ADAPTIVE CONTROL AND SYNCHRONIZATION OF CHUA’S CIRCUITS

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ABSTRACT

In this work, we address the adaptive control and synchronization problems of Chua’s circuits with unknown system parameters. Adaptive controllers with single state feedback are derived such that the trajectory of the Chua’s circuit is globally stabilized to an equilibrium point of the uncontrollable system. In addition, the proposed adaptive control schemes are then applied to achieve state synchronization of two identical Chua’s circuits. Furthermore, based on the Lyapunov approach, the closed-loop systems are proved to be globally exponentially or asymptotically stable. Numerical results and Pspice simulations demonstrate the proposed control scheme’s effectiveness.

KeyWords: Chaotic systems, Chua’s circuit, synchronization, adaptive control

I. INTRODUCTION

Chaos, an interesting phenomenon in nonlinear dynamical systems, has been thoroughly studied in different domains like physics, biology, mechanics, and electronics over the past two decades. Recently, after the pioneering work of Ott, Grebogi and Yorke (OGY) [16], several control strategies for stabilizing chaos have been proposed [3,11,14,15]. Generally speaking, there are two main approaches to controlling chaos: non-feedback control [12,13,17,19] and feedback control [4,5,9,21,24]. The feedback control approach offers many advantages, such as robustness and computational arduousness, over the nonfeedback control method.

On the other hand, synchronization of chaotic systems has also attracted much attention in recent years since it is highly promising for creating secure communication systems [8,20]. The concept of synchronization chaos involves making both the transmitter and the receiver, which are two chaotic systems, oscillate in a synchronized manner. Previous researches have demonstrated that the dynamical behaviors of two coupling chaotic systems can be synchronized after a transient period under certain conditions [2,23].

Several different approaches, including some conventional linear control techniques and advanced nonlinear control schemes, have already been successfully applied to the above problems. In these published works, it is essential to know the values of the system’s parameters for derivation of the controller. In practical situations, these parameters are unknown. Therefore, the derivation of an adaptive controller for controlling and synchronizing chaotic systems in the presence of unknown system parameters is an important issue [1,7,18,22]. In the work of Bernardo [1], an adaptive control scheme employing a discontinuous function for chaotic systems was proposed. The discontinuous control law induces an unfavorable chattering phenomenon, which must be averted in dynamical system design unless it has a desirable effect on some special applications. A modified scheme with a smooth function was also introduced in that paper. However, the synchronization error converges to a residual set whose size mainly relies on the magnitude of the designed parameter. On the other hand, in the work of Wu et al. [22], an adaptive control scheme was developed for controlling and synchronizing Chua’s circuits with mismatched parameters. However, a complex stability criterion for the overall system must be satisfied in advance.

Chua’s circuits are commonly used in analytical and experimental studies on chaos since it is rich in chaotic attractors and easy to implement in the laboratory. A typical Chua’s circuit and its physical meaning can be
found in Refs. [6,11] and [14]. After appropriate variable and parameter transformations, the set of nondimensional differential equations is given by

\[ \begin{align*}
\dot{x} &= \alpha (y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y - \gamma z,
\end{align*} \]  

and

\[ f(x) = bx + 0.5(a-b) \left[ x + 1 \right] \left[ x - 1 \right], \tag{2} \]

where \( x, y \) and \( z \) are state variables, and \( \alpha, \beta \) and \( \gamma \) are three positive real constants. Moreover, \( f(x) \) denotes a three-segment piecewise linear function, where \( a \) and \( b \) are two negative real constants and \( a < -1, -1 < b < 0 \). Figure 1 depicts the characteristic of \( f(x) \). From the figure, it can be easily verified that \(-x f(x) \leq |a| x^2\) holds for all \( x \in \mathbb{R} \). It has been shown that the chaotic motions of Chua’s circuits can be stabilized into a desired orbit or synchronized with each other if a particular coupling is introduced and some peculiar control schemes are applied.

The aim of this paper is to develop a simple and adaptive controller for resolving control and synchronization problems related to Chua’s circuits. We assume that the system’s parameters, \( \alpha, \beta, \gamma, a, \) and \( b \), are completely unknown, and that the state variables are available for implementing feedback controllers. All the derivations of adaptive controllers are based on the Lyapunov stability theory. Section 2 presents nonadaptive and adaptive controllers for controlling the chaos in a Chua’s circuit for the following four cases: (i) the state \( x \) is fed back into the first equation of (1); (ii) the states \( x \) and \( y \) are fed back into the first and second equations of (1), respectively; (iii) the states \( x \) and \( z \) are fed back into the first and third equations of (1), respectively; and (iv) the states \( x, y \) and \( z \) are all fed back into the first, second, and third equations of (1), respectively. Section 3 presents the state synchronization of two identical Chua’s circuits using a simple and adaptive control scheme. Section 4 presents numerical results that illustrate the proposed control scheme’s effectiveness. Section 5 is the conclusion.

\section{II. Adaptive Control of Chua’s Circuit}

For the purpose of controlling chaos using a feedback control approach, let us assume that the dynamical equations of the Chua’s circuit are given by

\[ \begin{align*}
\dot{x} &= \alpha(y - x - f(x)) + u_1, \\
\dot{y} &= x - y + z + u_2, \\
\dot{z} &= -\beta y - \gamma z + u_3,
\end{align*} \]

where \( u_1, u_2 \) and \( u_3 \) are external control inputs which will be suitably designed so as to drive the trajectory of the system, specified by \( (x, y, z) \), to any of the three equilibrium points of the uncontrolled (i.e. \( u_1 = u_2 = u_3 = 0 \)) system: \( X_0 \equiv (0, 0, 0) \), \( X_n = \left( \frac{(-a + b)(\beta + \gamma)}{\beta + b(\beta + \gamma)}, \frac{(b-a)\gamma}{\beta + b(\beta + \gamma)}, \frac{(a-b)\beta}{\beta + b(\beta + \gamma)} \right)^T \) and \( X_{-n} = -X_n \) by means of a state feedback control. We assume that \( \frac{(a-b)(\beta + \gamma)}{\beta + b(\beta + \gamma)} < 1 \) in this work such that these equilibrium points exist. For practical applications, a simple feedback controller is more desirable. In this work, we consider some special cases shown below:

(i) An external control with \( x \) as the feedback variable is added into the first equation in (3), i.e. \( u_1 = -k_1 x, u_2 = u_3 \equiv 0 \).

In this case, the trajectory of the controlled system can be stabilized to \( X_0 \) if \( k_1 > k_1^* \) is satisfied. Moreover, the convergent rate is in an exponential form if the system parameters are constant and known. To prove stability, the Lyapunov stability theorem is adopted herein. If a Lyapunov function candidate is chosen as

\[ V = \frac{1}{2} \left( \frac{1}{\alpha} x^2 + \beta y^2 + \gamma z^2 \right), \]

then the time derivative of \( V \) along the trajectory of the controlled system leads to

\[ \frac{dV}{dt} = \alpha(\gamma z + u_3) \left( \frac{1}{\alpha} x^2 + \beta y^2 + \gamma z^2 \right) \leq 0 \]
If one chooses \( k_1 > k'_1 = \alpha \left( \frac{1 + \beta}{2} \right)^2 + |a| - 1 \), then the \( 3 \times 3 \) matrix \( \Psi(k_1) \) in (5) is positive definite. This, in turn, implies that \( \dot{V} \leq 0 \), where \( 0 < \lambda_1 = \min (\Psi(k_1)) \) and \( \min (\Psi(k_1)) \) is the minimum eigenvalue of \( \Psi(k_1) \). Furthermore, from (4) we have \( \lambda_2 \dot{X} \leq V \leq \lambda_3 \dot{X} \), where \( 0 < \lambda_2 = \frac{1}{2} \min \left\{ \frac{1}{\alpha \beta}, 1 \right\} \) and \( 0 < \lambda_3 = \frac{1}{2} \max \left\{ \frac{1}{\alpha \beta}, 1 \right\} \). Therefore, the globally asymptotic stability of \( X = 0 \) is assured. Next, use of (5) yields \( \dot{V} \leq -\lambda_3 V \) and, hence, \( V(t) \leq V(0) \exp \left( -\frac{\lambda_3 t}{\lambda_1} \right) \), where \( V(0) \) is the initial Lyapunov function, which is bounded. Therefore, \( X \) converges to 0 at least exponentially [10].

(ii) An external control with \( y \) or \( z \) as the feedback variable is added into the second or third equation in (3), i.e., \( u_1 = u_3 = 0, u_2 = -k_2 y \) or \( u_1 = u_2 = 0, u_3 = -k_3 z \).

In this case, we can not conclude that stability of the controlled system exists via the Lyapunov stability theorem. Moreover, numerical simulations also verify that the controlled system can not be stabilized to 0 by means of the proposed control law for all \( k_2 \) or \( k_3 \).

(iii) External control inputs with \( x \) and \( y \) as the feedback variables are introduced into the first and second equations in (3), respectively, i.e., \( u_1 = -k_1 x, u_2 = -k_2 y \), and \( u_3 = 0 \).

In this case, the trajectory of the controlled system can be stabilized to \( X_n \) if \( k_1 > k'_1 \) and \( k_2 > k'_2 \) hold. Moreover, the convergence rate is in an exponential form if the system parameters are constant and known. Taking the derivation of \( V \) as (4) along the trajectory of the controlled system leads to

\[
V = -\begin{bmatrix} 1 - |a| + \frac{k_1}{\alpha \beta} & -\frac{1 + \beta}{2} & 0 \\ -\frac{1 + \beta}{2} & \beta \left( 1 + k_2 \right) & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv -X^T \Phi(k_1, k_2) X.
\]

If one chooses \( \frac{k_1}{\alpha \beta} + k_2 \geq 0 \), then the \( 3 \times 3 \) matrix \( \Phi(k_1, k_2) \) in (6) is positive definite. Similarly, based on the Lyapunov stability theorem and following some manipulations, we conclude that \( X \) converges to 0 at least exponentially under the proposed control law.

(iv) External control inputs with \( x \) and \( z \) as the feedback variables are introduced into the first and third equation in (3), respectively, i.e., \( u_1 = -k_1 x, u_2 = 0, \) and \( u_3 = -k_3 z \).

In this case, the trajectory of the controlled system can be stabilized to \( X_n \) if \( k_1 > k'_1 \) and \( k_2 > k'_2 \) hold. Moreover, the convergence rate is in an exponential form if the system parameters are constant and known. Taking the time derivative of \( V \) as (4) along the trajectory of the controlled system leads to

\[
V = -\begin{bmatrix} 1 - |a| + \frac{k_1}{\alpha \beta} & -\frac{1 + \beta}{2} & 0 \\ -\frac{1 + \beta}{2} & \beta \left( 1 + k_2 \right) & 0 \\ 0 & 0 & \gamma \left( 1 + k_3 \right) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv -X^T \Delta(k_1, k_3) X.
\]

If \( k_1 > k'_1 = \alpha \left( \frac{1 + \beta}{2} \right)^2 + |a| - 1 \) and \( k_3 > k'_3 = 0 \) are chosen, then the \( 3 \times 3 \) matrix \( \Delta(k_1, k_3) \) in (7) is positive definite. Similarly, using the Lyapunov stability theorem and some manipulations, we conclude that \( X \) converges to 0 at least exponentially under the proposed control law.

**Remark 1.** In addition to the above cases, there are many feedback control laws which can be used to stabilize the equilibrium point 0. For example, if \( u_1 = -k_1 x, u_2 = -k_2 y, u_3 = -k_3 z \), then \( X \) converges to 0 at least exponentially provided that \( \frac{k_1}{\alpha \beta} + k_2 \geq 0 \) and \( k_3 > k'_3 = 0 \).

**Remark 2.** The control for equilibrium point \( X_n \) or \( X_m \) can also be derived using the Lyapunov stability theorem. To save space, the derivations for this case are omitted here.

The feedback control laws derived thus far require that knowledge of the system’s parameters must be avail-
able a priori. However, in many real applications, it can be difficult to determine exactly the values of the system’s parameters. Consequently, the feedback gain \( k \) can not be appropriately chosen so as to guarantee the stability of the overall control system. If \( k \) is overestimated, then an expensive and overly conservative control effort is introduced. To overcome these drawbacks, adaptive techniques with state variable feedback control are derived to appropriately adjust the feedback gains, and in turn, to achieve the control objective. Again, we consider the following cases:

(i) An adaptive control with \( x \) as the feedback variable is added into the first equation in (3).

In this case, the feedback control laws are described by

\[
\begin{align*}
    u_1 &= -\hat{k}_1 x , \quad u_2 = u_3 \equiv 0 ,
\end{align*}
\]

where \( \hat{k}_1 \), an estimate of \( k_1^* \), is updated according to the following adaptive algorithm:

\[
\hat{k}_1 = \kappa x^2
\]

with \( \kappa \), an adaptation gain, determining the speed of the adaptation process. To prove stability, the Lyapunov stability theorem is used again. If a Lyapunov function candidate is chosen as

\[
V = \frac{1}{2} \left\{ \frac{1}{\alpha} x^2 + \beta y^2 + z^2 + \frac{1}{\alpha \kappa} (\hat{k}_1 - k_1^*)^2 \right\} ,
\]

then taking the time derivation of \( V \) along the trajectory of the controlled system leads to

\[
V = -\begin{bmatrix}
    x & y & z
\end{bmatrix}
\begin{bmatrix}
    \left( 1 - |a| + k_1^* \right) - \left( 1 + \beta / 2 \right) & 0 \\
    -\left( 1 + \beta / 2 \right) & \beta & 0 \\
    0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]

\[
\equiv - X^T \Psi(k_1^*) X \leq 0 .
\]

Since \( V \) is a positive and decrescent function and \( \Psi \) is negative semidefinite, it follows that the equilibrium point \( x = 0, y = 0, z = 0, \hat{k}_1 = k_1^* \) of systems (3) and (9) associated with (10) is uniformly stable, i.e., \( x(t), y(t), z(t) \in L_\infty \) and \( \hat{k}_1(t) \in L_\infty \). From (11), we can easily show that the squares of \( x(t), y(t) \) and \( z(t) \) are integrable with respect to time, i.e., \( x(t), y(t), z(t) \in L_2 \). Next, by Barbalat’s lemma [10], for any initial condition, (2) implies that \( \dot{x}(t), \dot{y}(t), \dot{z}(t) \in L_\infty \), which in turn implies that \( x(t), y(t) \) and \( z(t) \to 0 \) as \( t \to \infty \). This implies that the trajectory of the controlled Chua’s circuit is globally asymptotically stabilized to the equilibrium point \( X_0 \).

(ii) Adaptive control inputs with \( x \) and \( y \) as the feedback variables are introduced into the first and second equations in (3), respectively.

In this case, the feedback control laws are described by

\[
\begin{align}
    u_1 &= -\hat{k}_1 x , \quad u_2 = u_3 \equiv 0 , \\
    u_2 &= -\hat{k}_2 y ,
\end{align}
\]

where \( \hat{k}_1 \) and \( \hat{k}_2 \) are estimates of \( k_1^* \) and \( k_2^* \), respectively.

In addition, these estimates are updated according to the following adaptive algorithm:

\[
\hat{k}_1 = \kappa_1 x^2; \quad \hat{k}_2 = \kappa_2 y^2 \text{ with } \hat{k}_1(0) > 0; \quad \hat{k}_2(0) > 0 .
\]

To prove stability, the Lyapunov stability theorem is used again. If a Lyapunov function candidate is chosen as

\[
V = \frac{1}{2} \left\{ \frac{1}{\alpha} x^2 + \beta y^2 + z^2 + \frac{1}{\alpha \kappa_1} (\hat{k}_1 - k_1^*)^2 + \frac{1}{\alpha \kappa_2} (\hat{k}_2 - k_2^*)^2 \right\} ,
\]

then taking the time derivative of \( V \) along the trajectory of the controlled system leads to

\[
\dot{V} = -\begin{bmatrix}
    x & y & z
\end{bmatrix}
\begin{bmatrix}
    \left( 1 - |a| + k_1^* \right) - \left( 1 + \beta / 2 \right) & 0 \\
    -\left( 1 + \beta / 2 \right) & \beta & 0 \\
    0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]

\[
\equiv - X^T \Phi(k_1^*, k_2^*) X \leq 0 .
\]

Similarly, since \( V \) is a positive and decrescent function and \( \Phi \) is negative semidefinite, we can show that the trajectory of the controlled Chua’s circuit is asymptotically stabilized to the equilibrium point \( X_0 \).

(iii) Adaptive control inputs associated with state \( x \) and \( z \) feedback are introduced into the first and third equations in (3), respectively.

In this case, the feedback control laws are described by

\[
\begin{align}
    u_1 &= -\hat{k}_1 x , \quad u_2 \equiv 0 , \\
    u_3 &= -\hat{k}_3 z ,
\end{align}
\]
where $\hat{k}_1$ and $\hat{k}_3$ are estimates of $k_1^*$ and $k_3^*$, respectively. In addition, these estimates are updated according to the following adaptive algorithm:

$$\dot{\hat{k}}_1 = \kappa_1 \hat{z}_1^2; \quad \dot{\hat{k}}_3 = \kappa_3 \hat{z}_3^2 \quad \text{with} \quad \hat{k}_1(0) > 0 \quad \text{and} \quad \hat{k}_3(0) > 0 . \quad (17)$$

To prove stability, the Lyapunov stability theorem is used again. If a Lyapunov function candidate is chosen as

$$V = \frac{1}{2} \left( \frac{1}{\alpha} \hat{x}^2 + \beta \hat{y}^2 + \hat{z}^2 + \frac{1}{\alpha k_1} (\hat{k}_1 - k_1^*)^2 + \frac{1}{\alpha k_3} (\hat{k}_3 - k_3^*)^2 \right) ,$$

then taking the time derivative of $V$ along the trajectory of the controlled system leads to

$$\dot{V} = - X^T \Delta(k_1^*, k_3^*) X \leq 0 . \quad (19)$$

Similarly, since $\dot{V}$ is a positive and decrescent function and $\ddot{V}$ is negative semidefinite, we can show that the trajectory of the controlled Chua’s circuit is asymptotically stabilized to the equilibrium point $X_0$.

### III. ADAPTIVE SYNCHRONIZATION OF CHUA’S CIRCUITS

In order to observe the synchronization behavior in Chua’s circuits, we have two Chua’s circuits in which the drive system with three state variables denoted by the subscript 1 drives the response system, which has identical equations denoted by the subscript 2. However, the initial condition of the drive system is different from that of the response system. Therefore, the two Chua’s circuits are described, respectively, by the following equations:

$$\dot{x}_1 = \alpha(y_1 - x_1 - f(x_1)) ,$$
$$\dot{y}_1 = x_1 - y_1 + z_1 ,$$
$$\dot{z}_1 = - \beta y_1 - z_1 , \quad (20)$$

and

$$\dot{x}_2 = \alpha(y_2 - x_2 - f(x_2)) + u_1 ,$$
$$\dot{y}_2 = x_2 - y_2 + z_2 + u_2 ,$$
$$\dot{z}_2 = - \beta y_2 - z_2 + u_3 . \quad (21)$$

For the adaptive control approach derived for controlling chaos in the previous section, we have introduced three control inputs, $u_1, u_2$, and $u_3$, in (21). These inputs are determined for the purpose of synchronizing two identical Chua’s circuits with the same but unknown parameters, $\alpha, \beta, \gamma, \alpha$, and $b$, in spite of differences in initial conditions.

Let us define the state errors between the response system and the drive system as

$$e_x = x_2 - x_1; \quad e_y = y_2 - y_1; \quad e_z = z_2 - z_1 . \quad (22)$$

Subtracting (20) from (21) and using the notation (22) yield

$$\dot{e}_x = \alpha e_x - e_x - f(e_x + x_1) + f(x_1) + u_1 ,$$
$$\dot{e}_y = e_x - e_y + e_z + u_2 ,$$
$$\dot{e}_z = - \beta e_y - \gamma e_z + u_3 . \quad (23)$$

Clearly, the synchronization problem is now equivalent to the stabilizing problem of system (23) based on suitable choices of the control laws, $u_1, u_2$ and $u_3$. Let us now discuss the following three cases of control inputs $u_1, u_2$ and $u_3$:

(i) The state variable $x_1$ of the drive system is coupled to the first equation of the response system, and the external control with the state $x_2$ as the feedback variable is introduced into the first equation in (21).

Therefore, the feedback control law is described as

$$u_1 = - \hat{k}_1 e_x , \quad u_2 = u_3 \equiv 0 , \quad (24)$$

where $\hat{k}_1$ denotes an estimated feedback gain, which is updated according to the following adaptation algorithm:

$$\dot{\hat{k}}_1 = \kappa \hat{e}_x^2 ; \quad \hat{k}_1(0) = 0 . \quad (25)$$

Then, the resulting error dynamics can be expressed by

$$\dot{e}_x = \alpha e_x - e_x - f(e_x + x_1) + f(x_1) - \hat{k}_1 e_x ,$$
$$\dot{e}_y = e_x - e_y + e_z ,$$
$$\dot{e}_z = - \beta e_y - \gamma e_z ,$$
$$\dot{\hat{k}}_1 = \kappa \hat{e}_x^2 ; \quad \hat{k}_1(0) = 0 . \quad (26)$$

Consider a Lyapunov function candidate as

$$V = \frac{1}{2} \left( \frac{1}{\alpha} \hat{e}_x^2 + \beta \hat{e}_z^2 + \hat{e}_z^2 + \frac{1}{\alpha} (\hat{k}_1 - k_1^*)^2 \right) , \quad (27)$$

where $k_1^*$ is a positive constant which will be defined later.
Taking the time derivative of (27) and using (26) lead to

\[
V \leq -\left[|e_1|, |e_2|, |e_3|\right] \mathbf{P}(k_1^{**}) \left[|e_1|, |e_2|, |e_3|\right]^T.
\]  
\tag{28}

The first inequality is derived based on the fact that \(-e_i(f(e_i + x_1) - f(x_1)) \leq |a||e_i|^2\). The constant \(k_1^{**}\) has been appropriately chosen such that the 3 × 3 matrix \(\mathbf{P}(k_1^{**})\) in (28) is positive definite; then, \(V \leq 0\) holds. Since \(V\) is a positive and decrescent function and \(\dot{V} \leq 0\), it follows that the equilibrium point \((0, 0, 0, 0, 0)\) of system (26) is uniformly stable, and \(e(x) = 0\) is integrable with respect to time \(t\), i.e., \(e_1(t), e_2(t), e_3(t) \in L_2\) and \(\dot{\mathbf{e}}_i \in L_2\). From (28), we can easily show that the squares of \(e_1\), \(e_2\), and \(e_3\) are integrable with respect to time \(t\), i.e., \(e_1(t), e_2(t), e_3(t) \in L_2\). Next, by barbalat’s lemma, for any initial condition, (25) implies that \(\dot{e}_i(t), \dot{e}_i(t), \dot{\mathbf{e}}_i(t) \in L_2\), which in turn implies that \(e_1(t), e_2(t), e_3(t) \rightarrow 0\) as \(t \rightarrow \infty\). Thus, in the closed-loop system, \(x_2(t) \rightarrow x_1(t), y_2(t) \rightarrow y_1(t), z_2(t) \rightarrow z_1(t)\) as \(t \rightarrow \infty\). This implies that the two Chua’s circuits have been globally asymptotically synchronized under the control law (24) associated with (25).

(ii) The state variables \(x_1\) and \(y_1\) of the drive system are coupled to the first and second equations of the response system, and an external control with state \(y_2\) as the feedback variable is also introduced into the second equation in (21). Therefore, the feedback control law is described as

\[
u_1 = -\hat{k}_1 e_1, \quad u_2 = -\hat{k}_2 e_2, \quad u_3 = 0,
\]  
\tag{29}

where \(\hat{k}_1\) and \(\hat{k}_2\) comprise the estimated feedback gain, which is updated according to the following adaptation algorithm:

\[
\hat{k}_1 = \kappa_1 e_1^2; \quad \hat{k}_2 = \kappa_2 e_2^2; \quad \text{with} \quad \hat{k}_1(0) = 0; \quad \hat{k}_2(0) = 0.
\]  
\tag{30}

Then the resulting error dynamics can be expressed by

\[
\begin{align*}
\dot{e}_1 &= \alpha e_1^2 - e_1 - f(e_1 + x_1) + f(x_1) - \hat{k}_1 e_1, \\
\dot{e}_2 &= e_2 - e_1 + e_2 - \hat{k}_2 e_2, \\
\dot{e}_3 &= -\beta e_3 - \gamma e_3, \\
\hat{k}_1 &= \kappa_1 e_1^2; \quad \hat{k}_2 = \kappa_2 e_2^2; \quad \hat{k}_1(0) = 0; \quad \hat{k}_2(0) = 0.
\end{align*}
\]  
\tag{31}

Consider a Lyapunov function candidate as

\[
V = \frac{1}{2} \left[1 - a + \frac{k_1^{**}}{\alpha} - \frac{1 + \beta}{2}\right] 0 
\left[|e_1|, |e_2|, |e_3|\right]^T.
\]  
\tag{32}

where \(k_1^{**}\) and \(k_2^{**}\) are positive constants which will be defined later. Taking the time derivative of (32) and using (31) yield

\[
\begin{align*}
\dot{V} &= -\left[|e_1|, |e_2|, |e_3|\right] \Phi(k_1^{**}, k_2^{**}) - \left[|e_1|, |e_2|, |e_3|\right]^T. \\
&= -\left[|e_1|, |e_2|, |e_3|\right] \Phi(k_1^{**}, k_2^{**}).
\end{align*}
\]  
\tag{33}

The constants \(k_1^{**}\) and \(k_2^{**}\) have been appropriately chosen such that the 3 × 3 matrix \(\Phi(k_1^{**}, k_2^{**})\) in (33) is positive definite; then, \(V \leq 0\) holds. Similarly, since \(V\) is a positive and decrescent function and \(\dot{V} \leq 0\), it follows that the equilibrium point \((0, 0, 0, 0, 0)\) of system (26) is uniformly stable, and \(e(x) = 0\) is integrable with respect to time \(t\), i.e., \(e_1(t), e_2(t), e_3(t) \in L_2\). Next, by barbalat’s lemma, for any initial condition, (25) implies that \(\dot{e}_i(t), \dot{e}_i(t), \dot{\mathbf{e}}_i(t) \in L_2\), which in turn implies that \(e_1(t), e_2(t), e_3(t) \rightarrow 0\) as \(t \rightarrow \infty\). Thus, in the closed-loop system, \(x_2(t) \rightarrow x_1(t), y_2(t) \rightarrow y_1(t), z_2(t) \rightarrow z_1(t)\) as \(t \rightarrow \infty\). This implies that the two Chua’s circuits have been globally asymptotically synchronized under control law (29) together with (30).

(iii) The state variables \(x_1\) and \(z_1\) of the drive system are coupled to the first and third equations of the response system, and external control inputs with states \(x_2\) and \(z_2\) as the feedback variables are also introduced into the first and third equations in (21), respectively. Therefore, the feedback control law is described as

\[
u_1 = -\hat{k}_1 e_1, \quad u_2 = 0, \quad u_3 = -\hat{k}_3 e_3,
\]  
\tag{34}

where \(\hat{k}_1\) and \(\hat{k}_3\) are estimated feedback gains, which are updated according to the following adaptation algorithm:

\[
\hat{k}_1 = \kappa_1 e_1^2; \quad \hat{k}_3 = \kappa_3 e_3^2; \quad \hat{k}_1(0) = 0; \quad \hat{k}_3(0) = 0.
\]  
\tag{35}

Then the resulting error dynamics can be expressed by

\[
\begin{align*}
\dot{e}_1 &= \alpha e_1^2 - e_1 - f(e_1 + x_1) + f(x_1) - \hat{k}_1 e_1, \\
\dot{e}_2 &= e_2 - e_1 + e_2 - \hat{k}_2 e_2, \\
\dot{e}_3 &= -\beta e_3 - \gamma e_3, \\
\hat{k}_1 &= \kappa_1 e_1^2; \quad \hat{k}_2 = \kappa_2 e_2^2; \quad \hat{k}_1(0) = 0; \quad \hat{k}_2(0) = 0.
\end{align*}
\]  
\tag{36}

Consider a Lyapunov function candidate as
\[ V = \frac{1}{2} \left[ \frac{\alpha e_1^2 + \beta e_2^2 + e_3^2 + k_1^* (k_1 - k_1^*)^2 + k_3^* (k_3 - k_3^*)^2}{\alpha} \right], \]

where \( k_1^* \) is a positive constant which will be defined later. Taking the time derivative of (37) and using (36) yield

\[ \dot{V} = -\left[ e_1^T \right] \left[ \begin{array}{ccc} 1 & \frac{1+\beta}{2} & 0 \\ \frac{1+\beta}{2} & \beta & 0 \\ 0 & 0 & \gamma \left( 1 + k_3^* \right) \end{array} \right] \left[ e_1 \right] \]

\[ = \left[ e_1^T \right] \left[ \begin{array}{ccc} e_1^T & \frac{1}{\alpha} \Delta(k_1^*, k_3^*) \end{array} \right] \left[ e_1^T \right]^T. \]

The constants \( k_1^* \) and \( k_3^* \) have been appropriately chosen such that the \( 3 \times 3 \) matrix \( \Delta(k_1^*, k_3^*) \) in (38) is positive definite; then, \( V \leq 0 \) holds. Similarly, since \( V \) is a positive and decreasing function and \( V \) is negative semidefinite, we can conclude that the two Chua’s circuits have been globally asymptotically synchronized under control law (34) together with (35).

IV. NUMERICAL RESULTS AND PSPICE SIMULATIONS

In this section, we will present a series of numerical simulations to demonstrate the effectiveness of the proposed control schemes. All the simulation procedures were coded and executed using the software MATLAB. The fourth order Runge-Kutta integration method was used to solve the differential equations. In addition, the time step 0.001 was employed in all the simulations. The parameters of the Chua’s circuit were chosen as \( \alpha = 10 \), \( \beta = 13.14 \), \( \gamma = 0.07727 \), \( a = -1.28 \), \( b = -0.69 \) in all the simulations so that the Chua’s circuit would exhibit a chaotic behavior if no control inputs were applied. The initial states \( x(0) = 1.1, y(0) = 0 \), and \( z(0) = 0 \) were selected for use in the controlling chaos problem; however, \( x_1(0) = 1.1, y_1(0) = 0 \) and \( z_1(0) = 0 \) of the drive system and \( x_2(0) = 1.11, y_2(0) = 0 \), \( z_2(0) = 0 \) of the response system were chosen for the synchronization problem.

4.1 Controlling the chaos to equilibrium point \( X_0 \)

In the first simulation, we chose the control law as follows: \( u_1 = -k_1 x_1, u_2 = u_3 \equiv 0 \). As analyzed in the previous section, if the positive feedback gain \( k_1 \) was chosen such that \( k_1 > k_1^* \equiv \left( \frac{1}{2B} + \frac{1}{2} + |a| + \frac{B}{4} \right) = 49.889 \) held, then the chaos could be stabilized to the equilibrium point \( X_0 \).

Figure 2 depicts the state trajectories \( x(t) \), \( y(t) \) and \( z(t) \) of the controlled Chua’s circuit, in which the feedback control was activated at \( t = 20 \) sec and the feedback gain was selected as \( k_1 = 60 \). Next, we chose in the second simulation the states \( x \) and \( y \) as the feedback variables and the control laws as follows: \( u_1 = -k_1 x, u_2 = -k_2 y, u_3 = 0 \). If the feedback gains \( k_1, k_2 \) were chosen such that \( k_1 \sqrt{1 + k_2} + k_2 \left( 1 - |a| \right) > \left( \frac{1}{2B} + \frac{1}{2} + |a| + \frac{B}{4} \right)^2 \), then the chaos could be stabilized to \( X_0 \). Figure 3 shows the state trajectories \( x(t) \), \( y(t) \) and \( z(t) \) of the controlled Chua’s circuit with the feedback control activated at \( t = 20 \) sec and the feedback gains chosen as \( k_1 = 60 \) and \( k_2 = 10 \). To show the performance in terms of adaptive control of chaos, the following adaptive control laws: the control law (8) associated with (9) and the control law (12) incorporated with (13), respectively, were used to stabilize the controlled Chua’s circuit. Figure 4 and Fig. 5 show that the successful results obtained using these adaptive control schemes, respectively.

From these results, we observe that the state convergence of the second simulation is more rapid than that of the first simulation. Therefore, the control scheme for case (ii) developed in the previous section is preferred.

4.2 Adaptive synchronizing of two identical Chua’s circuits

In this case, we assumed that the drive system and the response system were two identical Chua’s circuits with different initial conditions. Figure 6 shows the evolutions of state synchronization and the history of the estimated feedback gain using feedback control law (25) associated with adaptation algorithm (26). Numerical results demonstrate that the two Chua’s circuits were asymptotically synchronized using the proposed adaptive control schemes.

4.3 Pspice simulations

We also performed a Pspice circuit simulation to observe the chaos synchronization of two identical Chua’s circuits and demonstrated the effectiveness of the proposed adaptive scheme. Figure 7 illustrates the electronic implementation of two Chua’s circuits and an adaptive controller. The electronic components with their values and types are given in Table 1. The values of the passive components, \( R_1 \), \( C_1 \), \( L_1 \), \( R_2 \), \( R_3 \), \( R_4 \), \( R_5 \), \( C_2 \), and \( N_1 \) were designed to rearrange the two Chua’s circuits in (1) by means of a normalized factor. Four operational amplifiers, \( A_1 \), \( A_2 \), associated with some passive components performed addition, substation and integration operations, and two multipliers \( X \) implemented two nonlinear items in the adaptive control law. Figure 8 depicts the Pspice simula-
Fig. 2. Simulation results of the controlled Chua’s circuit. The control law $u_1 = -60x$ is activated at $t = 20$ sec. (a) The trajectory of state $x$, (b) the trajectory of state $y$, (c) the trajectory of state $z$.

Fig. 3. Simulation results of the controlled Chua’s circuit. The control laws $u_1 = -60x$ and $u_2 = -10y$ are activated at $t = 20$ sec. (a) The trajectory of state $x$, (b) the trajectory of state $y$, (c) the trajectory of state $z$. 
Fig. 4. Simulation results of the controlled Chua’s circuit. The adaptive control law \( u_1 = -\dot{k}_1 x \) with \( \dot{k}_1 = 300x^2 \) is activated at \( t = 20 \) sec. (a) The trajectory of state \( x \); (c) the trajectory of state \( z \); (d) the history of the estimated gain \( \dot{k}_1 \).

Fig. 5. Simulation results of the controlled Chua’s circuit. The adaptive control laws \( u_1 = -\dot{k}_1 x \) and \( u_2 = -\dot{k}_2 y \) with \( \dot{k}_1 = 300x^2 \) and \( \dot{k}_2 = 10y^2 \) are activated at \( t = 20 \) sec. (a) The trajectory of state \( x \); (b) the trajectory of state \( y \); (c) the trajectory of state \( z \); (d) the history of the estimated gains \( \dot{k}_1 \) and \( \dot{k}_2 \).
Fig. 6. Simulation results of adaptive synchronization. The adaptive control law \( u_1 = -\dot{k}_1 (x_2 - x_1) \) with \( \dot{k}_1 = 300(x_2 - x_1)^2 \) is activated at \( t = 0 \) sec. (a) Synchronization of \( x_1 - x_2 \); (b) synchronization of \( y_1 - y_2 \); (c) synchronization of \( z_1 - z_2 \); (d) the history of the estimated gain \( \dot{k}_1 \).

Fig. 7. Circuit diagram of the state \( x \) synchronization of Chua’s circuits.
tion results of adaptive synchronization of two Chua’s circuits, in which the adaptive control law $u_1 = -\dot{k}_1(x_2 - x_1)$ with $\dot{k}_1 = 300(x_2 - x_1)^2$ was activated at $t = 10$ sec. From those results, we conclude that the proposed adaptive scheme works efficiently.

V. CONCLUSIONS

This paper has addressed the adaptive control and synchronization problems of Chua’s circuits. A high-gain state feedback controller approach has been derived for controlling chaos to the equilibrium point of the uncontrolled system. Adaptive controllers without knowledge of the system’s parameters have been proposed to resolve the chaos control problem. In addition, the proposed adaptive control scheme has been successfully used to asymptotically synchronize both the response Chua’s circuit and the drive circuit. All the results have been verified using the well-known Lyapunov stability theorem, in contrast to other methods for verifying chaos control and synchronization, which require numerical quantities of Lyapunov exponents and conditional Lyapunov exponents. The simulation results verify the effectiveness of the proposed schemes. Moreover, the control schemes are very simple and easy to implement. We believe that the proposed control schemes presented herein have potential for

Table 1. Electronic components with their values and device types.

<table>
<thead>
<tr>
<th>Component</th>
<th>Value or Device Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1, R2, R3, R4, R5</td>
<td>10K</td>
</tr>
<tr>
<td>R8, R9, R10, R11</td>
<td>100K</td>
</tr>
<tr>
<td>R7</td>
<td>1M</td>
</tr>
<tr>
<td>C</td>
<td>1n</td>
</tr>
<tr>
<td>R, R’</td>
<td>1.7K</td>
</tr>
<tr>
<td>R0, R0’</td>
<td>10</td>
</tr>
<tr>
<td>L, L’</td>
<td>22m</td>
</tr>
<tr>
<td>C1, C1’</td>
<td>100n</td>
</tr>
<tr>
<td>C2, C2’</td>
<td>10n</td>
</tr>
<tr>
<td>A1, A2, A3, A4</td>
<td>LF353</td>
</tr>
<tr>
<td>Analog Multipliers</td>
<td>AD632</td>
</tr>
</tbody>
</table>

Fig. 8. Pspice simulation results of adaptive synchronization. The adaptive control law $u_1 = -\dot{k}_1(x_2 - x_1)$ with $\dot{k}_1 = 300(x_2 - x_1)^2$ was activated at $t = 10$ sec. (a) Synchronization of $v_{C1} - v_{C1'}$; (b) synchronization of $v_{C2} - v_{C2'}$; (c) synchronization of $i_L - i_{L'}$. 

0V 2.0V 0V –2.0V
V(C1:2) (c)

–0.8V 0.4V 0.8V
V(C3:2) (e)

0A 5.0mA 0A
I(L2) (b)

–5.0mA 0A 5.0mA
I(L1) (b)
use in applications in the field of chaotic synchronization for secured communication systems.

**NOMENCLATURE**

\[ f(*) \quad \text{the piecewise-linear function of Chua’s circuit} \]

\[ k_1, k_2, k_3 \quad \text{feedback gains} \]

\[ k_i^* = 1, 2, 3 \quad \text{feedback gains such that } \Psi(k_i^*) > 0, \Phi(k_i^*, k_j^*) > 0 \text{ and } \Delta(k_i^*, k_j^*) > 0 \]

\[ R \quad \text{real number} \]

\[ u_1, u_2, u_3 \quad \text{control inputs} \]

\[ V \quad \text{Lyapunov function} \]

\[ x_0, x_m, X_n \quad \text{three equilibrium points of the Chua’s circuit} \]

\[ x, y, z \quad \text{state variables of the Chua’s circuit} \]

\[ \alpha, \beta, \gamma, a, b \quad \text{five parameters of the Chua’s circuit} \]

\[ \kappa_1, \kappa_2, \kappa_3 \quad \text{adaptation gains} \]

\[ \Psi(*) \quad \text{a } 3 \times 3 \text{ matrix} \]

\[ \Phi(*, *) \quad \text{a } 3 \times 3 \text{ matrix} \]

\[ \Delta(*, *) \quad \text{a } 3 \times 3 \text{ matrix} \]

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