APPLICATION OF $H_\infty$ CONTROL AND CLOSED LOOP IDENTIFICATION TO A MAGNETIC LEVITATION SYSTEM

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ABSTRACT

A systematic procedure for modeling and robust control of a multivariable magnetic levitation system is described. Our previous study revealed that an observer-based LQ controller can stabilize the system, but generates spillovers in the presence of an impulse disturbance. To solve this problem, we apply an $H_\infty$ control to suppress the spillovers caused by unmodeled dynamics which we estimate using closed loop identification. First, an exactly linearized model is obtained to compensate for nonlinearities in the system, followed by estimation of the unmodeled dynamics using closed loop identification. Second, this information is used to design a two degree of freedom $H_\infty$ servo system for suppressing the spillover while tracking a step reference input. Finally, the desired robust performance of the resulting servo system is confirmed theoretically by $\mu$-analysis and also experimentally.

Keywords: Multivariable magnetic levitation system, spillover, closed loop identification, mixed sensitivity problem of $H_\infty$ control, two degree of freedom control.

I. INTRODUCTION

Magnetic levitation systems have been recently studied as a promising noncontact transportation mechanism in various industries such as steel and semiconductor manufacturing plants [1,2]. In view of the academic background of control applications, a special issue on magnetic bearing control [3] has been published. We have been researching a multivariable control of a magnetic levitation system with a Y shape iron plate as a basic study of such mechanisms [4-9]. This system has three major obstacles to its stabilization such as input-output coupling, nonlinearities of electromagnetic forces, and destabilization of the closed loop system due to spillovers. In our previous study on an optimal servo system design for this system [5], we used exact linearization [10,11] to compensate for the nonlinearities, and obtained an exactly linearized model without coupling. This model successfully resulted in an observer-based LQ control system that achieves closed loop stability and good transient behavior. This control system, however, turned out to exhibit spillovers due to the unmodeled dynamics of vibrations in the presence of an impulse disturbance. As a result, the levitated plate instantly moved out of the limited measurable range of the gap-sensors. In order to solve this problem we need to design a robust stabilizing controller, which requires an estimation of the vibration characteristics in its uncertainty modeling. This is why we combine the $H_\infty$ control [12-15] and closed loop identification (CLID) [16-20], which is similar to what we have done in a recent study [6]. This paper is a continuation of our recent research. Our aim is to establish a practical and systematic procedure from modeling to synthesis for the design of robust control systems of magnetic levitation systems, and confirm its effectiveness in experiments.

As is well-known, the $H_\infty$ control theory has provided a sophisticated design method for robust control systems by explicitly incorporating modeling errors into design. Since its advent in the eighties [12], a variety of applications have been done particularly to control of flexible structures (see, e.g., [14,21-23]). Nevertheless, the potential difficulty we face in the application of this theory is uncertainty modeling; this is conspicuous in the case of unstable systems like a magnetic levitation system as treated here. In this paper, we pay special attention to this problem, and study the application of CLID to our magnetic levitation system as opposed to the research in [23] where a coprime factorization representation was used for...
uncertainty modeling for a large scale flexible structure which, however, is stable. This method has been studied since the seventies by many researchers as a means of identifying a model of unstable systems (see [19] for the survey). To our knowledge, however, no reports have been published on the application of this method to unstable systems nor have they included magnetic levitation systems.

The $H_{\infty}$ controller to be designed here necessitates high tracking performance because of the narrow range of measurement allowed for the gap-sensors. For this reason, we use a two degree of freedom (TDF) configuration which allows us to independently specify two frequency characteristics of disturbance attenuation and command following, thereby simultaneously suppressing spillovers and achieves good transient responses. For its justification, we analyze the control performance of the TDF servosystem with $\mu$-analysis, and show that its robust performance is guaranteed and even improved by tuning a time constant parameter which specifies a desired tracking property.

This paper is organized as follows. In section II, an experimental apparatus of the magnetic levitation system is explained. In section III, a modeling for robust control is described. First, in the physical modeling, exact linearization is applied to obtain a nominal model, while in uncertainty modeling, CLID is used to identify the characteristics of natural vibrations of the system. These two models are then combined to yield multiplicative uncertainties of the linearized model. In section IV, a robust stabilizing servo controller is designed with a mixed sensitivity problem of $H_{\infty}$ control, and the robust performance of the resulting control system is analyzed with $\mu$-analysis. The desired suppression of the spillovers and good transient behavior are verified in the experimental results. We conclude the paper with some remarks.

II. EXPERIMENTAL APPARATUS

Figure 1 depicts a diagram of the magnetic levitation system with its data processing units. This system consists of three components: a Y shape levitated object of 2kg, which is made of aluminum with small pieces of an iron plate mounted at the edges, and three electromagnets and gap-sensors. Figure 2 shows the positional relationship among these components, and the coordinate axes of the levitated object. Here mag, sen, p and O mean electromagnets, gap-sensors, acting points of electromagnetic forces, and the origin of three axes for the levitation,
respectively. The main objective of control is to levitate the plate at various reference positions and angles that may vary step wise. The control scheme for this system is: 1) the three electromagnets supply the attractive forces \( F_1, F_2, \) and \( F_3 \) to levitate the plate, 2) the resulting gap lengths \( r_1, r_2, \) and \( r_3 \) between the plate edges and the electromagnets are measured indirectly by the three eddy-current type gap sensors, which have a limited measurable range of 7.3 mm in a full scale of 10.0 mm, due to a physical constraint, 3) based on these measurements, the T805 transputer calculates the required digital command signals within a sampling time of 2.048 msec and 1.024 msec for H∞ and LQ controls, respectively, 4) these command signals in voltage and amperes which actuate the electromagnets and generate the corresponding attractive forces. Table 1 shows physical parameters of the magnetic levitation system.

### III. MODELING

In this section, we consider physical modeling using feedback linearization to obtain a linear and decoupled nominal model. We then apply CLID to obtain an uncertainty model for the vibration characteristics of the system under study.

#### A. A rigid model by means of exact linearization

Under several idealized assumptions, the equations of vertical, pitching, and rotating motions can be written respectively as

\[
M \ddot{x}_o = Mg - (F_1 + F_2 + F_3) \quad (1)
\]

\[
J_\theta \ddot{\theta}_\theta = F_1 l_{1g} - (F_2 + F_3) l_{2g} - Mgd \sin \theta_p \quad (2)
\]

\[
J_\theta \ddot{\theta}_r = (F_2 - F_3) l_{3g} - Mgd \sin \theta_r \quad (3)
\]

Here \( x_o \) is the vertical gap length between the electromagnet and the plate at the origin \( O \) right above the center of gravity \( G \), while \( \theta_\theta \) and \( \theta_r \) are the pitching and rotating angles, respectively (see Fig. 2 for the notations). Other notations are the total mass \( M \) of the plate, and moments of inertia \( J_\theta \) and \( J_\theta \) around the origin \( O \) in pitching and rotating directions \( X_\theta \) and \( X_r \), respectively; \( d \) is the distance between the origin \( O \) and the center \( G \), and \( l_{1g}, l_{2g}, l_{3g} \) are those between acting points \( p_1, p_2, p_3 \) and the axes. Gap lengths of the levitated plate at the three edges, \( r_1, r_2, \) and \( r_3 \), are written by

\[
r_1 = x_o - l_{1g} \tan \theta_p \quad (4)
\]

\[
r_2 = x_o + l_{2g} \tan \theta_p - l_{3g} \tan \theta_r \quad (5)
\]

\[
r_3 = x_o + l_{3g} \tan \theta_p + l_{3g} \tan \theta_r \quad (6)
\]

and the magnetic attractive forces can be written as a nonlinear function of input voltages \( u_1, u_2, \) and \( u_3 \) to the amplifier and the gap lengths:

\[
F_j := k_j \left( \frac{u_j}{r_j} \right)^2 \quad j = 1, 2, 3. \quad (7)
\]

Here we assume that the time delay between the input voltage and the output current is so small that it is neglected. Based on the above equations, we define the state \( x \), the input \( u \) and the output \( y \) by:

\[
x := \begin{bmatrix} x_o - x_o^* \theta_p \theta_r \end{bmatrix}^T \quad y := \begin{bmatrix} y_x \end{bmatrix}^T
\]

\[
u := \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T
\]

where \( x_o^* \) is the vertical gap length at an equilibrium state, the symbol * means its value at the equilibrium, and the output \( y \) is produced by the first three state variables which uniquely determine the position of the plate.

The nonlinearities of the system appear in the equations of rotation (2) and (3) as well as the magnetic attractive forces (7). To compensate for these nonlinearities, we use the exact linearization and decouple the input-output characteristics of the model. In the system under study, a feedback linearization is simply a transformation of the input \( u \) into a new input \( \nu \), as shown in Fig. 3:

### Table 1. Physical parameters.

<table>
<thead>
<tr>
<th>unit</th>
<th>value</th>
<th>unit</th>
<th>value</th>
<th>unit</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{1g} ) [m]</td>
<td>0.306</td>
<td>( M ) [kg]</td>
<td>1.93</td>
<td>( k_1 ) [Nm/V]</td>
<td>3.70 \times 10^{-4}</td>
</tr>
<tr>
<td>( l_{2g} ) [m]</td>
<td>0.203</td>
<td>( J_\theta ) [kgm²]</td>
<td>6.43 \times 10^{-2}</td>
<td>( k_2 ) [Nm/V]</td>
<td>1.03 \times 10^{-4}</td>
</tr>
<tr>
<td>( l_{3g} ) [m]</td>
<td>0.120</td>
<td>( J_\theta ) [kgm²]</td>
<td>1.82 \times 10^{-2}</td>
<td>( k_3 ) [Nm/V]</td>
<td>1.36 \times 10^{-4}</td>
</tr>
<tr>
<td>( l_{1q} ) [m]</td>
<td>0.341</td>
<td>( t ) [mm]</td>
<td>4.00</td>
<td>( F_1^* ) [N]</td>
<td>7.51</td>
</tr>
<tr>
<td>( l_{2q} ) [m]</td>
<td>0.238</td>
<td>( x_o^* ) [mm]</td>
<td>18.0</td>
<td>( F_2^* ) [N]</td>
<td>5.87</td>
</tr>
<tr>
<td>( c_1, c_2, c_3 ) [V/m]</td>
<td>1000</td>
<td>( d ) [mm]</td>
<td>3.24</td>
<td>( F_3^* ) [N]</td>
<td>5.87</td>
</tr>
</tbody>
</table>
Next, the definition of output yields the output equation:

\[ y = Cx \], 

where \( C = \begin{bmatrix} I_4 & 0_{3 \times 3} \end{bmatrix} \).

Finally, combining (10) with (11) yields the following linearized input-output decoupled model as a nominal model:

\[ G(s) = C(sI - A)^{-1}B = \frac{I_4}{s^2} \]  

While this model does not depend on the levitated positions of the plate, unlike the one linearized with Taylor’s series expansion, we need an exact measurement of \( y = \begin{bmatrix} x_u - x^*_u & \theta_p & \theta_r \end{bmatrix}^T \) to realize this model through the feedback structure of Fig. 3. This measurement can be obtained through the following relation:

\[ y_x - y^*_x = R \begin{bmatrix} x_u - x^*_u \\ \tan \theta_p \\ \tan \theta_r \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \left( 1 - \frac{1}{\cos \theta_p \cos \theta_r} \right), \]

where

\[ R = \begin{bmatrix} -c_1 & c_1 J_1/h & 0 \\ -c_2 & c_2 J_2/h & c_2 J_3/h \\ -c_3 & c_3 J_3/h & c_3 J_3/h \end{bmatrix}, \]  

and \( y_x = \) the displacement of vertical positions in voltage between three sensor heads and edges of the plate, and \( r \) is the thickness of each edge of the plate. Here \( c_1, c_2, \) and \( c_3 \) are coefficients for converting each gap between the plate and each sensor to its voltage, respectively. The second term of (13) can be neglected since its value is much smaller than the minimum resolution of the gap-sensors. This yields the following approximate equation of \( y \):

\[ \begin{bmatrix} x_u - x^*_u \\ \tan \theta_p \\ \tan \theta_r \end{bmatrix} \sim R^{-1}(y_x - y^*_x) = \begin{bmatrix} \frac{y_{x1}}{y_{x2}} \\ \frac{y_{x2}}{y_{x3}} \end{bmatrix} \Rightarrow \begin{bmatrix} x_u - x^*_u \\ \theta_p \\ \theta_r \end{bmatrix} = \begin{bmatrix} \frac{y_{x1}}{y_{x3}} \\ \frac{y_{x2}}{y_{x3}} \end{bmatrix}. \]  

**B. An uncertainty model by means of closed loop identification**

In our previous research the system under study was stabilized with an observer-based LQ controller for the exact linearized model obtained above. However, as shown in Fig. 4 by the FFT, the closed loop system has natural vibration modes in the lateral and longitudinal directions around 64 and 74 Hz, respectively; this is due to the the unmodeled dynamics in the physical modeling and sensor noises at 60 Hz of source frequency. Such model uncertainty instantly causes the spillovers as shown in [5], if an impulse disturbance is added in the system stabilized with the LQ controller. To estimate the vibration characteristics as accurately as possible, we use CLID since it is difficult to obtain an analytical model of our system because of its complicated shape.

As stated in [17-19], there are two methods for CLID which produces a true plant model \( G(s) \) in the closed-loop configuration of Fig. 5. One is the indirect method which indirectly produces it from the given two stabilizing controllers \( K_c(s) \) and \( K_n(s) \) together with the associated two time series models \( G_{c1}(s) \) and \( G_{c2}(s) \) of \( y \), which we can identify using stationary output data of the closed loop.
system. The other is the direct method which directly identifies it using an input $v$ to the plant and the output $y$ like ordinary open-loop system identification. Both methods have the same identifiability condition, which is satisfied by simply switching a stabilizing controller to another one [17]. However, the former one allows us to utilize a lot of prior knowledge such as the natural frequencies of 60, 64, and 74 Hz, unlike the latter one. For this reason we focus on the indirect method here. Regarding the direct method, we show only the results for comparison.

The procedure for obtaining the uncertainty model $\Delta_T(s)$ is as follows.

**Step 1. Identifiability**

We prepare two different controllers $K_1(s)$ and $K_2(s)$ that satisfy identifiability for CLID. In a square system, it is known [16,17] that a condition for identifiability is expressed by

$$\det \left( K_2(e^{j\omega}) - K_1(e^{j\omega}) \right) \neq 0 \text{ for } \forall \omega,$$

where an excitation signal input $r$ is set to 0. It will be shown later that this condition is satisfied with a special type of LQ controller called “ILQ controller”, which will be described later, if its design parameters are chosen properly.

**Step 2. Experiment for closed loop identification**

Using an ILQ controller $K_1$, we levitate the vehicle at the position $y_1$ of 3 mm above the equilibrium point $x^*_v$, where natural vibrations become large, and then switch the controller $K_1$ to another one $K_2$ under stable levitation in order to obtain the corresponding output data as in Fig. 6 (top: $K_1 \rightarrow K_2$, middle: $K_1$, bottom: $K_2$).

**Step 3a. Identification of time series model**

For each controller $K_1$, $K_2$ we identify two time series models $G_{yw1}(s)$ and $G_{yw2}(s)$ as indicated in Fig. 5 as three input/output AR models using Least Squares Method (LMS): $A(q)\hat{y}(t) = w(t)$ where $A(q) = I + A_1q^{-1} + \ldots + A_nq^{-n}$; $q^{-1}$ is a shift operator, $n$ is the degree of the shift, and $w$ is assumed to be a white noise. The frequency characteristics of these two models are shown in Fig. 7. These time series models $G_{yw}(s)$ are obtained in state space form, if we estimate their parameters based on *apriori* knowledge such as the three modes of the sensor noise at 60 Hz, and the spillovers at 64 and 74 Hz in Fig. 4. We set the orders of $G_{yw1}(s)$ and $G_{yw2}(s)$ to 18 (i.e., $n = 6$) and the resampling rate for the identification to $1.024 \times 5$ msec by using decimation. In addition, the data number is selected to be $1.5 \times 10^4$ samples in each controller by tuning it as large as possible to improve the signal/noise ratio.

**Step 3b. Determination of true plant model by indirect method**

Figure 8 shows the $\sigma$-plot of a true plant $\hat{G}(s)$ of 84th order that is indirectly determined by the following simple manipulation:

$$\begin{align*}
&\text{Fig. 4. Power spectral density of closed loop stationary output data [ILQ].} \\
&\text{Fig. 5. Configuration of CLID by indirect method.}
\end{align*}$$

Fig. 6. Experimental data under stable levitation using ILQ controllers $K_1(s)$ and $K_2(s)$. $K_1(s) : 0 \rightarrow 15.4$ [sec], $K_2(s) : 15.4 \rightarrow 30.8$ [sec]
\[
G_{yw1}(s) = [I + G(s)K_1(s)]^{-1} G_w(s),
\]
(16)
\[
G_{yw2}(s) = [I + \tilde{G}(s)K_2(s)]^{-1} G_w(s)
\]
\[
\Rightarrow \tilde{G}(s) = [G_{yw2}(s) - G_{yw1}(s)]
\]
\[
[K_1(s)G_{yw1}(s) - K_2(s)G_{yw2}(s)]^{-1}
\]
(17)

The corresponding result by the direct method (DM) is also shown for comparison. Note that the equations (15), (16), and \(K_1(I + \tilde{G}K_2)^{-1} = (I + K_1G)^{-1}K_1\) lead to the nonsingularity of \([K_1G_{yw1} - K_2G_{yw2}] = (I + K_1G)^{-1}(K_1 - K_2)\) \((I + \tilde{G}K_2)^{-1}G_w\) in (17). The point of this manipulation is that a noise model \(G_w(s)\) does not change after switching a controller to another one. It is empirically known that a frequency band of highly accurate identification lies in about 1 decade from \(1/100T_s\) to \(1/5T_s\) [24], i.e., from 10 Hz to 200 Hz due to \(T_s = 1.024\) msec of sampling time. As is expected from this empirical experience, the identified result turns out to be good in the same limited range, but suspicious outside this range due to systematic errors and the inaccuracy of identification in the lower frequency range due to the Least Squares Method [16].

**Step 4. Uncertainty model**

From the nominal model \(G(s)\) and a true plant \(\tilde{G}(s)\) we obtain a multiplicative uncertainty model \(\Delta_T(s)\) of 90th order by its definition:
\[
\Delta_T(s) = \tilde{G}(s)G^{-1}(s) - I
\]
(18)
together with a weighting function \(W_T(s)\) for robust stability such that \(\Delta_T(j\omega) \leq W_T(j\omega)\) for all \(\omega\), which will be used later in the \(H_\infty\) design in the next subsection IV.B. Their \(\sigma\)-plots are shown in Fig. 9 together with the corresponding result by the direct method (DM). The accuracy of the identification at lower frequencies is insufficient as stated above, although this is not a serious problem. In fact, we should estimate \(W_T(s)\) so as to have gains as low as possible in this frequency range as in Fig. 9, where the corresponding behavior is governed by the rigid-body motion. In the range of middle frequency, we use the identified result to select \(W_T(s)\) as accurately as possible to avoid causing spillovers. In the higher frequency range, we select \(W_T(s)\) to have relatively large gains since we cannot identify the characteristics in this range.

**IV. ROBUST CONTROL SYSTEM DESIGN**

In this section we apply a mixed sensitivity problem of \(H_\infty\) control [12,13] to design its servo controller [13-15] with a TDF configuration [27].

**A. Design results using ILQ control**

In our previous study [4-5], we obtained an observer-
based ILQ controller, which focuses on the tracking property of the servo system and aims to asymptotically achieve its desired transfer function in a decoupled form. The resulting closed loop system, which is shown in Fig. 10, has two pairs of transfer functions for tracking and disturbance attenuation. In the former pair, one of the two is from reference input \( r \) to output \( y \), i.e., \( \mathcal{G}_{\nu}(s) \), which has an interesting asymptotic property as the tuning parameter \( \Sigma \) increases:

\[
\mathcal{G}_{\nu}(s) = C_x(sI_3 - A_x)^{-1} B_x \rightarrow \frac{I_3}{(1 + Ts)^2} \ (\Sigma \rightarrow \infty),
\]

\[
A_x := \begin{bmatrix} A - B \Sigma K^0_3 \ B \Sigma K^0_1 \ - C \ 0_{3 \times 3} \end{bmatrix} , \quad B_x := \begin{bmatrix} 0_{3 \times 3} \end{bmatrix} , \quad C_x := \begin{bmatrix} C \ 0_{3 \times 3} \end{bmatrix}
\]

(19)

The other is from \( r \) to error \( e \), i.e., \( \mathcal{G}_{\nu}(s) = -C_x(sI_3 - A_x)^{-1} B_x + I_3 \). The latter pair are those from noise \( n \) to \( e \), i.e., \( \mathcal{G}_{\nu}(s) \) and \( \mathcal{G}_{\omega}(s) \), which are equivalent to a complementary sensitivity function \( T(s) \) and a sensitivity function \( S(s) \), respectively:

\[
\mathcal{G}_{\nu}(s) = -\mathcal{G}_{\omega}(s) \left( I_3 + s(\Sigma K^0_1)^{-1} \Sigma K^0_3 \left[ \begin{array}{c} A \ C \\ B \ D \end{array} \right] \right) \leftrightarrow T(s)
\]

(20)

\[
\mathcal{G}_{\omega}(s) = -\mathcal{G}_{\omega}(s) \left( I_3 - \frac{A - B \Sigma K^0_3}{C} B \Sigma K^0_3 \left[ \begin{array}{c} A \ C \\ B \ D \end{array} \right] \right) \leftrightarrow S(s)
\]

(21)

They are closely related with the observer used:

\[
\dot{w} = \dot{\bar{w}} + \bar{B}u + \bar{G} \bar{y} , \quad \dot{x} = \bar{C}w + \bar{D} \bar{y} .
\]

(22)

Here we use a minimal order observer and design \( \bar{A} \) so as to have stable triple eigenvalues. The observer-based ILQ design has two design parameters with clear physical meanings [25,26] other than the observer poles \( P \). One of them is a time constant parameter \( T \) specifying approximate output responses given by (19). The other is a gain tuning parameter \( \Sigma \) trading off between an asymptotic tracking property \( \mathcal{G}_{\nu}(s) \) and the magnitude of the control input \( u \) in Fig. 10. The ILQ feedback gains are written as

\[
K_r = \Sigma K^0_r , \quad K_t = \Sigma K^0_t , \quad \Sigma = \sigma I_3 (\sigma \geq \sigma)
\]

(23)

Here the gain tuning parameter \( \sigma \) must be no less than its lower bound \( \sigma \) in order to guarantee the closed loop stability. The nominal feedback gains \( K_r^0 \) and \( K_t^0 \) are analytically given in terms of \( T \) as follows in our system:

\[
K_r^0 = \left[ 2I_3 / T \ I_3 \right] , \quad K_t^0 = I_3 / T^2 .
\]

(24)

A desired objective of control is to make the step size of \( y_t \) at least more than 6 mm under levitation of the levitated vehicle at \( y_t = y_0 = 0 \) rad. Based on experimental results [4-5] and trial-and-error, the best values of the design parameters are selected as:

\[
T = 0.08 , \quad \sigma = 20 \times \sigma \ (\sigma = 50) , \quad P = -130 .
\]

(25)

As a result, we obtain an observer-based ILQ controller \( K(s) \) of 6th order in a decoupled form: \( K(s) = k(s)I_3 \). Hence, two such controllers \( K_1(s) = k_1(s)I_3 \), \( K_2(s) = k_2(s)I_3 \) satisfy the identifiability condition (15) if

\[
\left| k_i(j\omega) - k_j(j\omega) \right| > 0 \text{ for all } \omega .
\]

Then we see from a formula

\[
\left| k_i(j\omega) - k_j(j\omega) \right| > \left| \left| k_i(j\omega) \right| - \left| k_j(j\omega) \right| \right|
\]

that this inequality holds if the gain characteristics of \( k_i(s) \) and \( k_j(s) \), or the \( \sigma \)-plots of \( K_1(s) \) and \( K_2(s) \), do not cross. In fact, they do not cross each other for all frequencies, as shown in Fig. 11.

The step and impulse responses of the control system designed above are shown in Fig. 12 and Fig. 13, respectively. Figure 12 shows good and smooth tracking property of the output responses. However the real inputs \( u \) are disturbed by sensor noise, and eventually the unmodeled dynamics may cause spillovers if this oscillation is ampli-
fied by some commands or disturbances. In fact, we see in Fig. 13 that the unmodeled dynamics of the system gradually causes destabilization by addition of an impulse disturbance to the plate at the time indicated by the arrow; these are the spillovers we have repeatedly mentioned.

B. \( H_\infty \) control system design

To suppress the spillovers, we design an \( H_\infty \) control system, focussing on the design of a TDF servosystem that achieves both good tracking and robust stability.

B.1 Mixed sensitivity problem of \( H_\infty \) control

In view of the fact that the spillovers are generated by an impulse disturbance, we design an \( H_\infty \) controller \( K(s) \) with a trade-off between robust stabilization and disturbance attenuation in the mixed sensitivity problem of \( H_\infty \) control [12,13]. In this design we regard the flexible modes as multiplicative uncertainties \( \Delta T(s) \) and create a controller satisfying the following criterion:

\[
\begin{bmatrix}
\gamma_{opt} W_S(s) S(s) \\
W_T(s) T(s)
\end{bmatrix} < 1
\]  

Here \( S(s) \) and \( T(s) \) are sensitivity and complementary sensitivity functions, respectively, with the corresponding weighting functions \( W_S(s) \) and \( W_T(s) \) such that \( \sigma \{ \Delta \} \leq W_T(j\omega) \) for all \( \omega \).

B.2 Generalized plant for servo system

Although the exactly linearized model \( G(s) \) theoretically involves double-integrators, the real magnetic levitation system does not have such dynamics due to the existence of model uncertainties. By internal model principle, therefore, the servosystem should be designed so as to contain integrators in an \( H_\infty \) controller [13]. Thus, we consider a mixed sensitivity problem for an augmented plant with an internal model as in Fig. 14. The design procedure is as follows. First, we define a generalized plant \( P(s) \) for \( H_\infty \) design:

![Fig. 14. Generalized plant for servo system.](image)
where $\alpha(s)$ is defined by $\alpha(s) = \frac{s}{s + \alpha} \cdot I_s (\alpha > 0)$. Note that $P(s)$ does not satisfy the well-known standard assumption of the $H_\infty$ control since both $G(s)$ and $\alpha^{-1}(s)$ have poles at the origin. This problem can be resolved by an affine transformation $s = \alpha - \beta (\alpha > \beta > 0)$ used in a robust pole assignment [14]. This transformation is also introduced to shift the real parts of all stable poles further to the left of $-\beta$ by $H_\infty$ design, and $\beta$ can be utilized as an effective tuning parameter to quicken the convergence of error $e$ in $H_\infty$ norm. Therefore, stable triple poles $(\beta - \alpha)$ of $W_s(\pi)$ $\alpha(\pi)$ and a central controller $K_{mp}(\bar{s})$ are cancelled by the same zeros of $\alpha^{-1}(\pi)$ in Fig. 14 as follows:

$$K(\pi) = K_{mp}(\bar{s})\alpha^{-1}(\pi) = \frac{K_{mp}(\bar{s})}{\pi - \beta + \alpha} \cdot \frac{\pi - \beta - \alpha I_s}{\pi - \beta},$$

$$= K_{mp}(\bar{s}) \cdot \frac{I_s}{\pi - \beta}, \quad (28)$$

Last, by converting it into the frequency domain $s$ again, we obtain a final $H_\infty$ servor controller $K(s)$ with integrators but the stable zeros $-\alpha$ of $\alpha^{-1}(\pi)$ being deleted:

$$K(s) = K_{mp}(s) \cdot \frac{I_s}{\pi}. \quad (29)$$

**B.3 Two degree of freedom control system**

As stated in the introduction, we design the TDF control servosystem [27] shown in Fig. 15 to achieve good tracking as well as robust stability. With this configuration, we can independently achieve a desired input-output property $G_y(s)$ and a desired disturbance rejection property $G_n(s)$ unlike the observer-based LQ control system. These properties are expressed together with $G_n(s)$ and $G_m(s)$ by:

$$G_n(s) = N(s)H(s)$$

$$G_m(s) = [I_3 + G(s)K(s)]^{-1}H(s) = 0$$

$$T(s) := G_m(s) = [I_3 + G(s)K(s)]^{-1}G(s)K(s) = G_{ym}(s)$$

$$S(s) := G_m(s) = [I_3 + G(s)K(s)]^{-1}$$

given a coprime factorization of the nominal model $G(s) = N(s)H^{-1}(s) \in RH_\infty$ and a free feedforward parameter $H(s) \in RH_\infty$. As is well-known, this TDF configuration possesses the following properties:

- $G_y(s)$ is independent of the selection of $K(s)$ unless $G(s)$ has uncertainties.
- $K(s)$ works only when $y_c$ is not equal to $\hat{y}$ unless $G(s)$ has uncertainties.
- $H(s) \in RH_\infty$ can be set to $I_3$ if $G(s)$ is minimum phase.

In practice, however, the tracking property $G_y(s)$ of the real plant is subject to both model uncertainty and sensor noise, so the robust controller $K(s)$ always has to work for its compensation. It is thus important to evaluate its robust control performance with $\mu$-analysis, which will be described later.

**C. Selection of weighting functions**

In spite of a variety of $H_\infty$ design applications to robust control, it is difficult to select appropriate weighting functions covering uncertainties for robust stabilization. In this section, we show an interesting way to select a nonconservative weighting function $W_T(s)$ for robust stability. Our selection is based on the fact that the unstable system has been nominally stabilized with the observer-based LQ controller. In our selection, a key role is played by the gain property of $T(s)$, or equivalently, $\overline{\sigma}_n(s)$ for the nominally stabilized LQ control system. First, we observe that the gain plot of $T(s)$ in $H_\infty$ design is known to approach that of $W_T^{-1}(s)$ in the higher frequency range. This suggests a selection of $W_T(s)$ such that a gain plot of $W_T^{-1}(s)$ is similar to that of $\overline{\sigma}_n(s)$ in this range. In addition, $W_T(s)$ must be chosen so as to satisfy $\overline{\sigma}[\Delta_f(j\omega)] \leq \left| W_T(j\omega) \right|$ for all $\omega$, particularly in the middle frequency band of accurate identification of $\Delta_f(j\omega)$, as mentioned before. These observations suggest the following way to select $W_T(s)$:

$$W_T(s) = w_T(s)I_3,$$

where $w_T(s) = w_{Tn}(s) \cdot w_{Tf}(s)$.

Here $w_{Tn}(s)$ is a proper stable function that approximates the gain of $\overline{\sigma}_n(s)$, and $w_{Tf}(s)$ is the one such that $\overline{\sigma}[\Delta_f(j\omega)] \leq \left| w_{Tf}(j\omega) \right|$ for all $\omega$. Following this guideline together with a little trial-and-error, $W_T(s)$ is selected as:

![Two degree of freedom control system](image-url)
\[ W_F(s) = w_T(s)I_3, \]
\[
 w_T(s) = \frac{s^2 + 2 \cdot 1.4 \cdot (2\pi \cdot 60)s + (2\pi \cdot 60)^2}{1.1 \times (2\pi \cdot 60)^2} 
\cdot \frac{s^2 + 2 \cdot 1.4 \cdot (2\pi \cdot 10)s + (2\pi \cdot 10)^2}{s^2 + 2 \cdot 0.02 \cdot (2\pi \cdot 64)s + (2\pi \cdot 64)^2}, \tag{34} \]

which has a notch characteristics for suppressing the spillovers.

Following a similar guideline, the selection of \( W_S(s) \) as shown in Fig. 16, was made based on such important gain characteristics of \( G_{en}(s) \) as a roll-off rate of 60 dB/dec around 2 Hz as well as its peak gain around at 20 Hz. As a result, the following weighting function \( W_S(s) \) is selected after a little trial-and-error:

\[ W_S^{-1}(s) = w_S^{-1}(s) \cdot I_3, \]

where

\[
 w_S(s) = 0.47 \times \frac{s^2 + 2 \cdot 1.4 \cdot (2\pi \cdot 12)s + (2\pi \cdot 12)^2}{s^2 + 2 \cdot 1.0 \cdot (2\pi \cdot 0.4)s + (2\pi \cdot 0.4)^2}. \tag{35} \]

D. Design results using \( H_\infty \) control

The design procedure for an \( H_\infty \) servosystem with the TDF configuration of Fig. 15 is as follows.

Step 1. Characteristics of noise and disturbance attenuation \( G_{yn}(s) \)

An \( H_\infty \) servo controller \( K(s) \) is determined by solving \( \gamma \)-iteration for (26). The parameters \( \alpha \) and \( \beta \) with \( \alpha > \beta > 0 \) introduced in IV-B.2 are chosen as 1.01 and 1.00, respectively. As a result, the final \( H_\infty \) servo controller \( K(s) \), the complementary sensitivity function \( T(s) \), and the sensitivity \( S(s) \) are shown in Figs. 17, 18, and 16, respectively. From Fig. 17, it turns out that the \( H_\infty \) controller plays the role of a deep notch filter, by which the spillovers around 64 Hz can be suppressed. This provides a striking contrast to the ILQ controller.

Step 2. Characteristics of desired tracking \( G_{yr}(s) \)

In the TDF control system stated above, we can achieve a certain desired tracking property \( G_{yr}(s) \) by using the following precompensator.

\[
 N(s) = \frac{I_3}{(1 + Ts)^2} \in RH_\infty, \quad H(s) = I_3 \in RH_\infty \tag{36} \]
\[
 D(s) = \frac{s^2}{(1 + Ts)^2}I_3 \in RH_\infty \tag{37} \]

Fig. 17. The gain plots of \( K(s) \) by \( H_\infty \) control.

\[ \gamma_{op} = 1.00 \] of 21st ord.; \[ \cdot \cdot \cdot \] : [ILQ] of 6th ord.

Fig. 18. The gain plots of \( G_{yr}(s) \) and \( G_{yn}(s) \).

\[ \cdot \cdot \cdot : W_T^{-1}(s) \text{ of 12th ord.}, \quad \cdot \cdot \cdot : G_{yr}(s), \quad \cdot \cdot \cdot : G_{yr}(s) (T = 0.08) \]
which yields \( G_r(s) = N(s) \). Equation (36) follows from the fact that the nominal model \( G(s) \) is of a relative degree 2 and minimum phase. Equation (37) is then obvious from (12). Figure 18 shows the gain plot of the resulting \( G_r(s) \).

As a result, the final TDF configuration requires the desired robust controller to have a total of 33rd order.

E. Robust performance analysis

In this section, we use \( \mu \)-analysis [7] to analyze the robust control performance of the two degrees of freedom \( H_\infty \) control system in Fig. 19 as designed above. First, let us consider a one degree of freedom (ODF) \( H_\infty \) control system, which is indicated by the dotted line in Fig. 19. By the main loop theorem [13], we can characterize this performance as:

\[
\mu_\Delta(M(j\omega)) < 1 \quad \text{for all } \omega
\]

(38)

where

\[
M(s) = \begin{bmatrix}
W_c(s)S(s) - W_f(s)S(s) \\
W_f(s)T(s) - W_f(s)T(s)
\end{bmatrix} \in RH_\infty,
\]

\[
\Delta(s) = \text{diag} \{ \Delta_c(s), \Delta_r(s) \} \in BH_\infty
\]

(39)

The maximum value of \( \mu \) for this case (ODF) is shown in Table 2 and Fig. 20. As shown in Table 2 and in Fig. 20, the maximum value of \( \mu \) is 1.41. This means that robust performance is not obviously achieved. In fact, we were not able to increase the step size of the vertical position \( y_1 \) more than 0.3 mm without generating spillovers.

The so-called coefficient matrix \( M(s) \) for TDF, as defined in Fig. 19, is expressed by

\[
M(s) = \begin{bmatrix}
0_{y_1x} & -W_c(s)S(s) \\
W_f(s)N(s) - W_f(s)T(s)
\end{bmatrix} \in RH_\infty
\]

(40)

Table 2. Robust performance and selecting of time constant \( T \).

<table>
<thead>
<tr>
<th>Time constant ( T )</th>
<th>ODF</th>
<th>TDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T/4 )</td>
<td>1.41</td>
<td>1.08</td>
</tr>
<tr>
<td>( T/2 )</td>
<td>0.91</td>
<td>0.87</td>
</tr>
<tr>
<td>( T )</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>( 2T )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note the difference in the first column compared to (39), which leads to an improvement in the performance. The maximum values of \( \mu \) are shown in Table 2 for various values of \( T \) which specify the desired transfer function \( G_r(s) \). From this table we see that robust performance is achieved by selecting a \( T \) that decreases \( \mu \) to less than 1.

We select a small time constant \( T = 0.08 \) as used in the observer-based ILQ control, because the maximum value of \( \mu \) cannot be decreased further below 0.87 as shown in Table 2 and Fig. 20. In the TDF control system with \( G_r(s) \neq G_n(s) \), unlike one degree of freedom control systems with \( G_r(s) = G_n(s) \), each property can be tuned to the designer’s satisfaction so that the frequency band width of \( G_n(s) \) is far narrower than that of \( T(s) \). As a result, the maximum of \( \mu \) is essentially and substantially improved.

F. Experimental results and discussion

Figs. 21 and 22 show the experimental results of step and impulse responses, respectively, by using the \( H_\infty \) controller above. Here we examine the effectiveness of the design procedure. From the experimental results of the step response shown in Fig. 21, we observe that the two degree of freedom \( H_\infty \) control system shows the same good tracking and decoupling properties as that of the ILQ control system shown in Fig. 12. Both controllers can levitate the vehicle up to the limit of the maximum measurable range of the gap-sensors. While the \( H_\infty \) controller is a little inferior to ILQ controller with respect to its tracking
performance, it effectively suppressed the spillover in a position near the electromagnets as shown in Fig. 22. This means that the $H_{\infty}$ controller is superior to ILQ controller with respect to its ability to suppress spillovers. The difference in order between the final ILQ and $H_{\infty}$ controllers is 27. This is because the requirement for robustness necessitates the high accuracy of modeling, which leads to a high order $H_{\infty}$ controller. Therefore, it might be important to study an effective reduction of the robust controller such as the reduction of a performance-preserving frequency weighted controller reduction [13,28] in a future study.

V. CONCLUSION

In this paper we have discussed the detail of a consistent modeling and design procedure that suppresses the spillovers through $H_{\infty}$ control. The nice thing about this procedure is a unification of robust control and uncertainty identification. We have proposed an interesting technique of uncertainty modeling. The proposed control system for the magnetic levitation system with a TDF configuration possesses not only robust stability for suppressing the spillovers but also robust performance for a good tracking behavior. The experimental result is remarkably successful, which obviously proves the potential of this method in practice.

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REFERENCES


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