FAULT TOLERANT CONTROL OF FEEDBACK LINEARIZABLE SYSTEMS WITH STUCK ACTUATORS

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ABSTRACT

Fault tolerant control of feedback linearizable systems with stuck actuators is studied in this paper. Once stuck, the faulty actuators cannot respond to the control inputs and have fixed constant outputs. The fault considered here is severe in the sense that the faulty system is no longer feedback linearizable. By constructing a nonlinear transformation and extending Chen's disturbance observer to a multi-dimensional case, a generalized disturbance observer is developed for estimating the unknown constant outputs of the faulty actuators. Then, through integrating the generalized disturbance observer with a controller constructed by the famous cascade design method, a fault tolerant controller is obtained. It is proven that the fault tolerant controller ensures not only boundedness of the state of the faulty system but also satisfactory output performance. Finally, computer simulations are done using an inverted pendulum model to verify the effectiveness of the proposed method.

Key Words: Feedback linearizable systems, stuck actuators, fault tolerant control, disturbance observer.

I. INTRODUCTION

Increasing safety and performance requirements in modern industry lead to more and more complex control systems, which make faults inevitable. When a fault occurs in a control system such as a malfunction in some actuator or sensor, the controller may not act as expected, which usually causes performance degradation of the system or even makes the system unstable. So it is of practical interest to design controllers that can guarantee the system stability and maintain satisfactory performance when the fault occurs. This is the main objective of fault tolerant control (FTC) [1].

Developed in the 1970s, FTC has experienced great progress in the past three decades [1]. Generally, the fault information, such as where the fault is and how bad the fault is, can be very useful information in designing a fault tolerant controller. According to how this fault information is used, there are two types of FTC methods: passive and active methods [1]. Assuming the possible faults are known a priori, the passive method takes into account of all these possible faults in the design stage and does not change the controller when the fault occurs [2–7]. An active method usually uses a fault detection and isolation (FDI) unit to collect the fault information and changes the controller according to the fault [8–28]. Intuitively, the active method uses more information, thus, can achieve better performance than the passive method. One typical active FTC system widely used in the literature contains a normal controller, a FDI unit, and a fault estimation and controller redesign module [8, 9, 11–15, 19, 20, 23–28] as illustrated in Fig. 1. When a fault is detected, the FDI unit determines where the fault is, i.e., fault isolation, then the online fault estimation algorithm is activated. Based on this estimation, the controller is redesigned...
to compensate for the effect of the fault. In this paper, we also adopt this typical active FTC system.

One important fault scenario is that of stuck actuators, which is common in flight control and chemical processes. Once stuck, the faulty actuators can no longer respond to the control inputs and have fixed constant outputs. Thus, the faulty actuators not only reduce some freedom in the control ability but also act as persistent external disturbances [9]. The related FTC problem is very challenging in the sense that the remaining healthy actuators must compensate for the effect of the faulty ones and maintain the original performance as much as possible at the same time. Generally speaking, there are mainly two difficulties in the related FTC problem. Due to the additional cost for extra components, in practice, the outputs of actuators are usually not measured. Thus, the constant outputs of the stuck actuators, i.e. stuck positions, are difficult to obtain. This is the first difficulty. The second one is: given the stuck positions, designing a fault tolerant controller is still not an easy task. This is especially true when the structure of the original system is dramatically changed by the faulty actuators. Boskovic [8] and Tao [21] solved the FTC problem by adaptive control techniques. Chen [9] presented an iterative learning observer to estimate the unknown stuck positions. All of them assume a matching condition, i.e., the correct outputs of the stuck actuators can be completely generated by the remaining healthy ones, which makes the fault tolerant controller design very easy when the stuck positions are available. When this matching condition does not hold, their methods cannot be applied.

Feedback linearizable systems (FLS) are a simple class of nonlinear systems, which can be turned into linear systems by appropriate feedback control [29]. Based on the linear systems, the conventional linear controller design techniques can be used. Therefore, the control problem of FLS is relatively easier compared with that of the general nonlinear systems. Although simple, FLS is a very important class of nonlinear systems, which is common in physical nonlinear systems such as robotics, electric machines, autopilot control, and flight mechanics [30]. Thus, the FTC problem for FLS with stuck actuators is of practical interest. To solve this problem, Tao [22] presented an adaptive compensation framework which required no FDI unit. Global stability is guaranteed by adjusting the controller’s parameters online. Liang [4] proposed a passive approach. Separating the actuators into a potentially faulty part and non-faulty part, the main idea is to design a basic controller by the use of the non-faulty ones. When the potentially faulty actuators are healthy, they improve the performance; and when they are stuck, their effects can be compensated for by the basic controller. Another method for resolving the same problem using optimization techniques was also investigated in [10]. However, none of these methods considered the two above mentioned difficulties simultaneously. Reference [10] assumed that the stuck positions were known, which made the first difficulty trivial, while references [4] and [22] assumed that the faulty system is still feedback linearizable, i.e. satisfies the matching condition, making the second difficulty trivial. Therefore, further work on this problem is necessary.

In this paper, we consider an FTC problem for FLS with stuck actuators. The two above mentioned difficulties are considered simultaneously, i.e., we assume that the stuck positions are unknown and the faulty system is no longer feedback linearizable. The typical active FTC system depicted in Fig. 1 is used. Due to the great development of fault detection and diagnosis techniques, in this paper, we assume that the FDI unit has been successfully designed. The aim of this paper is to design the online fault estimation algorithm and redesign the controller for the faulty system to compensate for the effect of the stuck actuators. Since the faulty system is not feedback linearizable, the controller design becomes difficult. To deal with the difficulty, a nonlinear transformation is first constructed using geometric methods, which can transform the faulty system into a cascade system. Under an assumption of the feasibility of the famous cascade design method [31], a controller is then designed for the faulty system. On the other hand, based on the nonlinear transformation, a generalized disturbance observer (GDO) is presented by the nonlinear observer design techniques to estimate the unknown stuck positions. Finally, integrating the controller with the GDO, a fault tolerant controller is obtained. Compared with the existing work already reported in the literature, the contribution of this paper is mainly in the following three aspects:

1. A GDO is newly developed which can estimate the stuck positions accurately. Moreover, its de-
sign is systematic and simple. As a matter of fact, the proposed GDO is a multi-dimensional extension of Chen’s disturbance observer [32]. This extension is nontrivial and greatly enlarges the application field of Chen’s result.

2. The two difficulties in FTC for control systems with stuck actuators are considered simultaneously, which makes the proposed approach less conservative and more practical for real systems.

3. The boundedness of the state and the satisfactory output performance of the resultant FTC system are rigorously proved.

The remaining parts of this paper are organized as follows. Section II introduces the problem formulation. Section III contains the main results of this paper including the construction of the nonlinear transformation, the design of the GDO, the controller design by the cascade design method with the true stuck positions and the fault tolerant controller design as well as the theoretical proofs of its performance. Section IV presents simulation results while Section V concludes the paper with final remarks.

Notations. $A^T$ and $A^{-1}$ are, respectively, the transpose and the inverse of a matrix $A$. $I_n$ denotes an identity matrix of dimension $n$. $0^{m	imes n}$ stands for an $m 	imes n$ zero matrix. $\| \cdot \|$ denotes the Euclidean norm or its induced norm. $\text{col}(x, y)$ implies that the two vectors $x$ and $y$ are stacked to produce a new vector. $\text{diag}(A_1, \ldots, A_k)$ represents a block diagonal matrix with $k$ matrices $A_i$ $(1 \leq i \leq k)$ on its main diagonal.

II. PROBLEM FORMULATION

2.1 A canonical form of FLS and the normal control objective

As is commonly known, FLS can be transformed to a canonical form by a standard nonlinear transformation [29]. Thus, in this paper, we consider the canonical form as follows:

$$
\dot{x} = Ax + B(b(x) + a(x)u) \\
y = Cx
$$

(1)

where $x \in D_0 \subseteq R^n$ is the state, $u = \text{col}(u_1, \ldots, u_m) \in R^m$ is the control input, and $y = \text{col}(y_1, \ldots, y_m) \in R^m$ is the controlled output. $b(x): D_0 \rightarrow R^m$ and $a(x): D_0 \rightarrow R^{m \times n}$ are sufficiently smooth over an open set $D_0$ with $0 \in D_0$, $b(0) = 0$ and $a(x)$ is nonsingular for all $x \in D_0$. Hence, the origin $x = 0$ is an equilibrium of the open-loop system (1) when $u = 0$. $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{m \times n}$ have the following structure,

$$
A = \text{diag}(A_1, A_2, \ldots, A_m) \\
B = \text{diag}(B_1, B_2, \ldots, B_m) \\
C = \text{diag}(C_1, C_2, \ldots, C_m).
$$

The linear subsystem $(A_i, B_i, C_i)$ $(1 \leq i \leq m)$ takes the Brunovsky canonical form [29]:

$$
A_i = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{r_i \times r_i}
$$

$$
B_i = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}_{r_i \times 1}
$$

$$
C_i = [1 0 0 \cdots 0]_{1 \times r_i}, \quad \sum_{i=1}^{m} r_i = n.
$$

The canonical form (1) can also be written in a more compact form:

$$
\dot{x} = f(x) + g(x)u \\
y = Cx
$$

(2)

where $f(x) = Ax + Bb(x)$ and $g(x) = Ba(x) = (g_1(x), \ldots, g_m(x))$.

Note that $y$ is the output to be controlled. It doesn’t mean that only $y$ is measurable. Throughout the present paper, the following condition is assumed.

Assumption 1. The state $x$ of system (1) is measurable.

Given the normal system (1), the normal control objective is to design a controller such that:

1. The state of the closed-loop system consisting of the controller and the normal system (1) is bounded; The controlled output of the closed-loop system satisfies $\lim_{t \rightarrow \infty} y(t) = y_c$.

$y_c = \text{col}(y_{1c}, \ldots, y_{mc})$ is the constant setpoint. The subscript $c$ means ‘constant’. The normal control objective can be easily achieved by the feedback linearization techniques. Thus, no details of the normal controller design will be given in this paper.

2.2 The faulty system description and the FTC objective

The aim of this paper is to design the fault estimation and controller redesign module. Throughout the paper, the FDI unit is assumed to be successfully designed. Without loss of generality, we suppose that the
first $p$ ($1 \leq p < m$) actuators are stuck at the time $t = t_f$ while the remaining $(m - p)$ actuators are healthy. The faulty system can be modeled as:

$$\begin{align*}
\dot{x} &= f(x) + g_F(x)\omega + g_H(x)u_H \\
y &= Cx
\end{align*}$$

(3)

where $\omega = \text{col}(\omega_1, \ldots, \omega_p) \in \Omega \subseteq R^p$ is an unknown constant vector which represents the constant outputs of the stuck actuators. $g_F(x) = (g_1(x), \ldots, g_p(x))$, $g_H(x) = (g_{p+1}(x), \ldots, g_m(x))$ and $u_H = \text{col}(u_{p+1}, \ldots, u_m)$. The subscript $F$ means ‘faulty’ while $H$ means ‘healthy’.

One necessary condition of a nonlinear system to be feedback linearizable is that the number of the independent control inputs equals the number of the independent controlled outputs [29]. When the first $p$ actuators are stuck, the number of the independent control inputs is less than $m$. Therefore, the faulty system is no longer feedback linearizable with the $m$ independent controlled outputs. The normal control objective is no longer achievable due to this dramatic structure change. Only most $(m - p)$ priority outputs can be arbitrarily controlled while $p$ secondary outputs have to be given up as analyzed in [10]. Without loss of generality, we suppose that the last $(m - p)$ outputs of system (3) are priority outputs.

Given the faulty system (3), the **FTC objective** is to design a controller such that:

1. The state of the closed-loop system consisting of the controller and the faulty system (3) is bounded; The last $(m - p)$ controlled outputs $y_H$ of the closed-loop system satisfies $\lim_{t \to \infty} y_H(t) = y_{H_c}$ where $y_{H_c} = \text{col}(y_{(p+1)c}, \ldots, y_{mc})$.

**Remark 1.** The number of the stuck actuators greatly influences the feasibility of the FTC objective. Particularly, when all the actuators are stuck, the faulty system is no longer controllable and no FTC methods will be feasible. Setting the FTC objective as the above, we do not mean that the objective can always be achieved. Alternatively, we just aim to find out when the FTC objective is achievable and how to achieve it. Controllers that can achieve this FTC objective are said to be fault tolerant.

If the FTC objective is achieved, the equilibrium of the closed-loop system consisting of the fault tolerant controller and the faulty system (3) must be in the following form:

$$\begin{align*}
x_{f}(\omega) &= \text{col}(y_1(\omega), 0^{p-1}, \ldots, y_p(\omega), 0^{p-1}, \\
y_{(p+1)c}, 0^{p-1}, \ldots, y_{mc}, 0^{m-1})
\end{align*}$$

(4)

where $y_i(\omega) : R^p \to R$ ($1 \leq i \leq p$) is any function such that $x_{f}(\omega)$ is an equilibrium. If a controller is designed such that the state of the closed-loop system consisting of the controller and the faulty system (3) is bounded and $\lim_{t \to \infty} x(t) = x_{f}(\omega)$, then the FTC objective will be achieved. Motivated by this observation, all the efforts in the sequence are devoted to achieving this goal.

### III. MAIN RESULTS

#### 3.1 Construction of a nonlinear transformation

In this subsection, under an involutive condition, a nonlinear transformation is constructed, which can transform the faulty system (3) to a cascade system. The aim of this transformation is twofold. First, based on the transformation, Chen’s disturbance observer [32] can be extended to a multi-dimensional situation for constant disturbance, which plays a key role in the active FTC system. Second, the faulty system is no longer feedback linearizable, which makes the controller design difficult. A cascade form makes the cascade design possible, thus facilitates the controller design. To begin with, an involutive condition on $g(x)$ is supposed.

**Assumption 2.** The distribution $\Delta_i = \text{span}\{g_i(x), \ldots, g_m(x)\}$ is involutive over $D_0$ for all $i$ ($2 \leq i \leq p+1$). (For the definition of involutive, see A1.)

**Lemma 1.** If Assumption 2 holds, then there exists a nonlinear transformation $z = \text{col}(z_1, z_2) = \Phi(x) = \text{col}(\Phi_1(x), \Phi_2(x))$ defined on a compact set $D \subseteq D_0$, which transforms the faulty system (3) to a cascade system:

$$\begin{align*}
\dot{z}_1 &= f_1(z) + g_1(z)\omega \\
\dot{z}_2 &= f_2(z) + g_2F(z)\omega + g_2H(z)u_H \\
y &= h(z)
\end{align*}$$

(5)

where $z_1 = \Phi_1(x) \in R^{n-m+p}$ and $z_2 = \Phi_2(x) \in R^{m-p}$, $f_1(z) \in R^{n-m+p}$, $f_2(z) \in R^{m-p}$, $g_1(z) \in R^{(n-m+p) \times p}$, $g_2F(z) \in R^{(m-p) \times p}$ and $g_2H(z) \in R^{(m-p) \times (m-p)}$ are as follows:

$$\begin{bmatrix}
f_1(z) \\
f_2(z)
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Phi_1(x)}{\partial x} f(x) \\
\frac{\partial \Phi_2(x)}{\partial x} f(x)
\end{bmatrix} \bigg|_{x = \Phi^{-1}(z)}$$

(6)
\[
\begin{bmatrix}
g_1(z) & 0 
g_2F(z) & g_2H(z)
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial \Phi(x)}{\partial x} g(x)
\end{bmatrix}_{x=\Phi^{-1}(z)}
= \begin{bmatrix}
\frac{\partial \Phi_1(x)}{\partial x} g_F(x) & \frac{\partial \Phi_1(x)}{\partial x} g_H(x) 
\frac{\partial \Phi_2(x)}{\partial x} g_F(x) & \frac{\partial \Phi_2(x)}{\partial x} g_H(x)
\end{bmatrix}_{x=\Phi^{-1}(z)}
\]

(7)

Then, (10) holds by (12)

Moreover, \(g_2H(z)\) is nonsingular for all \(z \in \Phi(D)\) and \(g_1(z)\) has the following form:

\[
g_1(z) = \begin{bmatrix}
0 
g_{TF}(z)
\end{bmatrix}
\]

(9)

where \(g_{TF}(z) \in R^{p \times p}\) is a low triangular nonsingular matrix over \(\Phi(D)\).

**Proof.** To prove the first part of the lemma, a nonlinear transformation \(\Phi(x)\) satisfying the following equation is constructed.

\[
\frac{\partial \Phi_1(x)}{\partial x} g_H(x) = 0
\]

(10)

Since \(a(x)\) is nonsingular, \(g(x)\) is full column rank. By the definition of nonsingular distributions (see A2; note the difference from nonsingular matrices), \(A_i (2 \leq i \leq p + 1)\) is nonsingular. Together with Assumption 2, we conclude that \(A_i\) is involutive and nonsingular with rank \((m - i + 1)\) over \(D_0\) for all \(i (2 \leq i \leq p + 1)\).

Consider the following equations with an unknown function \(T(x) : R^n \rightarrow R\).

\[
\begin{cases}
\frac{\partial T(x)}{\partial x} [g_1(x), \ldots, g_m(x)] = 0 \\
\frac{\partial T(x)}{\partial x} [g_2(x), \ldots, g_m(x)] = 0 \\
\vdots \\
\frac{\partial T(x)}{\partial x} [g_{p+1}(x), \ldots, g_m(x)] = 0.
\end{cases}
\]

(11)

It is obvious that the solutions to the \(i\)th equation in (11) are also solutions to the \((i + 1)\)th equation.

Let \(\bar{x} \in R^{n-m}\) denote all the entries of \(x\) except the last state of every subsystem \((A_i, B_i, C_i) (1 \leq i \leq m)\) of system (1). By the special structure of system (1), we have:

\[
\frac{\partial \bar{x}}{\partial x} [g_1(x), \ldots, g_m(x)] = 0.
\]

(12)

Hence, the entries of \(\bar{x}\) are \((n - m)\) solutions to the first equation in (11).

Next, for the second equation in (11), \(\Delta_2\) is involutive and nonsingular with rank \((m - 1)\). From Frobenius theorem (see A3), the equation has \((n - m + 1)\) independent solutions (For independent functions, see A3). Note that the entries of \(\bar{x}\) are its \((n - m)\) independent solutions. One more independent solution \(T_1(x)\) exists over a set \(D_1 \subseteq D_0\).

Continue in this way until the last equation in (11). Since \(\Delta_i\) is involutive and nonsingular with rank \((m - i + 1)\), the \(i\)th equation in (11) must have \((n - m + i - 1)\) independent solutions. One more solution \(T_{i-1}(x)\) \((3 \leq i \leq p + 1)\), which is independent of the \((n - m + i - 2)\) solutions to the \((i - 1)\)th equation, i.e., the entries of \(\bar{x}\) and \(T_1(x), \ldots, T_{i-2}(x)\), exists over a set \(D_{i-1} \subseteq D_{i-2}\).

Define \(T(x) := \text{col}(T_1(x), \ldots, T_p(x))\). By construction, the entries of \(T(x)\) are independent over \(D_p\).

Moreover,

\[
\frac{\partial T_i(x)}{\partial x} g_j(x) = 0 \quad (\forall i, j, 1 \leq i \leq p, 1 \leq j \leq m).
\]

(13)

On the other hand, a function \(S(x) : R^n \rightarrow R^{m-p}\) can always be found such that the derivative of \(S(x) = \text{col}(\bar{x}, T(x))\) is nonsingular over a compact set \(D \subseteq D_p\). Define a nonlinear transformation \(z = \text{col}(z_1, z_2) = \Phi(x) = \text{col}(\Phi_1(x), \Phi_2(x))\) over \(D\) where \(z_1 = \Phi_1(x) = \text{col}(\bar{x}, T(x)) \in R^{n-m+p}\) and \(z_2 = \Phi_2(x) = \text{col}(\bar{x}) \in R^{m-p}\). Then, (10) holds by (12) and (13). Thus, the first part of the lemma is proved.

By the full column rank of \(g(x)\) as well as the low triangular form of the left part in (7), \(g_1(z)\) and \(g_2H(z)\) are full column rank over \(\Phi(D)\). Hence, \(g_2H(z)\) is nonsingular. As for \(g_1(z)\), the zero submatrix of \(g_1(z)\) in (9) can be confirmed by (12). Together with the full column rank of \(g_1(z)\), the nonsingularity of \(g_{TF}(z)\) can be inferred. Moreover, notice that

\[
g_{TF}(z) = \begin{bmatrix}
\frac{\partial T(x)}{\partial x} g_F(x)
\end{bmatrix}_{x=\Phi^{-1}(z)}
= \begin{bmatrix}
\frac{\partial T_i(x)}{\partial x} g_j(x)
\end{bmatrix}_{i \times p \times p}_{x=\Phi^{-1}(z)}.
\]

(14)

Its low triangularity can be verified by (13).
3.2 Design of the generalized disturbance observer

Chen [32] proposed a disturbance observer for one-dimensional disturbance. In this subsection, based on the cascade system (5) in Section 3.1, we extend this result to a multi-dimensional situation and propose a GDO to estimate the stuck positions of the faulty actuators. This extension is nontrivial, as will be shown in what follows.

Let \( z_T \) denote the vector consisting of the last \( p \) entries of \( z_1 \) in (5). Its dynamics is:

\[
\dot{z}_T = f_T(z) + g_T(z)\omega
\]

where \( f_T(z) \) is the vector consisting of the last \( p \) entries of \( f_1(z) \) and \( g_T(z) \) is given by (9).

Design the GDO as follows:

\[
\dot{\eta} = -A g_T F(z) \eta - A [g_T F(z) A z_T + f_T(z)] \\
\dot{\omega} = \eta + A z_T
\]

where \( \eta \in \mathbb{R}^p \) is the state of the GDO, \( \dot{\omega} \in \mathbb{R}^p \) is the estimate of \( \omega \) and \( A \in \mathbb{R}^{p \times p} \) is a constant observer gain which will be designed later. The estimation error \( \hat{\dot{\omega}} := \dot{\omega} - \dot{\omega} \) satisfies:

\[
\hat{\dot{\omega}} = -A g_T F(z) \hat{\dot{\omega}}
\]

**Lemma 2.** If Assumptions 1–2 hold and the state \( z \) of the cascade system (5) is bounded by \( \Phi(D) \), then a constant observer gain \( A \) exists such that the estimation error of the GDO (16) exponentially converges to zero.

**Proof.** If a constant matrix \( A \) exists such that:

\[
Q := A g_T F(z) + g_T^T F(z) A^T > \mu_p I_p
\]

where \( \mu_p > 0 \) is a pre-chosen positive constant, then the error system (17) will be exponentially convergent to zero. To prove Lemma 2, such a constant matrix \( A \) is constructed. In the rest of the proof, we only consider the case of constant diagonal matrices, i.e.,

\[
A = \text{diag}(A_1, A_2, \ldots, A_p)
\]

Let \( g_{TF}^{ij}(z) \) denote the \((i, j)\)th element of \( g_T F(z) \). Since \( g_T F(z) \) is a low triangular nonsingular matrix on \( \Phi(D) \), as pointed out in Lemma 1, its diagonal elements are nonzero on \( \Phi(D) \). Without loss of generality, suppose \( g_{TF}^{ij}(z) > 0 \). Thus, they can be divided as:

\[
g_{TF}^{ij}(z) = d_i^0 + d_i^1(z), \quad 1 \leq i \leq p
\]

where the constant \( d_i^0 > 0 \) and \( d_i^1(z) \geq 0 \). \( d_i^0 \) can be chosen as the minimum of \( g_{TF}^{ij}(z) \) on \( \Phi(D) \), which cannot be zero due to the compactness of \( \Phi(D) \).

Define \( g_{TF}^i(z) := \text{col}(g_{TF}^{1i}(z), \ldots, g_{TF}^{(i-1)i}(z)) \in \mathbb{R}^{i-1} \) and \( \Lambda_i := \text{diag}(\lambda_1, \ldots, \lambda_i) \). Let \( g_{TF}^i(z) \) \((1 \leq i \leq p)\) denote the \( i \)th principal submatrix of \( g_T F(z) \). Thus, \( g_T F(z) \) and \( Q \) can be rewritten as follows:

\[
g_T F(z) = \begin{bmatrix}
g_{TF}^{p-1}(z) & 0 \\
(g_{TF}^p(z))^T & g_{TF}^p(z)
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
Q_{p-1} & \lambda_p g_{TF}^p(z) \\
\lambda_p(g_{TF}^p(z))^T & 2 \lambda_p g_{TF}^p(z)
\end{bmatrix}
\]

where \( Q_{p-1} = \Lambda_{p-1} g_{TF}^{p-1}(z) + (\Lambda_{p-1} g_{TF}^{p-1}(z))^T \). Using Schur complements (see A4) for (20), we have:

\[
Q > \mu_p I_p \iff 2 \lambda_p g_{TF}^p(z) > \mu_p
\]

Choosing \( \lambda_p = \mu_p/d_i^0 \), from (19) we have:

\[
\lambda_p g_{TF}^p(z) \geq \mu_p.
\]

By \( \lambda_p = \mu_p/d_i^0 \) and (22), the following sufficient condition can be obtained:

\[
Q > \mu_p I_p \iff Q_{p-1} > \mu_p I_{p-1} + \frac{\mu_p}{d_i^0} g_{TF}^p(z)(g_{TF}^p(z))^T.
\]

From the boundedness of \( g_{TF}^p(z) \) on \( \Phi(D) \), a constant \( \mu_{p-1} \) exists such that \( \mu_{p-1} I_{p-1} > \mu_p/(d_i^0)^2 g_{TF}^p(z)(g_{TF}^p(z))^T \). Consequently,

\[
Q > \mu_p I_p \iff Q_{p-1} > (\mu_{p-1} + \mu_p) I_{p-1}.
\]

Hence, to make (18) hold, it is sufficient to design \( \Lambda_{p-1} \) such that \( Q_{p-1} > (\mu_{p-1} + \mu_p) I_{p-1} \).

Continue in this way until \( Q_1 = 2 \lambda_1 g_{TF}^{1i}(z) \in \mathbb{R} \). Taking \( \lambda_i = (1/d_i^0)^{\sum_{k=i}^p \mu_k} (2 \leq i \leq p) \), we have:

\[
Q_i > \sum_{k=i}^p \mu_k I_i \iff Q_{i-1} > \sum_{k=i-1}^p \mu_k I_{i-1}, \quad 2 \leq i \leq p
\]
where $Q_i = \Lambda_i g_{TF}^T(z) + (\Lambda_i g_{TF}^T(z))^T$. \{\mu_i\} \ (1 \leq i \leq p)$ is a constant sequence satisfying

$$\mu_{i-1} I_{i-1} > \frac{\mu^i_{TF}(z) (g^i_{TF}(z))^T}{(d^0_i)^2} \sum_{k=i}^p \mu_k, \quad 2 \leq i \leq p \tag{26}$$

From (25), it follows that $Q > \mu_P I_P$ holds if $Q > \sum_{k=1}^p \mu_k$. The latter can always be satisfied by choosing $\Lambda_1$ sufficiently large. Thus, an observer gain $\Lambda$ has been constructed such that (18) holds, which completes the proof. \hfill \Box

**Remark 2.**

(i) It can be verified that the GDO’s estimation error satisfies $\|\hat{\omega}(t)\| \leq \|\hat{\omega}(0)\| \exp(-\mu_P t/2)$. A larger $\mu_P$ leads to faster convergence speed. On the other hand, it also leads to a larger $\Lambda_i$ (1 $\leq i \leq p$), which results in high sensitivity to the measurement noise. Therefore, a trade-off should be made when choosing $\mu_P$.

(ii) In the proof, the existence of a feasible observer gain is proved by constructing a diagonal one. Nevertheless, better performance may be achieved by relaxing the diagonal form restriction, which deserves further research.

### 3.3 Controller design with the true stuck positions and stability analysis

Consider the cascade system (5). Define $z_e(\omega) : = \text{col}(z_{1e}(\omega), z_{2e}(\omega)) = \Phi(z_e(\omega))$ where $z_{1e}(\omega) \in R^{n-m+p}$ and $z_{2e}(\omega) \in R^{m-p}$. Throughout this paper, we assume that $z_e(\omega)$ is in the interior of $\Phi(D)$. If the state $z$ of the cascade system (5) is controlled to $z_e(\omega)$ and is bounded, then the state $x$ of the faulty system (3) is controlled to $x_e(\omega)$, and the FTC objective will be achieved as analyzed in Section II. In this subsection, based on the cascade system (5), a controller is designed to achieve this goal under an assumption of the feasibility of the cascade design method [31]. The true stuck positions are used which will be replaced by their estimates in Section 3.4.

Define $\tilde{z} := z - z_e(\omega) = \text{col}(\tilde{z}_1, \tilde{z}_2)$ where $\tilde{z}_1 = z_1 - z_{1e}(\omega)$ and $\tilde{z}_2 = z_2 - z_{2e}(\omega)$. The transformed faulty system (5) can be rewritten as:

$$\begin{align*}
\dot{\tilde{z}}_1 &= f_1(\tilde{z} + z_e(\omega)) + g_1(\tilde{z} + z_e(\omega))\omega \\
\dot{\tilde{z}}_2 &= f_2(\tilde{z} + z_e(\omega)) + g_{2F}(\tilde{z} + z_e(\omega))\omega + g_{2H}(\tilde{z} + z_e(\omega))u_H. \quad (27)
\end{align*}$$

**Assumption 3.** There exist a continuous function $\rho(\tilde{z}_1, \omega): R^{n-m+p} \times R^p \rightarrow R^{m-p}$ and two continuously differentiable functions $W(\tilde{z}_1, \omega): R^{n-m+p} \times R^p \rightarrow R$ and $K(\tilde{z}_1, \omega): R^{n-m+p} \times R^p \rightarrow R^{m-p}$ with $K(0, \omega) = 0$ for all $\{\omega|z_1e(\omega), z_2 + z_2e(\omega)\} \in \Phi(D)$ and all $\omega \in \Omega$ such that the following inequalities hold:

$$\begin{align*}
&c_1 \|\tilde{z}_1\| \leq W(\tilde{z}_1, \omega) \leq c_2 \|\tilde{z}_1\|^2 \tag{28} \\
&\left\| \frac{\partial W(\tilde{z}_1, \omega)}{\partial \tilde{z}_1} \right\| \leq c_3 \|\tilde{z}_1\| \tag{29} \\
&\frac{\partial W(\tilde{z}_1, \omega)}{\partial \omega} \leq c_4 \|\tilde{z}_1\| \tag{30} \\
&\leq -c_5 \|\tilde{z}_1\|^2 + \|\tilde{z}_2 - K(\tilde{z}_1, \omega)||\rho(\tilde{z}_1, \omega)\| \tag{31}
\end{align*}$$

where $c_1, c_2, c_3, c_4$ and $c_5$ are all positive constants.

**Remark 3.**

(i) Equations (28), (29) and (31) in Assumption 3 mean that the subsystem $\tilde{z}_1$ of (27) can be exponentially stabilized to zero regarding $\tilde{z}_2$ as a virtual control input. This assumption is widely used with a linear growth assumption of interconnection terms as a feasibility condition of the cascade design [31]. Nevertheless, (31) replaces the linear growth assumption with a generalized matching condition [33]. Actually, there are many other feasibility conditions for cascade design. We point out that how to design a better controller based on the cascade form (5) is out of the scope of this research; on the contrary, here, we just use the existing methods. The main topics of this paper are how to transform the faulty system (3) to a cascade form, how to estimate the stuck positions and how to use the estimates to design a fault tolerant controller.

(ii) Equations (28), (29) and (31) are reasonable in the sense that if subsystem $\tilde{z}_1$ cannot be controlled by $\tilde{z}_2$, then it will become uncontrollable. If this is true, no FTC scheme is feasible. The main restriction imposed by this is that $\tilde{z}_2$ not only exponentially stabilizes $\tilde{z}_1$ but also satisfies a generalized matching condition. Despite the conservativeness, it covers many applications [31], e.g., (28), (29) and (31) always hold if system (27) is linear and controllable.

(iii) Equation (30) is a condition we need to guarantee the stability of the closed-loop system when
Lemma 3. Suppose, for the normal system (1), the first $p$ ($1 \leq p < m$) actuators are stuck at the time $t_f$. Assumptions 1–3 hold and the normal controller switches to the controller (32) at the time $t_s$ ($t_s \geq t_f$). If $\varepsilon$ is chosen sufficiently small and $z(t_s) \in M$, then the corresponding trajectory of the closed-loop system (27) and (32) is bounded. Moreover, $z(t) \rightarrow z_\varepsilon(\omega)$ as $t \rightarrow \infty$.

Proof. The proof is similar to the one in Qu [33]. We include it here for the completeness of the paper. The derivative of $V(\tilde{z})$ along the trajectory of (27) and (32) is given below.

$$
\dot{V}(\tilde{z}) = \frac{\partial W(\tilde{z}_1, \omega)}{\partial \tilde{z}_1} \left[ f_1(z) + g_1(z)\omega \right] + (\tilde{z}_2 - K(\tilde{z}_1, \omega))^T \left[ \tilde{z}_2 - \frac{\partial K(\tilde{z}_1, \omega)\tilde{z}_1}{\partial \tilde{z}_1} \right]
$$

By (27), (31) and (32), we have:

$$
\dot{V}(\tilde{z}) \leq -c_5\|\tilde{z}_1\|^2 + \|\tilde{z}_2 - K(\tilde{z}_1, \omega)\|\|\rho(\tilde{z}_1, \omega)\| \\
- (\tilde{z}_2 - K(\tilde{z}_1, \omega))^T (\tilde{z}_2 - K(\tilde{z}_1, \omega)) \\
- (\tilde{z}_2 - K(\tilde{z}_1, \omega))^T (\tilde{z}_2 - K(\tilde{z}_1, \omega))\|\rho(\tilde{z}_1, \omega)\|^2 \\
\frac{\|\tilde{z}_2 - K(\tilde{z}_1, \omega)\|\|\rho(\tilde{z}_1, \omega)\| + \varepsilon\varphi(t)}{\|\tilde{z}_2 - K(\tilde{z}_1, \omega)\|\|\rho(\tilde{z}_1, \omega)\| + \varepsilon\varphi(t)}.
$$

It can be verified that the last term in (36) is less than $\varepsilon\varphi(t)$. Therefore,

$$
\dot{V}(\tilde{z}) \leq -c_5\|\tilde{z}_1\|^2 - (\tilde{z}_2 - K(\tilde{z}_1, \omega))^T (\tilde{z}_2 - K(\tilde{z}_1, \omega)) \\
\times (\tilde{z}_2 - K(\tilde{z}_1, \omega)) + \varepsilon\varphi(t).
$$

On the boundary of $M$, $\|\tilde{z}_1\|$ and $\|\tilde{z}_2 - K(\tilde{z}_1, \omega)\|$ cannot be zero at the same time. By choosing $\varepsilon$ sufficiently small, $\dot{V}(\tilde{z}) < 0$ can be obtained on the boundary of $M$. Thus, the corresponding trajectory is bounded. By (37) and Lemma 4 (in the appendix), the attractiveness property holds, i.e., $z(t) \rightarrow z_\varepsilon(\omega)$ as $t \rightarrow \infty$. 

Remark 4. The stability result in Lemma 3 is a local one. Actually, $M$ is an estimate of the region of attraction. That is, at the switching time $t_s$, if the state $z(t_s)$ is beyond the set $M$, then the boundedness and attractiveness of the closed-loop system (27) and (32) may not be guaranteed.
Define \( \dot{\hat{z}} := z - z_e(\hat{\omega}) = \text{col}(\hat{z}_1, \hat{z}_2) \) where \( \hat{z}_1 = z_1 - z_{1e}(\hat{\omega}) \) and \( \hat{z}_2 = z_2 - z_{2e}(\hat{\omega}) \). The cascade system (5) can also be rewritten as:

\[
\begin{align*}
\dot{\hat{z}}_1 &= f_1(z) + g_1(z)\hat{\omega} + g_1(z)\hat{\omega} - \frac{\partial z_{1e}(\hat{\omega})}{\partial \hat{\omega}} \hat{\omega} \\
\dot{\hat{z}}_2 &= f_2(z) + g_2F(z)\hat{\omega} + g_2F(z)\hat{\omega} + g_2H(z)u_H - \frac{\partial z_{2e}(\hat{\omega})}{\partial \hat{\omega}} \hat{\omega}.
\end{align*}
\]

The resultant fault tolerant controller is:

\[
u^{FTC}(\hat{z}, \hat{\omega}) = u_H(\hat{z}, \hat{\omega}).
\]

A Lyapunov function candidate can be taken as follows for the closed-loop system (16), (38) and (39).

\[
V_1(\hat{z}, \hat{\omega}) = W(\hat{z}_1, \hat{\omega}) + \frac{1}{2}(\hat{z}_2 - K(\hat{z}_1, \hat{\omega}))^T \\
\times (\hat{z}_2 - K(\hat{z}_1, \hat{\omega})) + \beta \hat{\omega}^T \hat{\omega}.
\]

where \( \beta > 0 \) is a constant to be chosen. Define:

\[
M_{1b} := \{ (\hat{z} + z_e(\hat{\omega}), \omega - \hat{\omega}) | V_1(\hat{z}, \hat{\omega}) \leq b, b > 0, b \in R \}
\]

\[
M_1 := \max\{M_{1b} | M_{1b} \subseteq \Phi(D) \times \Omega \}.
\]

The following theorem gives a stability analysis of the closed-loop system consisting of the cascade system, the GDO and the fault tolerant controller.

**Theorem 1.** Suppose that, for the normal system (1), the first \( p \) (\( 1 \leq p < m \)) actuators are stuck at the time \( t_f \). Assumptions 1–3 hold, the GDO is properly designed, and the normal controller switches to the fault tolerant controller (39) at time \( t_\epsilon \) (\( t_\epsilon \geq t_f \)). If \( \epsilon \) is chosen sufficiently small and \( (z(t_\epsilon), \hat{\omega}(t_\epsilon)) \in M_1 \), then the corresponding trajectory of the closed-loop system (16), (38) and (39) is bounded. Furthermore, \( z \to z_e(\omega) \) and \( \hat{\omega}(t) \to \omega \) as \( t \to \infty \).

**Proof.** Since Assumption 3 is supposed to hold for all \( \{z_1(z_1 + z_{1e}(\omega), \hat{z}_2 + z_{2e}(\omega)) \in \Phi(D) \} \) and all \( \omega \in \Omega \), it necessarily holds for \( (z, \hat{\omega}) \in M_1 \subseteq \Phi(D) \times \Omega \). If the observer gain \( \Lambda \) of the GDO is properly designed such that (18) holds, by Lemma 1 and Assumption 3, the derivative of \( V_1(\hat{z}, \hat{\omega}) \) along the trajectory of the closed-loop system (16), (38) and (39) will satisfy the following inequality:

\[
\begin{align*}
\dot{V}_1(\hat{z}, \hat{\omega}) &\leq -c_5 \| \hat{z}_1 \|^2 - (\hat{z}_2 - K(\hat{z}_1, \hat{\omega}))^T \\
&\times (\hat{z}_2 - K(\hat{z}_1, \hat{\omega}))) + P(\hat{\omega}, \hat{\omega}) \\
&- \beta \mu_p \hat{\omega}^T \hat{\omega} + \varepsilon \varphi(t)
\end{align*}
\]

where \( P(\hat{\omega}, \hat{\omega}) \) is given as follows:

\[
P(\hat{\omega}, \hat{\omega}) = \partial W(\hat{z}_1, \hat{\omega}) \partial \hat{z}_1 \\
+ \partial W(\hat{z}_1, \hat{\omega}) \partial \hat{\omega} + \partial \hat{\omega} \partial \hat{\omega} + \partial \hat{\omega} \partial \hat{\omega}.
\]

By (29) and (30) in Assumption 3, over the bounded set \( M_1 \), it follows that

\[
\| P(\hat{\omega}, \hat{\omega}) \hat{\omega} \| \leq c_6 \| \hat{z}_1 \| \| \hat{\omega} \| + c_7 \\
\| \hat{z}_2 - K(\hat{z}_1, \hat{\omega}) \| \| \hat{\omega} \|, \ \forall (\hat{z}, \hat{\omega}) \in M_1(44)
\]

for some positive constants \( c_6 \) and \( c_7 \). Using Lemma 5 (in the appendix) yields:

\[
\| P(\hat{\omega}, \hat{\omega}) \hat{\omega} \| \leq c_6 \delta_1 \| \hat{z}_1 \|^2 + \frac{c_6}{\delta_1} \| \hat{\omega} \|^2 + c_7 \delta_2 \\
\times \| \hat{z}_2 - K(\hat{z}_1, \hat{\omega}) \|^2 + \frac{c_7}{\delta_2} \| \hat{\omega} \|^2
\]

for positive constants \( \delta_1 \) and \( \delta_2 \). Choosing \( \delta_1 \) and \( \delta_2 \) sufficiently small while \( \beta \) sufficiently large such that

\[
c_6 \delta_1 \leq \frac{c_5}{2}, \quad c_7 \delta_2 \leq \frac{1}{2}, \quad \delta_1 \leq \frac{c_6}{\delta_1}, \quad \delta_2 \leq \frac{\beta \mu_p}{2}
\]

we have

\[
\begin{align*}
\dot{V}_1(\hat{z}, \hat{\omega}) &\leq -\frac{c_5}{2} \| \hat{z}_1 \|^2 - \frac{1}{2} (\hat{z}_2 - K(\hat{z}_1, \hat{\omega}))^T \\
&\times (\hat{z}_2 - K(\hat{z}_1, \hat{\omega}))) + \frac{\beta \mu_p}{2} \hat{\omega}^T \hat{\omega} + \varepsilon \varphi(t) \\
&\forall (\hat{z}, \hat{\omega}) \in M_1
\end{align*}
\]

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If ε sufficiently small, by Lemma 4 (in the appendix), the boundedness and attractiveness properties can be proven.

IV. SIMULATION STUDY

An inverted pendulum system [34] as shown in Fig. 2 is employed to demonstrate the effectiveness of the proposed FTC method. It consists of two identical inverted pendulums with length l coupled by a spring and subject to two distinct inputs $u_1$ and $u_2$. The spring constant is $k$ and the position of the spring is $a$. The normal length of the spring equals the distance of the two pivots $p_1$ and $p_2$.

Choosing the states $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$, the dynamics of the system can be modeled as follows in a neighborhood of zero with some approximation:

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{g}{l} \sin x_1 + \frac{ka^2}{ml^2}(-\sin x_1 + \sin x_3) + \frac{\cos x_1}{ml} u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{g}{l} \sin x_3 + \frac{ka^2}{ml^2}(\sin x_1 - \sin x_3) + \frac{\cos x_3}{ml} u_2 
\end{align*}$$

(47)

where $g$ is the gravity constant. As a numerical example, set

$$\frac{g}{l} = 0.5, \quad \frac{a}{l} = \frac{1}{\sqrt{2}}, \quad \frac{k}{m} = 2, \quad \frac{1}{ml} = 4.$$ 

Choosing $y_1 = x_1$ and $y_2 = x_3$ the controlled outputs, the numerical model is

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.5 \sin x_1 + \sin x_3 + 4u_1 \cos x_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -0.5 \sin x_3 + \sin x_1 + 4u_2 \cos x_3 \\
y_1 &= x_1, \quad y_2 = x_3. 
\end{align*}$$

(48)

The normal control objective is to control the outputs $y_1$ and $y_2$ to zero, which can be easily achieved by the following controller:

$$\begin{align*}
u_1 &= (4 \cos x_1)^{-1}(-\sin x_2 - \sin x_3) \\
u_2 &= (4 \cos x_3)^{-1}(-\sin x_1 - \sin x_4) 
\end{align*}$$

(49)

Fig. 3 displays the simulation results of the closed-loop system (48) and (49).

Suppose that the first actuator is stuck at $t = t_f$, then

$$u_1(t) = \omega, \quad t \geq t_f$$

(50)

where $\omega$ is an unknown constant.

After the fault (50) occurs, only one output can be arbitrarily controlled. Thus, the normal control objective must degrade. The FTC objective here is to control $y_2$ to zero while maintaining the boundedness of the faulty system (48) and (50). This can be achieved by steering its state to $x_c(\omega) = (\tan^{-1}(8\omega), 0, 0, 0)^T$ which is the equilibrium of the faulty system. Define $\bar{x}_1 = x_1 - \tan^{-1}(8\omega)$.

Notice that the faulty system is already a cascade system. Partition the faulty system (48) and (50) into two subsystems $(x_1, x_2, x_3)$ and $x_4$. A Lyapunov function candidate can be taken as below for the subsystem $(x_1, x_2, x_3)$ over a neighborhood of zero. It can be
verified that the function is positive definite:

\[ V(x_1, x_2, x_3, \omega) = (6\theta + 1)(1 - \cos \tilde{x}_1) + 3x_2^2 \]
\[ + x_2 \sin \tilde{x}_1 + 0.5(x_2 + \sin x_3)^2 \]  

(51)

where \( \theta = (1/\sqrt{1 + 64\omega^2})(0.5 + 32\omega^2) > 0 \).

Regarding \( x_4 \) as a virtual control input, by the well known backstepping technique, a virtual controller \( K(x_1, x_2, x_3, \omega) = -(1/\cos x_3)[7x_2 + 2 \sin x_3 + (1 - \theta) \sin \tilde{x}_1] \) can be designed for the subsystem \((x_1, x_2, x_3)\). Moreover,

\[ \dot{V}(x_1, x_2, x_3) \leq -\theta \sin^2 \tilde{x}_1 - 5x_2^2 - (x_2 + \sin x_3)^2 \]
\[ + (x_4 - K)(x_2 + \sin x_3) \cos x_3. \]  

(52)

It can be verified that the subsystem satisfies Assumption 3. As a result, a controller can be designed following the steps in Section III,

\[
u_2 = \frac{1}{4 \cos x_3} \left[ -0.5 \sin x_3 - \sin x_1 + \frac{\partial K}{\partial x_1} x_2 \right. \]
\[ + \frac{\partial K}{\partial x_2} (-\theta \sin \tilde{x}_1 + \sin x_3) + \frac{\partial K}{\partial x_3} x_4 - 3x_4 + 3K \]
\[ - \frac{(x_4 - K)(x_2 + \sin x_3) \cos x_3}{\|x_4 - K\| \|x_2 + \sin x_3\| \cos x_3 + 0.2e^{-t}} \].  

(53)

The GDO is designed as

\[
\dot{\eta} = -4\eta \cos x_1 \\
- (4x_2 \cos x_1 + \sin x_3 - 0.5 \sin x_1) \]
\[
\dot{\omega} = \eta + x_2. \]

Replacing \( \omega \) with its estimate \( \hat{\omega} \) in (52), a fault tolerant controller is obtained:

\[
u_{FTC} = \frac{1}{4 \cos x_3} \left[ -0.5 \sin x_3 - \sin x_1 + \frac{\partial \hat{K}}{\partial \hat{x}_1} x_2 \right. \]
\[ + \frac{\partial \hat{K}}{\partial \hat{x}_2} (-\theta \sin \hat{x}_1 + \sin x_3) \]
\[ + \frac{\partial \hat{K}}{\partial \hat{x}_3} x_4 - 3x_4 + 3\hat{K} \]
\[ - \frac{(x_4 - \hat{K})(x_2 + \sin x_3) \cos x_3}{\|x_4 - \hat{K}\| \|x_2 + \sin x_3\| \cos x_3 + 0.2e^{-t}} \].  

(55)

where \( \hat{x}_1 = x_1 - \tan^{-1}(8\hat{\omega}), \hat{K} = K(\hat{x}_1, x_2, x_3, \hat{\omega}) \) and \( \hat{\theta} = (1/\sqrt{1 + 64\hat{\omega}^2})(0.5 + 32\hat{\omega}^2) \).

For simulation, set the initial state as \( x(0) = [0.2, 0.2, 0.1, 0.1]^T \). Suppose that the first actuator is stuck at \( t_f = 1s \) and the controller switches to the fault tolerant controller at \( t_s = 2s \).

It implies that FTC is necessary.
Suppose at $t_f = 2s$, FDI is accomplished. Then, the GDO (54) is activated with $\eta(t_f) = 0$ and the normal controller switches to the fault tolerant one (55). Fig. 5 exhibits the results. The first output $y_1$ is controlled to a new work point and the second output $y_2$ to zero. All the states are bounded. So the effectiveness of the proposed FTC method is demonstrated.

V. CONCLUSIONS

An active FTC method for FLS with stuck actuators is proposed in this paper. The considered fault is severe in the sense that the faulty system is no longer feedback linearizable, which makes the controller design difficult. Under an involutive assumption, a nonlinear transformation is first constructed, which transforms the faulty system into a cascade system. Under an assumption of the feasibility of the famous cascade design method, a controller is designed for the faulty system. On the other hand, a GDO is presented to estimate the unknown stuck positions. Integrating the GDO with the controller constructed by the cascade design method, a fault tolerant controller is obtained. The resultant fault tolerant controller guarantees not only boundedness of the state of the faulty system but also satisfactory output performance. The design details of the nonlinear transformation and the GDO are given. An illustrative example verifies the effectiveness of the method. Due to the difficulty of nonlinearity, the feasibility of the cascade design method which is supposed in Assumption 3 is crucial to the proposed method. Perhaps with further developments in nonlinear control, the conservativeness introduced by the assumption can be reduced.

APPENDIX

A1. A distribution $\Delta(x)$ is involutive if the Lie bracket $[\tau_1, \tau_2]$ of any pair vector fields $\tau_1$ and $\tau_2$ belonging to $\Delta(x)$ is a vector field which belongs to $\Delta(x)$, i.e., $[\tau_1, \tau_2] \in \Delta(x)$. Lie bracket is defined as

$$[\tau_1, \tau_2] = \frac{\partial \tau_2}{\partial x} \tau_1 - \frac{\partial \tau_1}{\partial x} \tau_2.$$ 

A2. A distribution $\Delta(x)$, defined on an open set $U$, is nonsingular if there exists an integer $d$ such that $\dim(\Delta(x)) = d$ for all $x$ in $U$ [29]. Note the difference from nonsingular matrices.

A3. (Frobenius Theorem [29]) Consider a nonsingular $d$-dimensional distribution $\Delta(x) = \text{span}\{f_1(x), \ldots, f_d(x)\}$, defined on an open set $U$ of $\mathbb{R}^n$. The vector fields $f_1(x), \ldots, f_d(x)$ are independent. There exist $(n - d)$ independent solutions defined on $U^* \subseteq U$ to the following differential equation if and only if the distribution $\Delta(x)$ is involutive.

$$\frac{\partial T}{\partial x} (f_1(x) \cdots f_d(x)) = 0$$

By ‘independent’, we mean that the row vectors

$$\frac{\partial T_1}{\partial x}(x), \ldots, \frac{\partial T_{n-d}}{\partial x}(x)$$

are independent at each $x$.

A4. (Schur complements [35]) For a symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{22} \end{bmatrix}$ where $S_{11} = S_{11}^{T} \in \mathbb{R}^{r \times r}$ and $S_{22} = S_{22}^{T} \in \mathbb{R}^{(n-r) \times (n-r)}$, the following inequalities are equivalent.

(i) $S > 0$

(ii) $S_{22} > 0, S_{11} - S_{12}S_{22}^{-1}S_{12}^{T} > 0$

Consider a nonlinear system:

$$\dot{x} = f(x, t), \quad x(t_0) = x_0$$

where $f(x, t) : D \times \mathbb{R}^+ \to \mathbb{R}^n$ is piecewise continuous with respect to $x$ and $t$. Moreover, $f(0, t) = 0$.

Lemma 4. For the above system, suppose that, for any bounded set $D \subseteq \mathbb{R}^n$, the range of function $f(x, t)$ is a bounded set in $\mathbb{R}^n$. If there exists a continuously differentiable positive definite function $V_2(x) : D \to \mathbb{R}$ satisfying:

$$\dot{V}_2(x) = \frac{\partial V_2}{\partial x} f(x, t)$$

$$\leq -W_2(x) + \delta(t) \quad \forall (x, t) \in D \times \mathbb{R}^+$$

where $W_2(x)$ is a positive definite continuous function over $D$ and $\delta(t)$ is a nonnegative continuous function for all $t \in [t_0, \infty)$ satisfying $\int_{t_0}^{\infty} \delta(s) ds < \beta$ which is sufficiently small, and define $M_{2b} := \{x | V_2(x) < b, b > 0, b \in \mathbb{R}\}$ and $M_2 := \max(M_{2b}, M_{2b} \subseteq D)$, then for any initial value $x_0 \in M_2$, the corresponding trajectory $x(t)$ is bounded. Furthermore, $x(t)$ is attractive, i.e., $x(t) \to 0$ as $t \to \infty$.

Proof. The lemma can be directly deduced from corollary 2.18 in [33]. Thus, the details are omitted.
Lemma 5. (Cheng and Zhao [2]) For any two compatible real vectors or matrices $X$ and $Y$, and for any positive constant $\alpha$, the inequality $X^T Y + Y^T X \leq 2X^T X + (1/\alpha) Y^T Y$ holds.

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