the sliding surface can be guaranteed by choosing appropriate \( \sigma^T \), the sliding mode prediction surface will be stable, accordingly.

According to nominal dynamic (3) and sliding mode prediction function (6), the following can be given:

\[ s_m(k + 1) = \sigma^T [Ae(k) + bu(k) + \hat{\xi}_d(k)] - \alpha^{k+1} s(0) \]  \hspace{1cm} (7)

then a \( p \) step ahead predictor can be derived by continuing Eq. (7),

\[ s_m(k + p) = \sigma^T A^p e(k) + \sum_{i=1}^{p} \sigma^T A^{p-i} bu(k + p - i) \]

\[ + \sum_{i=1}^{p} \sigma^T A^{p-i} \hat{\xi}_d(k + p - i) - \alpha^{k+p} s(0) \]  \hspace{1cm} (8)

where \( p \) is arbitrary positive integer.

Therefore, when time \( k \rightarrow p \) is a starting point, SMPM predictive value \( s_{mp}(k \mid k - p) \) can be calculated by (8):

\[ s_{mp}(k \mid k - p) = \sigma^T A^p e(k - p) + \sum_{i=1}^{p} \sigma^T A^{p-i} bu(k - i) \]

\[ + \sum_{i=1}^{p} \sigma^T A^{p-i} \hat{\xi}_d(k - i) - \alpha^p s(0) \]  \hspace{1cm} (9)

3.2 The design of the control law

3.2.1 Feedback correction for SMPM

In practice, SMPM will inevitably have some errors because of time-variance, nonlinearity, or disturbances and so forth; therefore, the modeled output will not be the same as the real output. In predictive control strategy, a common way to solve the above problem is the feedback correction approach [14,15]. Here, the error between practical sliding mode value \( s(k) \) and SMPM predictive value \( s_{mp}(k \mid k - p) \) is used to make feedback correction for SMPM output \( s_m(k + p) \), therefore, the closed-loop output of SMPM \( \hat{s}_m(k + p) \) is:

\[ \hat{s}_m(k + p) = s_m(k + p) + \xi p [s(k) - s_{mp}(k \mid k - p)] \]  \hspace{1cm} (10)

where \( \xi_p \in \mathbb{R} \) is the correction coefficient, that is the weighted feedback correction. From the viewpoint of practice, usually, \( \xi_p = 1, 0 < \xi_p < 1 \) \( (p \neq 1) \). The action of feedback correction will reduce with the decrease of \( \xi_p \) [15].

3.2.2 Sliding mode reference trajectory (SMRT) design

According to predictive control approach, the value of the sliding mode function \( s(k) \) should move to desired value \( s_d \) along a reference trajectory with respect to \( k [15] \). Consequently, the following sliding mode reference trajectory (SMRT) is given:

\[ s_r(k + p) = \beta s_r(k + p - 1) + (1 - \beta) s_d \quad s_r(0) = s(0) \]  \hspace{1cm} (11)

where \( 0 < \beta < 1 \).

Obviously, the smaller \( \beta \) is, the faster SMRT will reach desired value \( s_d \). Since the control objective is to drive error states to the sliding surface, namely, \( s(e) = 0 \), the desired value of sliding mode function should be \( s_d = 0 \).

Consequently, (11) deduces to:

\[ s_r(k + p) = \beta^{k+p} s(0) \]  \hspace{1cm} (12)

3.2.3 Receding horizon optimization to obtain control law

Now, the performance index is given as:

\[ J = \sum_{i=1}^{N} [s_r(k + i) - \hat{s}_m(k + i)]^2 + \sum_{j=0}^{M-1} \lambda_j [u(k + j)]^2 \]  \hspace{1cm} (13)

where \( N \) and \( M \) are positive integers, called prediction horizon and control horizon, respectively. \( N \) influences the
stability and convergence of the closed-loop system greatly. When the stability is not good, increasing $N$ is needed; whereas, when the convergence is bad, decreasing $N$ is required. $M$ should satisfy $0 < M \leq N$, and $u(k + j) = u(k + M - 1)$, $M \leq j < N$. Increasing $M$ can improve the dynamic response of the closed-loop system, but will weaken the stability and robustness of it. Besides, reducing $M$ can decrease calculation complexity. $\lambda_j$ is weight coefficient, which adjusts the relation between the tracking errors and the control. If the control signal changes drastically, to increase $\lambda_j$ will do a favor to gain better control performance [15].

Define

$$S_m = [s_m(k + 1), s_m(k + 2), \ldots, s_m(k + N)]^T,$$

$$S_r = [s_r(k + 1), s_r(k + 2), \ldots, s_r(k + N)]^T,$$

$$\hat{S}_m = [\hat{s}_m(k + 1), \hat{s}_m(k + 2), \ldots, \hat{s}_m(k + N)]^T,$$

$$\hat{S} = [s(k) - s_{mp}(k | k - 1), s(k) - s_{mp}(k | k - 2),$$

$$\ldots, s(k) - s_{mp}(k | k - N)]^T = [\hat{S}(1), \hat{S}(2), \ldots, \hat{S}(N)]^T.$$

$$U = [u(k), u(k + 1), \ldots, u(k + M - 1)]^T,$$

$$\hat{X}_d = [\hat{x}_d(k), \hat{x}_d(k + 1), \ldots, \hat{x}_d(k + N - 1)]^T,$$

$$L_1 = [\alpha_1^A, \alpha_2^A, \ldots, \alpha_N^A]^T,$$

$$L_2 = [\beta_1^A, \beta_2^A, \ldots, \beta_N^A]^T,$$

$$\Xi = \text{diag}[\xi_1, \xi_2, \ldots, \xi_N],$$

$$\Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_M],$$

$$F = [\sigma^T A, \sigma^T A^2, \ldots, \sigma^T A^N]^T,$$

$$D = \begin{bmatrix}
\sigma^T & 0 & \ldots & 0 \\
\sigma^T A & \sigma^T & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^T A^{N-1} & \sigma^T A^{N-2} & \ldots & \sigma^T 
\end{bmatrix},$$

$$G = \begin{bmatrix}
\sigma^T b & 0 & \ldots & 0 \\
\sigma^T Ab & \sigma^T b & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^T A^{N-2}b & \sigma^T A^{N-3}b & \ldots & \sigma^T A^{N-M}b \\
\sigma^T A^{N-1}b & \sigma^T A^{N-2}b & \ldots & \sigma^T A^{N-M+1}b \\
\end{bmatrix},$$

$$\text{then (8), (10), (12), respectively, can be described in vector form as follows:}

$$S_m = Fe(k) + GU + D\hat{X}_d - L_1\alpha^t s(0) \quad (14a)$$

$$\hat{S}_m = S_m + \Xi \hat{S} \quad (14b)$$

$$S_\tau = L_2\beta^t s(0) \quad (14c)$$

The corresponding vector form of the performance index (13) is:

$$J = (S_m - \hat{S}_m)^T(S_m - \hat{S}_m) + \Lambda U^T U$$

where

$$H = (L_1\alpha^t + L_2\beta^t) s(0) - Fe(k) - D\hat{X}_d - \Xi \hat{S}$$

The solution of minimizing (15) gives the vector of control actions $U$. The first element of $U$ is used as control input signal for the real process, while other elements are not used for control, but can serve as initial values for the next optimization. The minimization of (15) and the calculation of the necessary matrices are repeated at every integration step.

$$U = (G^T G + \Lambda)^{-1}G^T H. \quad (16)$$

The first element of $U$ is:

$$u(k) = [1, 0, \ldots, 0]^T (G^T G + \Lambda)^{-1}G^T H \quad (17)$$

The control signal $u(k)$ for the closed-loop system is now obtained.

### 3.3 Robustness analysis

Consider error system (2) and sliding mode function (5). The following is given:

$$s(k + 1) = \sigma^T e(k + 1)$$

$$= \sigma^T [Ae(k) + bu(k) + \dot{\hat{x}}(k) + d(k)] \quad (18)$$

A $p$ step ahead predictor is derived by continuing (18):

$$s(k + p) = \sigma^T A^p e(k) + \sum_{i=0}^{p-1} \sigma^T A^{i-1}bu(k + p - i) + \sum_{i=0}^{p} \sigma^T A^{i-1}[\hat{x}_d(k + p - i) + d(k + p - i)] \quad (19)$$
Define:
\[ S = [s(k+1), s(k+2), \ldots, s(k+N)]^T \]
\[ V = [d(k), d(k+1), \ldots, d(k+N-1)]^T \]
\[ \dot{X}_d = [\dot{x}_d(k), \dot{x}_d(k+1), \ldots, \dot{x}_d(k+N-1)]^T \]

The corresponding vector form of Eq. (19) is:
\[ S = F e(k) + G U + D \dot{X}_d + D V \]

Under the control law (16), the vector form of practical sliding mode motion for the closed-loop system is:
\[ S = F e(k) + G (G^T G + \Lambda)^{-1} G^T H + D \dot{X}_d + D V \] (20)

According to performance index (15), the action of weight coefficient matrix \( \Lambda \) is used to limit the control input \( U \). Thus, for the sake of clarity, it is sound to suppose that \( \Lambda = 0 \), i.e., there is no limitation for control input \( U \).

Then, (20) can be written as:
\[ S = F e(k) + H + D \dot{X}_d + D V \]

Besides, in terms of (19), counting time \( k - p \) as the start point yields:
\[ s(k) = \sigma^T A^p e(k-p) + \sum_{i=1}^{p} \sigma^T A^{i+1} b u(k-i) \]
\[ + \sum_{i=1}^{p} \sigma^T A^{i-1} [\dot{x}_d(k-i) + d(k-i)] \] (22)

therefore,
\[ \tilde{S}(p) = \sum_{i=1}^{p} \sigma^T A^{i+1} d(k-i) + \alpha^k s(0), \quad p = 1, 2, \ldots, N \]

thus
\[ \tilde{S} = \tilde{D} \dot{V} + Q \alpha^k s(0) \] (24)

where
\[ \tilde{D} = \begin{bmatrix} \sigma^T & 0 & \cdots & 0 \\ \sigma^T & \sigma^T A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^T & \sigma^T A & \cdots & \sigma^T A^{N-1} \end{bmatrix}, \]
\[ \tilde{V} = [d(k-1), d(k-2), \ldots, d(k-N)]^T, \]
\[ Q = [1, 1, \ldots, 1]^T. \]

Due to receding horizon optimization, only the present control input signal is implemented. The practical sliding mode motion of the closed-loop system can be described as:
\[ s(k+1) = [1, 0, \ldots, 0] [(L_1 \alpha^k + L_2 \beta^k) s(0) - \Xi \bar{S} + D V] \]
\[ = [1, 0, \ldots, 0] [(L_1 \alpha^k + L_2 \beta^k) s(0) \]
\[ - \Xi (\bar{D} \dot{V} + Q \alpha^k s(0)) + D V] \] (25)

On account of \( \xi_1 = 1 \), (25) reduces to:
\[ s(k+1) = (\beta^{k+1} + \alpha^{k+1} - \alpha^k) s(0) + \sigma^T [d(k) - d(k-1)] \] (26)

**Theorem 1.** If the change rate of disturbances is bounded, the following inequality holds:
\[ |\sigma^T [d(k) - d(k-1)]| \leq \mu \] (27)
where \( \mu \) is a positive constant. Then, the closed-loop control system, which is constructed by (2), (5), and (17), is robustly stable.

**Proof.** From (26), it is clear that \( s(k+1) \) is composed of two parts:
\[ s_1(k+1) = (\beta^{k+1} + \alpha^{k+1} - \alpha^k) s(0) \]
\[ s_2(k+1) = \sigma^T [d(k) - d(k-1)] \]
Since \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \), the following inequalities hold:
\[ |2 \alpha - 1| < 1 \] (29a)
\[ |\beta + \alpha - 1| < 1 \] (29b)

For (28a), the following analysis can be given:
1. When \( \beta \leq \alpha \),
\[ |s_1(k+1)| = |\alpha^{k+1} + (\alpha - 1) \alpha^k| \]
\[ = |2 \alpha - 1| \alpha^k | \leq |2 \alpha - 1| \alpha^k \] (30)

Due to (29a) and \( 0 < \alpha < 1 \), (30) turns into \( |s_1(k+1)| < \alpha^k \).
2. When \( \beta > \alpha \),
\[ |s_1(k+1)| = |\beta^{k+1} + (\alpha - 1) \beta^k| \]
\[ = |(\beta + \alpha - 1) \beta^k| = |\beta + \alpha - 1| \beta^k \] (31)

Due to (29b) and \( 0 < \beta < 1 \), (31) turns into \( |s_1(k+1)| < \beta^k \).

Consequently,
\[ |s_1(k+1)| < \gamma^k \] (32)

Obviously, \( 0 < \gamma < 1 \), thus \( \forall \varrho > 0 \), \( \exists k_0 < \infty \) such that \( |s_1(k+1)| < \varrho \) when \( k > k_0 \).

For (28b), due to hypothesis (27), \( |s_2(k+1)| \leq \mu \) can always be satisfied.
As a result, when \( k > k_0 \):

\[
|s(k + 1)| = |s_1(k + 1) + s_2(k + 1)| \leq |s_1(k + 1)| + |s_2(k + 1)| < \rho + \mu = \varepsilon
\]  

(33)

In terms of (26), \( s(k + 1) \) has no relation with \( s(k) \), while due to (33), the conditions (4) in Definition 2 are satisfied, namely, the system (2) satisfies the reaching condition of the quasi-sliding mode in the \( \varepsilon \) vicinity of the sliding surface \( s(k) = 0 \). Consequently, the practical sliding mode motion of the closed-loop system will definitely converge to a \( \varepsilon \) vicinity of sliding surface and stay on it subsequently. Besides, since the stability and dynamic performance of the sliding surface has been guaranteed, the closed-loop control system constructed by (2), (5), and (17), is robustly stable.

**Remark 1.** when \( \alpha = \beta = \frac{1}{2} \), according to (28a), it yields \( s_1(k + 1) \), therefore, (33) reduces to \( |s(k + 1)| = \mu \), which means the width of quasi-sliding mode band (QSMD) decreases from \( \varepsilon \) to \( \mu \), that is QSMD width only depends on the change rates of external disturbances and parameter uncertainties.

**Remark 2.** Usually, external disturbances and parameter uncertainties have bounded change rates, and sometimes they are constant, i.e., generally speaking, inequality (27) can be satisfied. Especially, for slowly varying external disturbances and parameter uncertainties, or as sample period \( T \to 0 \), the QSMD width will be tiny.

### IV. CASE STUDY

As far as we know, plenty of examples in research on DSMC have been done for stabilization control, not tracking control, see e.g. [7-12]. One of the reasons is that the stabilization problem can be viewed as a special tracking problem when \( x_d = 0 \), thus the stabilization control results can show the tracking control performance. In order to indicate the properties of the proposed control law design method, a numerical simulation comparison with [12] will be made in Case 1, where all conditions are the same as that of in [12] with \( x = [0, 0]^T \). In Case 2, the results of a cart-pendulum experiment with a square wave serial as one of the desired state trajectories are illustrated.

**Case 1.** Consider the following second order discrete-time system, which is the example in [12]:

\[
\begin{align*}
x_1(k + 1) &= 1.5 x_1(k) + 0.3 x_2(k) \\
x_2(k + 1) &= x_1(k) + 0.6 x_2(k) + u(k)
\end{align*}
\]

Choose the sliding surface and initial states that are the same as that in [12], namely, \( \sigma^T = [2.5 \ 1] \) and \( x(0) = [4 \ 4]^T \), respectively. The desired state vector is set as \( x_d = [0 \ 0]^T \). According to predictive control strategy [15], it is suitable to select parameters in the proposed algorithm as \( \alpha = 0.1, \beta = 0.1, N = 2, M = 1, \lambda = 0.001, \Xi = \text{diag}[1, 0.5, 0.25, 0.125, 0.0625] \). In [12], parameters \( q^T = 0.25, \epsilon^T = 0.5 \) were needed, thus the same are used here as well. Besides, we suppose the external disturbance vector is \( w(k) = [-3\sin(8k) \ 2\cos(15\delta)]^T \), parameter uncertainty matrix is \( \Delta A = 0.5 \times A \), and they influence the system at step \( k = 51 \).

The results under the algorithm in this paper and in [12] are illustrated in Figs. 1 and 2, respectively. The detailed comparison is shown in Table 1.
Table 1. Performance comparison.

<table>
<thead>
<tr>
<th>Performance comparison</th>
<th>nominal system</th>
<th>uncertain system</th>
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<tbody>
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<td>convergence step</td>
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<td>14</td>
<td>32</td>
</tr>
</tbody>
</table>

It is obvious that chattering phenomena is eliminated in Fig. 1. Compared with Fig. 2, when there is no uncertainty, the system states in Fig. 1 have smaller peak values and both the sliding mode trajectory and input signal converge faster; when uncertainty appears, the system states still have smaller peak value and the state $x_2$ has faster convergence. Also, the input signal and the sliding trajectory converge faster. Therefore, the system with the algorithm in this paper possesses stronger robustness, faster convergence, and chatter elimination. Although the input signal in Fig. 1 is a bit larger than that in Fig. 2 at the very start, the proposed algorithm in this paper is better on the whole.

Case 2. Here, we apply the algorithm to a Cart-Pendulum experiment system. Assume there is no friction in the system, the simplified model of the system is shown in Fig. 3, and system parameters are shown in Table 2.

![Cart-Pendulum system simplified model](image)

Fig. 3 Cart-Pendulum system simplified model.

Table 2. System parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of cart</td>
<td>$m_c$</td>
<td>0.455</td>
<td>Kg</td>
</tr>
<tr>
<td>mass of pendulum</td>
<td>$m_p$</td>
<td>0.104</td>
<td>Kg</td>
</tr>
<tr>
<td>length of pendulum</td>
<td>$l_p$</td>
<td>0.3048</td>
<td>m</td>
</tr>
<tr>
<td>constant of gravitation</td>
<td>g</td>
<td>9.8</td>
<td>N/Kg</td>
</tr>
<tr>
<td>motor torque constant</td>
<td>$K_t$</td>
<td>0.07</td>
<td>Nm/Amp</td>
</tr>
<tr>
<td>motor pinion diameter</td>
<td>$r_p$</td>
<td>0.00635</td>
<td>m</td>
</tr>
<tr>
<td>amplifier gain</td>
<td>$G_{amp}$</td>
<td>5</td>
<td>$G_{amp}$/Volt</td>
</tr>
<tr>
<td>allowable cart displacement</td>
<td>$x_m$</td>
<td>± 0.3</td>
<td>m</td>
</tr>
<tr>
<td>allowable input voltage</td>
<td>$u_a$</td>
<td>± 3</td>
<td>Volt</td>
</tr>
</tbody>
</table>

Suppose the cart moving to the right side is the positive direction and the pendulum rotating clockwise has a positive angle, then when the pendulum is at vertical down position, the following linear model holds:

$$
\begin{bmatrix}
\dot{x} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
\theta
\end{bmatrix} +
\begin{bmatrix}
0 \\
K
\end{bmatrix} u,
$$

where $x$ is cart displacement, $\theta$ is pendulum angle, $\dot{x}$ is cart velocity, $\dot{\theta}$ is pendulum angle velocity, $\ddot{x}$ is cart acceleration, $\ddot{\theta}$ is pendulum angle acceleration, $u$ is the input voltage, $y$ is the system output. $l_p = \frac{l}{2}$ is the length from the pendulum gravity center to the pendulum bottom, namely, half of full length. $K = \frac{K_t \times G_{amp}}{r_p}$ is the gain from the input voltage $u$ to the input force $F$, namely, $F = K \times u$.

As the sample period of the Cart-Pendulum system is $T = 0.03$ second, and there exists zero-order hold, the corresponding discrete-time system is:

$$
\begin{align*}
\begin{bmatrix}
x(k+1) \\
\theta(k+1)
\end{bmatrix} &=
\begin{bmatrix}
1.0000 & -0.0003 & 0.0300 & -0.0000 \\
0 & 0.9695 & 0 & 0.0297
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\theta(k)
\end{bmatrix} +
\begin{bmatrix}
0.0139 \\
0.0905 \\
0.9233 \\
5.9998
\end{bmatrix} u(k),
\end{align*}
$$

$$
\begin{align*}
x(k+2) &=
\begin{bmatrix}
0 & -0.0169 & 1.0000 & -0.0003 \\
0 & -2.0204 & 0 & 0.9695
\end{bmatrix}
\begin{bmatrix}
x(k+1) \\
\theta(k+1)
\end{bmatrix},
\end{align*}
$$

The objective is to let the cart track a specified locus, while the pendulum returns to a vertical down position as quickly as possible. The specified locus is:

$$
x_d(kT) = \begin{cases} 0.1 & 0 < kT \leq 3 \text{ sec} \\ -0.1 & 3 \text{ sec} < kT \leq 6 \text{ sec} \end{cases},
$$

$$
x_d(kT + T_d) = x_d(kT), \ T_d = 6 \text{ sec}, \ \theta_d(kT) = 0.
$$

At the beginning of the experiment, we put the cart at
x = 0.05m and the pendulum at the right side horizontal position choosing the parameters in the algorithm presented in this paper as \( N = 5, M = 1, \Xi = \text{diag}[1, 0.5, 0.25, 0.125, 0.0625], \) and \( \lambda = 1 \) and placing the poles of the sliding surface at \([0.01, 0.02, 0.03]\). In addition, in order to show the strong robustness of the controlled closed-loop system, we touch the pendulum during the process, it happens almost at the fifteenth second. The results are shown in Fig. 4.

Obviously, the input force is smooth and appropriate, the cart tracks the specified locus very well, the pendulum returns to vertical down position quickly, and it has strong robustness to external disturbance. Consequently, the cart-pendulum experiment results verify the effectiveness of the algorithm in this paper.

![Cart-Pendulum experiment](image)

Fig. 4 Cart-Pendulum experiment.

V. CONCLUSIONS

In this paper, the strategy of predictive control is introduced into the discrete-time variable structure control method. Based on a constructed sliding mode prediction model, combined with feedback correction and receding horizon optimization approaches, a novel variable structure control law for discrete-time system is obtained. The robustness of the obtained control law is proven to be effective, and the common boundary condition for the uncertainty is not required. Numerical simulation results and cart-pendulum experiment results verify that the presented control law design approach is effective.

REFERENCES

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