OPTIMAL CONTROL AND ROBUST STABILITY OF UNCERTAIN IMPULSIVE DYNAMICAL SYSTEMS

Bin Liu, Kok Lay Teo, and Xinzhi Liu

ABSTRACT

This paper studies the problem of optimal feedback control and robust stability for uncertain impulsive dynamical systems. By using algebraic inequalities, Riccati and Hamilton-Jacobi inequalities, the conditions are derived under which not only the uncertain impulsive dynamical system has robust asymptotic stability but also the value of the optimal hybrid performance functional can be estimated. An example is also given to illustrate our results.

Key Words: Uncertain impulsive dynamical systems, hybrid performance functional, robust stability, Hamilton-Jacobi/Riccati inequality.

I. INTRODUCTION

The theory of impulsive dynamical systems has been developed quite rapidly in recent years, see [1, 2] and references cited therein. Such systems arise in many applied fields, such as control technology, communication networks, and biological management. Since impulsive dynamical systems provide a natural framework for mathematical modeling of many physical phenomena, their study is assuming a greater importance. On the other hand, the problem of robust stability has received considerable attention lately because in many practical problems system uncertainty may occur due to modeling errors, measurement inaccuracy, linear approximation, and so on. In recent years, many results of stability for impulsive dynamical systems have been proposed in [1–5, 12, 15–18] and references therein. But so far very few robust stability results for uncertain impulsive dynamical systems have been reported. More recently, [7–14] discussed robust stability and its applications of a general impulsive dynamical system and obtained some criteria on robust stability.

In this paper, we study the optimal control and robust stability problem for uncertain impulsive dynamical systems with a hybrid performance function. Since there exist uncertainties in the system, it is difficult to calculate explicitly the value of the hybrid performance functional. Hence, it is important to estimate its bounds. By employing the results developed in the [3, 8], we derive some criteria under which the uncertain impulsive dynamical system is robust asymptotic stable and the bounds of the hybrid performance functional can be estimated. These criteria will be less conservative than those in [5] and more general than those in [13, 14]. Meanwhile, we also study the optimal feedback control problem for controlled uncertain impulsive dynamical systems.

The rest of this paper is organized as follows. In Section II, we make some preliminaries. In Section III, we make the estimation of the bounds for hybrid performance functional. Based the results established in Section III, in Section IV, we derive the conditions of both robust stability and the estimation of bounds of the
Let \( R^n \) denote the \( n \)-dimensional Euclidean space. Let \( R_+ = [0, +\infty) \), \( N = \{1, 2, \cdots\} \). Denote by \( \times \) the class of functions \( \phi : R_+ \rightarrow R_+ \), which are continuous, strictly increasing and \( \phi(0) = 0, x_0 \) the class of continuous functions \( \psi : R_+ \rightarrow R_+ \) such that \( \psi(s) = 0 \) if and only if \( s = 0 \) and \( PC \) the class of functions \( \lambda : R_+ \rightarrow R_+ \), where \( \lambda \) is continuous everywhere except \( t_k (k \in N) \) at which \( \lambda \) is left continuous and the right limit \( \lambda(t_k^+) \) exists. Let \( \| \cdot \| \) stand for the Euclidean norm in \( R^n \).

Consider the uncertain impulsive dynamical systems of the form
\[
\begin{align*}
\dot{x} &= f_c(t, x), \quad t \in (t_{k-1}, t_k] \\
\Delta x &= f_d(t_k, x(t_k)), \quad t = t_k, \ k \in N
\end{align*}
\]
where \( 0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \rightarrow \infty \) as \( k \rightarrow \infty \), and \( x \in R^n \), \( \Delta x = (t_k^+ - t_k) (x(t_k^+) - x(t_k)) \), \( f_c \in F_c \in \{ f_c : R_+ \times R^n \rightarrow R^n, f_c(t, 0) \equiv 0 \} \), \( f_c(t, x) \) is Lipschitz continuous in \( x \), \( f_d \in F_d \in \{ f_d : R_+ \times R^n \rightarrow R^n, f_d(t, 0) \equiv 0 \} \), \( f_d(t, x) \) is continuous.

Denote \( F = F_c \times F_d \). Denote by \( x(t, x_0) \) the solution of (1) satisfying the initial condition \( x(t_0^+) = x_0 \).

Let \( x(t_f) = x_f \), where \( t_0 < t_f < \infty \) or \( t_f = \infty \). The hybrid performance functional can be described as
\[
J^{t_f}_{(f_c, f_d)}(x_0) = \int_{t_0}^{t_f} L_c(t, x(t)) dt + \sum_{t_0 \leq t < t_f} L_d(t_k, x_k)
\]
where \( x_k = x(t_k) \), and \( L_c : R_+ \times R^n \rightarrow R \) and \( L_d : R_+ \times R^n \rightarrow R \) are given.

**Definition 1.** The uncertain impulsive dynamical system (1) is called robust stable, robust asymptotic stable, and robust exponential stable, respectively, if for any \( f_c \in F_c, f_d \in F_d \), the equilibrium \( x = 0 \) of (1) is stable, asymptotic stable, and exponential stable, respectively.

**III. THE BOUNDING OF** \( J^{t_f}_{(f_c, f_d)}(x_0) \)

Since functions \( f_c \) and \( f_d \) are uncertain, it is all almost impossible to calculate explicitly the hybrid performance functional. Therefore, it is significant to estimate its bounds.

**Theorem 1.** Assume that there exist functions \( \Gamma_c : R_+ \times R^n \rightarrow R, \Gamma_d : R_+ \times R^n \rightarrow R \), and \( V : R_+ \times R^n \rightarrow R_+ \), where \( V \) is \( C^1 \) function, such that
\[
\begin{align*}
(i) \quad & \text{for all } t \in (t_{k-1}, t_k], \ k \in N, \\
& V(t, x) f_c(t, x) \leq V(t, x) f_c(t, 0) + \Gamma_c(t, x), \\
& V(t, x) f_d(t, x) \leq V(t, x) f_d(t, 0) + \Gamma_d(t, x), \\
& V(t, x) \leq V(0, x) = 0;
\end{align*}
\]
\[
(ii) \quad & \text{for all } t \in (t_{k-1}, t_k], \ k \in N, \\
& L_c(t, x) + V(t, x) f_c(t, x) \leq L_c(t, x) + V(t, x) f_c(t, 0) + \Gamma_c(t, x) + \Gamma_d(t, x), \\
& L_d(t, x) \leq L_d(t, x) + V(t, x) f_d(t, x) \leq V(t, x) f_d(t, 0) + \Gamma_d(t, x), \\
& L_d(t, x) \leq V(t, x) f_d(t, 0) + \Gamma_d(t, x), \\
& \text{sup}_{(f_c, f_d) \in F} J^{t_f}_{(f_c, f_d)}(x_0) \leq L(t_f, x_f) \leq V(t_0, x_0),
\end{align*}
\]
where
\[
J^{t_f}_{(f_c, f_d)}(x_0) = \int_{t_0}^{t_f} L_c(t, x(t)) + \Gamma_c(t, x(t)) dt + \sum_{t_0 \leq t < t_f} L_d(t_k, x_k)
\]
where in (8), \( x(t), t \neq t_0, t_1, \cdots, t_f \) is a solution to system (1) with \( (f_c, f_d) = (f_c0, f_d0) \).
Proof. Let \( x(t), t \neq t_0 \) be the solution to system (1). Then, for \( t \in (t_{k-1}, t_k], k \in N \),
\[
D^+ V(t, x) = V_\ell(t, x) + V_\delta(t, x) f(t, x) \quad (9)
\]
\[
\Delta V(t_k, x_k) = V(t_k, x_k + f_d) - V(t_k, x_k). \quad (10)
\]
By conditions (i)–(ii) and (9), we get
\[
L_c(t, x(t)) \leq -D^+ V(t, x). \quad (11)
\]
By conditions (iii)–(iv) and (10), we have
\[
L_d(t_k, x_k) \leq -\Delta V(t_k, x_k). \quad (12)
\]
Now, integrating over the interval \([t_0, t_f]\) with \( \{ k \in N : t_0 \leq t_k < t_f \} = \{1, 2, \ldots, m\} \), by (11)–(12), we have
\[
J_{(f_c, f_d)}^{t_f}(x_0) \leq -\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} D^+ V(t, x(t))dt
\]
\[
- \int_{t_0}^{t_f} D^+ V(t, x(t)) dt
\]
\[
- \sum_{i=1}^{m} \Delta V(t_i, x_i)
\]
\[
= V(t_0, x_0) - V(t_f, x_f)
\]
\[
\leq V(t_0, x_0). \quad (13)
\]
Next, if \( x(t) \) is a solution to (1) with \((f_c, f_d) = (f_{c0}, f_{d0})\), then, by conditions (ii) and (iv), for all \( t \in (t_{k-1}, t_k], k \in N \), we get
\[
L_c(t, x(t)) + \Gamma_c(t, x(t)) = -D^+ V(t, x(t)) \quad (14)
\]
\[
L_d(t_k, x_k) + \Gamma_d(t_k, x_k) = -\Delta V(t_k, x_k). \quad (15)
\]
Hence, integrating over the interval \([t_0, t_f]\) with \( \{ k \in N : t_0 \leq t_k < t_f \} = \{1, 2, \ldots, m\} \), by (14)–(15) and similar to (13), we have
\[
\tilde{I}_{(f_{c0}, f_{d0})}^{t_f}(x_0) = V(t_0, x_0) - V(t_f, x_f). \quad (16)
\]
Thus, by \( V(t_f, x_f) \geq 0 \) and (13), we obtain that the results of this theorem hold and the proof is complete. \( \square \)

Theorem 2. Assume that there exist functions \( \tilde{\Gamma}_c : R_+ \times R^n \to R, \tilde{\Gamma}_d : R_+ \times R^n \to R, \) and \( \tilde{V} : R_+ \times R^n \to R, \) where \( \tilde{V} \) is \( C^1 \) function, such that
(i) for all \( t \in (t_{k-1}, t_k], k \in N, \)
\[
\tilde{V}_\ell(t, x) f_c(t, x) \geq \tilde{V}_\ell(t, x) f_{c0}(t, x)
\]
\[
-\tilde{\Gamma}_c(t, x); \quad (17)
\]
(ii) for all \( t \in (t_{k-1}, t_k], k \in N, \)
\[
L_c(t, x) + \tilde{V}_\ell(t, x) f(t, x) f_{c0}(t, x)
\]
\[
= 0; \quad (18)
\]
(iii) for all \( k \in N, \)
\[
\tilde{V}(t_k, x_k + f_d) \geq \tilde{V}(t_k, x_k) - \tilde{\Gamma}_d(t_k, x_k); \quad (19)
\]
(iv) for any \( k \in N, \)
\[
L_d(t_k, x_k) + \tilde{V}(t_k, x_k + f_{d0}(t_k, x_k))
\]
\[
- \tilde{V}(t_k, x_k) - \tilde{\Gamma}_d(t_k, x_k) = 0. \quad (20)
\]
Then,
\[
\inf_{(f_c, f_d) \in F} J_{(f_{c0}, f_{d0})}^{t_f}(x_0) \geq \tilde{I}_{(f_{c0}, f_{d0})}^{t_f}(x_0)
\]
\[
= \tilde{V}(t_0, x_0) - \tilde{V}(t_f, x_f) \quad (21)
\]
where
\[
\tilde{I}_{(f_{c0}, f_{d0})}^{t_f}(x_0) = \int_{t_0}^{t_f} [L_c(t, x(t)) - \tilde{\Gamma}_c(t, x(t))]dt
\]
\[
+ \sum_{t_0 \leq t_k < t_f} [L_d(t_k, x_k) - \tilde{\Gamma}_d(t_k, x_k)] \quad (22)
\]
where in (22), \( x(t), t \neq t_0, \) is a solution to system (1) with \((f_c, f_d) = (f_{c0}, f_{d0}).\)

Proof. By similar proof of Theorem 1, we can obtain that this theorem also holds. The details are omitted here. \( \square \)

Remark 1. Equations (6) and (8) in Theorem 1, and (18) and (20) in Theorem 2 are strong and conservative with respect to functions \( \Gamma_c, \Gamma_d, \tilde{\Gamma}_c, \) and \( \tilde{\Gamma}_d. \) If we replace “\( = \)" in (6) and (8) by “\( \leq \)”, and “\( = \)" in (18) and (20) by “\( \geq \)”, then, we can relax these conditions to less conservative inequalities.

Corollary 1. Assume that \((f_c, f_d) \equiv (f_{c0}, f_{d0}),\) and there exists a function \( W : R_+ \times R^n \to R, \) where \( W \) is \( C^1 \) function, such that
(i) for all \( t \in (t_{k-1}, t_k], k \in N, \)
\[
L_c(t, x) + W_\ell(t, x) f(t, x) f_{c0}(t, x)
\]
\[
= 0; \quad (23)
\]
(ii) for all $k \in N$,
\[ L_d(t_k, x_k) + W(t_k, x_k + f_{d0}(t_k, x_k)) - W(t_k, x_k) = 0. \]  
(24)

Then,
\[ J_{(f_c, f_d)}^f(x_0) = W(t_0, x_0) - W(t_f, x_f). \]  
(25)

Proof. It is the direct consequence of Theorem 1 and Theorem 2 with $W = V = \tilde{V}$, $\Gamma_c = \tilde{\Gamma}_c$, $\Gamma_d = \tilde{\Gamma}_d = 0$, and all the inequalities in Theorem 1 and Theorem 2 are changed into equations.

IV. ANALYSIS OF $J_{(f_c, f_d)}^f(x_0)$ AND ROBUST STABILITY

In this section, we analyze the hybrid performance functional $J_{(f_c, f_d)}^f(x_0)$ and robust stability criteria for system (1).

Theorem 3. (a) Assume that there exist functions $\Gamma_c : R_+ \times R^n \rightarrow R$, $\Gamma_d : R_+ \times R^n \rightarrow R$, and $V \in \mathcal{V}_0$, where $V$ is $C^1$ function, such that the conditions (i), (ii) and (iv) of Theorem 1 hold. Furthermore, the following conditions are satisfied:

(i) there exist $a, b \in \mathbb{R}$, such that for $(t, x) \in R_+ \times R^n$,
\[ b(||x||) \leq V(t, x) \leq a(||x||); \]  
(26)

(ii) there exist $c \in \mathbb{R}, p \in PC$ such that for all $t \in (t_{k-1}, t_k], k \in N$,
\[ V_l(t, x) + V_k(t, x)p_{e0}(t, x) + \Gamma_c(t, x) \leq p(t)c(V(t, x(t))); \]  
(27)

(iii) there exist $\psi_k \in \mathcal{V}_0, k \in N$, such that
\[ V(t_k, x_k + f_{d0}(t_k, x_k)) \leq V(t_k, x_k + f_{d0}(t_k, x_k)) + \Gamma_d(t_k, x_k) \leq \psi_k(V(t_k, x_k)); \]  
(28)

(iv) there exists a constant $\sigma > 0$ such that for all $z \in (0, \sigma)$
\[ \int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\phi_k(z)} \frac{ds}{c(s)} \leq -r_k, k \in N; \]  
(29)

where $r_k \geq 0$ and $\sum_{k=1}^{\infty} r_k = +\infty$.

Then, the zero solution $x(t) = 0$ of system (1) is robust asymptotically stable for all $(f_c, f_d) \in F$, and
\[ \sup_{(f_c, f_d) \in F} J_{(f_c, f_d)}^f(x_0) \leq I_{(f_c, f_d)}^f(x_0) \leq V(t_0, x_0). \]  
(30)

Specially,
\[ \sup_{(f_c, f_d) \in F} J_{(f_c, f_d)}^\infty(x_0) \leq I_{(f_c, f_d)}^\infty(x_0) = V(t_0, x_0) \]  
(31)

where $I_{(f_c, f_d)}^f(x_0)$ is defined as in (8) of Theorem 1, in which $x(t), t \neq t_0$, is a solution to system (1) with $(f_c, f_d) = (f_{c0}, f_{d0})$.

(b) Assume all the conditions of (a) hold and furthermore, suppose that there exist functions $\tilde{\Gamma}_c : R_+ \times R^n \rightarrow R$, $\tilde{\Gamma}_d : R_+ \times R^n \rightarrow R$, and $W : R_+ \times R^n \rightarrow R$, where $W$ is $C^1$ function, such that

(v) for all $t \in (t_{k-1}, t_k], k \in N$,
\[ W_x(t, x) f_c(t, x) \geq W_x(t, x) f_{c0}(t, x) - \tilde{\Gamma}_c(t, x); \]  
(32)

(vi) for all $t \in (t_{k-1}, t_k], k \in N$,
\[ L_c(t, x) + W_l(t, x) + W_k(t, x)p_{e0}(t, x) - \tilde{\Gamma}_c(t, x) = 0; \]  
(33)

(vii) for all $k \in N$,
\[ W(t_k, x_k + f_{d0}) \geq W(t_k, x_k + f_{d0} - \tilde{\Gamma}_d(t_k, x_k)); \]  
(34)

(viii) for all $k \in N$,
\[ L_d(t_k, x_k) + W(t_k, x_k + f_{d0}(t_k, x_k)) = -W(t_k, x_k) - \tilde{\Gamma}_d(t_k, x_k) = 0. \]  
(35)

Then,
\[ \inf_{(f_c, f_d) \in F} J_{(f_c, f_d)}^f(x_0) \geq \tilde{J}_{(f_c, f_d)}^f(x_0) = W(t_0, x_0) - W(t_f, x_f) \]  
(36)

where $\tilde{J}_{(f_c, f_d)}^f(x_0)$ is defined as in (22) of Theorem 2, in which $x(t), t \neq t_0$, is a solution to system (1) with $(f_c, f_d) = (f_{c0}, f_{d0})$. 

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Proof. (a) By conditions (ii) and (3), we get for 
\[ t \in (t_{k-1}, t_k), \ k \in N, \]
\[ D^+ V(t, x(t)) \leq p(t)c(V(t, x(t))), \]
and from condition (iii), for \( k \in N, \)
\[ V(t_k, x_k + f_{dk}(t_k, x_k)) \leq \psi_k(V(t_k, x_k)). \]
Thus, from (26), (29), (37) and (38) we obtain that all conditions of theorem 1 in [8] are satisfied and hence the zero solution of (1) is robust asymptotic stable for all \( (f_c, f_d) \in F. \)

The proof is complete.□

Remark 2. (a) By Theorem 2 in [8], if all the conditions of Theorem 3 hold except the conditions (iii) and (vii), in which \( p(t) \) is replaced by \(-p(t), \) then all results of Theorem 3 still hold.

(b) The conditions of Theorem 3 are less conservative than that in [5], where the system is time-invariant and needs stronger conditions:
\[ V_k(x) f_{d0}(x) + \Gamma_c(x) < 0, \]
\[ V(x_k + f_{d0}(x_k)) + \Gamma_d(x_k) \leq V(x_k). \]

(c) In [14], the stability result is based on the comparison principle, where the comparison system is a time-varying linear impulsive system. In Theorem 3, if we consider the stability by comparison system, then we can get the comparison system as:
\[ \dot{w}(t) = p(t)c(w(t)), \quad t \neq t_k, \]
\[ w(t_k^+) = \psi_k(w(t_k)), \]
\[ w(t_0) = 0 \geq 0. \]

It is clearly that system (41) stands for more general case than linear system studied in [14]. Hence, the stability result in Theorem 3 is more general than that in [14].

(d) For conditions in Theorem 3a and Theorem 3b, similar discussion as in Remark 1 will relax the conservativeness of these conditions. Moreover, similar discussion will relax the conservative nature of corresponding conditions in the following Corollaries 2–3. The details are omitted here.

(e) It should be noticed that the results in Theorem 3 and Remark 2a can be used to stabilize and synchronize chaotic systems and complex dynamical networks. Moreover, by using the results in [19], less conservative conditions may be derived for the synchronization where the driving system and the derived system are not necessary to be identical.

In the following, we specialize Theorem 3 to two cases:

Case 1. Consider the nonlinear uncertain impulsive dynamical systems of the form
\[ \dot{x} = f_{c0}(t, x) + g_c(t, x), \quad t \neq t_k, \]
\[ \Delta x = f_{d0}(t, x(t)) + g_d(t, x(t)), \quad t = t_k, \ k \in N, \]
where \( g_c, g_d : R_+ \times R^n \to R^n. \) The functions \( g_c, g_d \) represent structural uncertainty or uncertain perturbation characterized by
\[ g_c \in \Omega_{c_e} = \{ g_c : g_c = e_c \cdot \delta_c, \ |\delta_c|| \leq ||m_c|| \} \]
\[ g_d \in \Omega_{c_d} = \{ g_d : g_d = e_d \cdot \delta_d, \ |\delta_d|| \leq ||m_d|| \} \]
where \( e_c, e_d : R_+ \times R^n \to R^n \times m \) are known matrix functions whose entries are smooth functions of the state, and \( \delta_c, \delta_d \) are unknown vector-valued functions whose norm are bounded, respectively, by the norm of vector-valued functions \( m_c, m_d, \) respectively. Here, \( m_c, m_d : R_+ \times R^n \to R^m \) are given smooth functions.

Corollary 2. Assume that there exists \( V \in v_0, \) where \( V \) is \( C^1 \) function, such that the condition (i) of Theorem 3 holds. Suppose further that
(i) there exist functions \( P_{1k} : R_+ \times R^n \to R^{1 \times m}, \)
\[ P_{2k} : R_+ \times R^n \to R^{m \times m} \] with \( P_{2k}(t, x) \geq 0, \) and for \( t \in R_+, \ x \in R^n, \ y \in R^m, \ k \in N, \)
\[ V(t, x + f_{d0}(t, x) + e_d(t, x)y) \leq V(t, x) + f_{d0}(t, x) + P_{1k}y + y^TP_{2k}y; \]

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(ii) there are positive constants \( \varepsilon_k, (k \in N) \) such that
\[
V(t_k^+ \cdot x_k + f_{d0}(t_k, x_k)) + \varepsilon_k^{-1} P_{tk} P_{tk}^T
+ (\varepsilon_k + \lambda_{\max}(P_{2k})) m_{dk}^T m_{dk}
\leq \psi_k(V(t_k, x_k))
\]
(44)
where \( \psi_k \in \mathcal{X}, P_{tk} = P_{tk}(t_k, x_k), P_{2k} = P_{2k}(t_k, x_k), m_{dk} = m_{dk}(t_k, x_k), k \in N; \)
(iii) there exist \( c \in \mathcal{X}, p \in \mathcal{P} \) and \( \lambda_k \in C[R_+, R_+] \) such that Hamilton-Jacobi inequalities are satisfied
\[
V_t + V_x f_{c0} + \frac{\lambda_k^2}{2} V_x e e_c^T V_x + \frac{1}{2\lambda_k^2} m_c^T m_c
\leq p(t) \cdot c(V)
\]
(45)
where \( f_{c0} = f_{c0}(t, x(t)), e_c = e_c(t, x(t)), m_c = m_c(t, x(t)), (t, x) \in (t_k, t_{k+1}] \times \mathcal{S}_p, k \in N; \)
(iv) there exists a constant \( \sigma > 0 \) such that for all \( z \in (0, \sigma) \) such that (29) holds:
(v) for all \( t \in (t_{k-1}, t_k],
L_c(t, x) + V_t(t, x) + V_x(t, x) f_{c0}(t, x)
+ \frac{\lambda_k^2}{2} V_x e e_c^T V_x + \frac{1}{2\lambda_k^2} m_c^T m_c = 0;
\]
(46)
(vi) for all \( k \in N,
L_d(t_k, x_k) + V(t_k, x_k + f_{d0}(t_k, x_k))
- V(t_k, x_k) + \varepsilon_k^{-1} P_{tk} P_{tk}^T
+ (\varepsilon_k + \lambda_{\max}(P_{2k})) m_{dk}^T m_{dk} = 0.
\]
(47)
Then, the zero solution \( x(t)=0 \) of system (42) is robust asymptotic stable for all \( (g_c, g_d) \in \Omega_{g_c} \times \Omega_{g_d}, \)
and
\[
\sup_{(f_c, f_d) \in F} J_{(f_c, f_d)}^{1f}(x_0) \leq I_{(f_{c0}, f_{d0})}^{1f}(x_0)
\leq V(t_0, x_0).
\]
(48)
Specially,
\[
\sup_{(f_c, f_d) \in F} J_{(f_c, f_d)}^{\infty}(x_0) \leq I_{(f_{c0}, f_{d0})}^{\infty}(x_0)
\]
= \( V(t_0, x_0) \)
(49)
where \( I_{(f_{c0}, f_{d0})}^{1f}(x_0) \) is defined as in (8) of Theorem 1, where,
\[
\Gamma_c(t, x) \triangleq \frac{\lambda_k^2}{2} V_x e e_c^T V_x + \frac{1}{2\lambda_k^2} m_c^T m_c,
\]
(50)
\[
\Gamma_d(t_k, x_k) \triangleq \varepsilon_k^{-1} P_{tk} P_{tk}^T
+ (\varepsilon_k + \lambda_{\max}(P_{2k})) m_{dk}^T m_{dk},
\]
(51)
and \( t_f < \infty \) or \( t_f = \infty, x(t), t \neq t_0, \) is a solution to system (42) with \( (g_c, g_d) = (0, 0). \)

**Proof.** Since \( f_c = f_{c0} + e_c \delta_c \) and \( f_d = f_{d0} + e_d \delta_d, \) then, for some \( \lambda_k \in C[R_+, R_+] \) and all \( t \in (t_{k-1}, t_k], k \in N,
\]
\[
V(t_k, x_k) f_c(t, x) \leq V(t_k, x_k) + \Gamma_c.
\]
(52)
Thus, by (52) and condition (iii) and condition (v), we get
\[
V_t + V_x f_{c0} + \Gamma_c \leq p(t) \cdot c(V),
\]
(53)
\[
L_c + V_t + V_x f_{c0} + \Gamma_c = 0.
\]
(54)
Let \( f_d = f_{d0} + e_d \delta_d, \) then, by conditions (i)–(ii), for some \( \varepsilon_k > 0, \) we have
\[
V(t_k, x_k + f_d(t_k, x_k)) \leq \psi_k(V(t_k, x_k)).
\]
(55)
Thus, (55) and condition (vi) imply that
\[
V(t_k, x_k + f_d) \leq V(t_k, x_k + f_{d0}) + \Gamma_d
\leq \psi_k(V(t_k, x_k)),
\]
(56)
\[
L_d(t_k, x_k) + V(t_k, x_k + f_{d0})
- V(t_k, x_k) + \Gamma_d = 0.
\]
(57)
Therefore, by (52)–(57), we obtain that all conditions of Theorem 1 hold and the proof is complete.

**Remark 3.** By Remark 2a, if in Corollary 2, \( p(t) \) is replaced by \( -p(t), \) then all results of Corollary 2 still hold.

**Case 2:** Consider the linear uncertain impulsive dynamical systems of the form
\[
\dot{x} = A_c(t)x + B_c(t)x, \quad t \neq t_k,
\]
\[
\Delta x = A_d(t)x + B_d(t)x, \quad t = t_k,
\]
\[
x(t_0^+) = x_0, \quad k \in N,
\]
(58)
with the hybrid quadratic performance functional:

\[ J_{(B_c, B_d)}^{1f}(x_0) = \int_{t_0}^{t_f} x^T(t)R_c(t)x(t)dt + \sum_{t_0 \leq t < t_f} x_k^T R_d x_k, \quad (59) \]

where \( A_c(t), A_d(t) \in R^{n \times n} \) are known matrices, and \( B_c(t), B_d(t), k \in N \) are interval matrices (see [8]): \( B_c(t) \in N[P_c(t), Q_c(t)], B_d(t) \in N[P_{dk}(t), Q_{dk}(t)] \), where \( P_c(t), P_{dk}(t), Q_c(t), Q_{dk}(t), k \in N \) are known matrices. \( R_c(t) \in P^{n \times n}, R_d \in N^{n \times n}, \) where \( P^{n \times n} \) denote the set of \( n \times n \) positive definite matrices, and \( N^{n \times n} \) denote the set of nonnegative definite matrices.

By lemma 3.1 in [8], for any interval matrix \( B(t) \in N[P(t), Q(t)] \), where \( P = (p(t)_{ij})_{n \times n} \) and \( Q = (q(t)_{ij})_{n \times n} \) are known matrices, then, we have:

\[ B(t) = B(t_0) + E(t) \Sigma(t) F(t), \quad (60) \]

where \( B(t_0) = \frac{1}{2}(P(t) + Q(t)), H(t) = (h(t)_{ij})_{n \times n} = \frac{1}{2}(Q(t) - P(t)), \Sigma(t) \in \Sigma^* := \{ \Sigma(t) \in R^{n \times n^2} : \Sigma^T(t) \Sigma(t) \leq I_n, \text{ and } E(t)E(t)^T = \text{diag}(\sum_{j=1}^n h(t)_{1j}, \ldots, \sum_{j=1}^n h(t)_{nj}) \in R^{n \times n}, \ F(t)F(t) = \text{diag}(\sum_{j=1}^n h(t)_{1j}, \ldots, \sum_{j=1}^n h(t)_{nj}) \in R^{n \times n} \} \).

Denote \( A_{c0}(t) = A_c(t) + B_c(t), A_{d0}(t) = A_{d0}(t) + B_{d0}(t), B_c(t) = B_{c0}(t) + E_c(t) \Sigma_c(t) F_c(t), \text{ and } B_d(t) = B_{d0}(t) + E_d(t) \Sigma_d(t) F_d(t) \), where \( \Sigma_c(t), \Sigma_d(t) \in \Sigma^* \).

**Corollary 3.** Assume that there exist scalar functions \( \lambda_k(t) > 0, \eta(t) > 0, k \in N, \) and an uniformly positive definite matrix function bounded from above \( X(t) \) such that

(i) \( X(t) \) is differentiable at \( t \neq t_k \) and the Riccati inequality holds for all \( t \in (t_k - 1, t_k) \), \( k \in N, \)

\[ \dot{X} + XA_{c0} + A_{c0}^TX + 2\lambda_k^2XE_cE_c^TX + \frac{1}{2\lambda_k^2}F_c^TF_c < -\eta X, \quad (61) \]

where \( X = X(t), A_{c0} = A_{c0}(t), E_c = E_c(t), F_c = F_c(t), \lambda_c = \lambda_k(t), \eta = \eta(t), \text{ and } \)

\[ R_c + \dot{X} + XA_{c0} + A_{c0}^TX + 2\lambda_k^2XE_cE_c^TX + \frac{1}{2\lambda_k^2}F_c^TF_c = 0 \quad (62) \]

(ii) there exist some \( r_k \geq 0 \) with \( \sum_{k=1}^{+\infty} r_k = +\infty \), and \( \epsilon_k > 0, k \in N \) such that

\[ f - \int_{t_k}^{t_{k+1}} \eta(s)ds + \ln \beta_k \leq -r_k, \quad \text{for all } k \in N, \quad (63) \]

where \( \beta_k = \lambda_{\max}(X_k^{-1}[(I + A_{d0_k})^T X_k + \epsilon_k^{-1}X_kE_{dk} E_{dk}^T X_k]^T(I + A_{d0_k}) + [\epsilon_k + \lambda_{\max}(E_{dk}^T X_k E_{dk})]F_{dk}^TF_{dk}) \), \( X_k = X(t_k), E_{dk} = E_{dk}(t_k), \text{ and } F_{dk} = F_{dk}(t_k); \)

(iii) for all \( k \in N, \)

\[ R_d + (I + A_{d0_k})^T X_k + \epsilon_k^{-1}X_kE_{dk} E_{dk}^T X_k \times (I + A_{d0_k}) - X_k + [\epsilon_k + \lambda_{\max}(E_{dk}^T X_k E_{dk})]F_{dk}^TF_{dk} = 0. \quad (64) \]

Then, the zero solution \( x(t) = 0 \) of system (58) is robust asymptotic stable for all \( (B_c, B_d) \in \Sigma = \Sigma^* \times \Sigma^* \), and

\[ \sup_{(B_c, B_d) \in \Sigma} \int_{(B_c, B_d)}^{t_f} x(t) \, dt \leq \int_{(B_c, B_d)}^{t_f} \beta_{(A_{c0}, A_{d0_k})} \, x_0 \, dt \leq x_0 \, x_0. \quad (65) \]

Specially,

\[ \sup_{(B_c, B_d) \in \Sigma} \int_{(B_c, B_d)}^{t_f} x(t) \, dt \leq \int_{(B_c, B_d)}^{t_f} \beta_{(A_{c0}, A_{d0_k})} \, x_0 \, dt = x_0 \, x_0. \quad (66) \]

where \( X_0 = X(t_0) \), and,

\[ I_{(A_{c0}, A_{d0_k})}^{1f}(x_0) = \int_{t_0}^{t_f} x^T \left[ R_c + 2\lambda_k^2X_cE_c^T E_c^T X \right. \]

\[ + \frac{1}{2\lambda_k^2}F_c^TF_c \left. \right] x(t)dt + \sum_{t_0 \leq t < t_f} x_k[R_d \]

\[ + \epsilon_k^{-1}(I + A_{d0_k})^T X_kE_{dk} E_{dk}^T X_k(I + A_{d0_k}) \]

\[ + [\epsilon_k + \lambda_{\max}(E_{dk}^T X_k E_{dk})]F_{dk}^TF_{dk} [X_k, \quad (67) \]

where \( t_f < \infty \) or \( t_f = \infty \), and in (67), \( x(t), t \neq t_0, \) is a solution to system (58) with \( (B_c, B_d) = (0, 0), k \in N \).

**Proof.** Let \( V = V(t, x) = x^T X(t)x. \) We can test all the conditions of Theorem 3 and Remark 2 (a). The details are omitted here.

**V. OPTIMAL ROBUST CONTROL**

In this section, we consider the optimal robust control problem for uncertain impulsive dynamical controlled systems.
Consider general uncertain impulsive controlled systems:

\[
\begin{align*}
\dot{x}(t) &= \tilde{f}_c(t, x(t), u_c(t)), \quad t \in (t_{k-1}, t_k] \\
\Delta x &= \tilde{f}_d(t, x(t), u_d(t)), \quad t = t_k, \quad k \in N \\
x(t_0^+) &= x_0
\end{align*}
\]  

(68)

where \( u_c : R_+ \rightarrow U_c \subset R^{n_c} \), \( u_d : R_+ \rightarrow U_d \subset R^{n_d} \), with \( u_c(0) = 0, u_d(0) = 0 \), here \((U_c, U_d) \subset R^{n_c} \times R^{n_d}\) is the class of admissible hybrid control inputs; \( \tilde{f}_c \in \tilde{F}_c = \{ \tilde{f}_c : R_+ \times R^n \times R^{n_c} \rightarrow R^n, \tilde{f}_c(t, y, 0) = 0 \} \), where \( \tilde{f}_c(t, y, 0) \) is Lipschitz continuous in \( (x, y) \), \( \tilde{f}_d \in \tilde{F}_d = \{ \tilde{f}_d : R_+ \times R^n \times R^{n_d} \rightarrow R^n, \tilde{f}_d(t, 0, 0) = 0 \} \), where \( \tilde{f}_d \) is continuous. \( \tilde{F}_c \) and \( \tilde{F}_d \) denote the set of non-linear uncertain functions with \( f_c \in \tilde{F}_c \) and \( f_d \in \tilde{F}_d \) defining the nominal nonlinear functions of \( f_c \in F_c \) and \( f_d \in F_d \), respectively. Denote \( \bar{F} = \tilde{F}_c \times \tilde{F}_d \).

Similarly to (2), the hybrid performance functional of system (68) can be described as

\[
J^{ij}_{(f_c, f_d)}(x_0, u_c, u_d) = \int_{t_0}^{t_j} L_c(t, x(t), u_c(t))dt + \sum_{t_{k-1} \leq t < t_j} L_d(t_k, x(t_k), u_d(t_k)),
\]

(69)

where \( L_c : R_+ \times R^n \times R^{n_c} \rightarrow R \) and \( L_d : R_+ \times R^n \times R^{n_d} \rightarrow R \) are given.

The robust optimal control problem of (68) with hybrid performance functional (69) is to design the state feedback control law \((u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))\), where \( \phi_c : R^n \rightarrow R^{n_c}, \phi_d : R^n \rightarrow R^{n_d} \) with \( \phi_c(0) = 0, \phi_d(0) = 0 \), such that the closed-loop system is robust asymptotic stable, and the bounds of \( J^{ij}_{(f_c, f_d)}(x_0, \phi_c(x), \phi_d(x)) \) can be estimated. Moreover, \((u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))\) can minimize the bounds of \( J^{ij}_{(f_c, f_d)}(x_0, u_c, u_d) \), which is the hybrid performance functional of the nominal system of (68).

**Theorem 4.** Assume that there exist functions \( \Gamma_c : R_+ \times R^n \times R^{n_c} \rightarrow R, \Gamma_d : R_+ \times R^n \times R^{n_d} \rightarrow R, V \in V_0 \), where \( V \) is \( C^1 \) function satisfying (26), and \( \phi_c : R^n \rightarrow R^{n_c}, \phi_d : R^n \rightarrow R^{n_d} \) with \( \phi_c(0) = 0, \phi_d(0) = 0 \), such that

1. for all \( t \in (t_{k-1}, t_k], k \in N \),
\[
\begin{align*}
V_c(t, x) + V_x(t, x)\tilde{f}_c(t, x, \phi_c(x)) + \Gamma_c(t, x, \phi_c(x)) \\
&\leq V_c(t, x)\tilde{f}_c(t, x, \phi_c(x)) + \Gamma_c(t, x, \phi_c(x))
\end{align*}
\]

(70)

2. there exist \( c \in x, p \in PC \) such that for all \( t \in (t_{k-1}, t_k] \),
\[
V_c(t, x) + V_x(t, x)\tilde{f}_c(t, x, \phi_c(x)) + \Gamma_c(t, x, \phi_c(x)) \leq p(t)c(V(t, x));
\]

(71)

3. there exist \( \psi_k \in x_0, k \in N \),
\[
V(t_k, x_k) + V_x(t_k, x_k)\tilde{f}_d(t_k, x_k, \phi_d(x_k)) + \Gamma_d(t_k, x_k, \phi_d(x_k)) \leq \psi_k(V(t_k, x_k));
\]

(72)

4. there exists a constant \( \sigma > 0 \) such that for all \( z \in (0, \sigma) \) such that (29) holds;

5. \( H_c(t, x, \phi_c(x)) = 0 \), for all \( t \in (t_{k-1}, t_k] \),
\[
H_d(t_k, x_k, \phi_d(x_k)) = 0, \quad k \in N,
\]

(73)

(74)

(75)

(76)

where

\[
\begin{align*}
H_c(t, x, u_c) &= L_d(t, x, u_c) + V(t, x) + \tilde{f}_c(t, x, u_c) + \Gamma_c(t, x, u_c), \\
H_d(t, x, u_d) &= L_d(t, x, u_d) + V(t, x) + \tilde{f}_d(t, x, u_d) + \Gamma_d(t, x, u_d)
\end{align*}
\]

(77)

(78)

Then, with the control law \((u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))\), \( t \geq t_0, k \in N \), the zero solution \( x(t) = 0 \) of the closed-loop system (68) is robust asymptotic stable for all \((\tilde{f}_c, \tilde{f}_d) \in \bar{F})\), and

\[
\sup_{(f_c, f_d) \in \bar{F}} J^{ij}_{(f_c, f_d)}(x_0, \phi_c(x), \phi_d(x)) \leq \sup_{(f_c, f_d) \in \bar{F}} J^{ij}_{(f_c, f_d)}(x_0, \phi_c(x), \phi_d(x)) \leq V(t_0, x_0).
\]

(79)

Specially,

\[
\sup_{(f_c, f_d) \in \bar{F}} J^{\infty}_{(f_c, f_d)}(x_0, \phi_c(x), \phi_d(x)) \leq V(t_0, x_0).
\]

(80)
Hence, (82) holds and the proof is complete.

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where

\[ I_{(j_0, j_0)}^{j_f}(x_0, u_c, u_d) \]

\[ = \int_{j_0}^{j_f} [L_e(t, x(t), u_c) + \Gamma_e(t, x(t), u_c)]dt \]

\[ + \sum_{t_j \leq t_k < j_f} [L_d(t_k, x_k, u_d(t_k)) \]

\[ + \Gamma_d(t_k, x_k, u_d(t_k))], \quad (81) \]

where \( t_j < \infty \) or \( t_j = \infty \), and in (81), \( x(t), t \neq t_0 \), is a solution to system (68) with \((\bar{f}_c, \bar{f}_d) = (\bar{f}_0, \bar{f}_d)\).

Moreover, the hybrid feedback control law \((u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))\), \( t \geq t_0, k \in \mathbb{N} \), minimizes \( I_{(j_0, j_0)}^{\infty}(x_0, u_c, u_d) \) in sense that

\[ I_{(j_0, j_0)}^{\infty}(x_0, \phi_c(x), \phi_d(x)) \]

\[ = \min_{(u_c, u_d) \in (U_c, U_d)} I_{(j_0, j_0)}^{\infty}(x_0, u_c, u_d). \quad (82) \]

**Proof.** By applying Theorem 3, we only need to prove that the hybrid feedback control law \((u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))\), \( t \geq t_0, k \in \mathbb{N} \), minimizes \( I_{(j_0, j_0)}^{\infty}(x_0, u_c, u_d) \) in sense of (102).

Let \((\bar{f}_c, \bar{f}_d) = (\bar{f}_0, \bar{f}_d)\), then for all \( t \in (t_k, t_{k+1}], k \in \mathbb{N} \),

\[ L_e(t, x(t), u_c(t)) + \Gamma_e(t, x(t), u_c(t)) \]

\[ = -D^+V(t, x(t)) + H_e(t, x(t), u_c(t)), \quad (83) \]

and for all \( k \in \mathbb{N} \),

\[ L_d(t_k, x_k, u_d(t_k)) + \Gamma_d(t_k, x_k, u_d(t_k)) \]

\[ = -\Delta V(t_k, x_k) + H_d(t_k, x_k, u_d(t_k)). \quad (84) \]

Thus, using (75) and (76), and the fact that system (68) with \((\bar{f}_c, \bar{f}_d) = (\bar{f}_0, \bar{f}_d)\) is asymptotic stable, we have

\[ I_{(j_0, j_0)}^{\infty}(x_0, \phi_c(x), \phi_d(x)) \]

\[ = \lim_{t \to \infty} V(t, x(t)) + V(t_0, x_0) \]

\[ + \int_{t_0}^{\infty} H_e(t, x(t), u_c(t))dt + \sum_{k=1}^{\infty} H_d(t_k, x_k, u_d(t_k)) \]

\[ \geq V(t_0, x_0) = I_{(j_0, j_0)}^{\infty}(x_0, \phi_c(x), \phi_d(x)). \quad (85) \]

Hence, (82) holds and the proof is complete. \( \square \)

**Remark 3(a).** In Theorem 4, if \( p(t) \) is replaced by \(-p(t)\), then the results of Theorem 4 still hold.

(b) By (73)–(76), if nonnegative functions \( H_e(t, x, u_c), H_d(t, x, u_d) \) are continuously differentiable in \( u_c \) and \( u_d \), respectively, then,

\[ \frac{\partial H_e(t, x, u_c)}{\partial u_c}
\]

\[ \bigg|_{u_c = \phi_c(x)} = 0, \]

\[ \frac{\partial H_d(t, x, u_d)}{\partial u_d}
\]

\[ \bigg|_{u_d = \phi_d(x)} = 0 \]

which can be used to derive the hybrid state feedback control law \((u_c, u_d) = (\phi_c(x), \phi_d(x))\).

(c) Similar to Theorem 3b, we can derive the below bound of \( I_{(j_0, j_0)}^{j_f}(x_0, u_c, u_d) \).

Now, we specialize Theorem 4 to linear uncertain impulsive dynamical system:

\[ \dot{x} = (A_c(t) + B_c(t))x + D_c(t)u_c(t), \quad t \neq t_k, \]

\[ \Delta x = (A_{dk}(t) + B_{dk}(t))x + D_{dk}u_d(t), \quad t = t_k, \]

\[ x(t_0^+) = x_0, \quad k \in \mathbb{N}, \]

with the hybrid quadratic performance functional:

\[ J_{(j_0, j_0)}^{j_f}(x_0, u_c, u_d) \]

\[ = \int_{t_0}^{j_f} [x^T R_c x + u_c^T S_c u_c]dt \]

\[ + \sum_{t_0 \leq t \leq j_f} [x_k^T R_d x_k + u_k^T S_d x_k], \quad (87) \]

where \( \bar{A}_c(t) = A_c(t) + B_c(t), \bar{A}_{dk}(t) = A_{dk}(t) + B_{dk}(t), \)

and \( A_c(t), A_{dk}(t), B_c(t), B_{dk}(t), (k \in \mathbb{N}), \quad R_c(t) \in P^{n \times n}, \) and \( R_d \in P^{n \times n} \), are defined as in (58)–(59), and where \( D_c(t) \in R^{n \times n_c}, D_{dk}(t) \in R^{n \times n_d}, S_c(t) \in P^{n_c \times n_c}, \)

\( S_d(t) \in P^{n_d \times n_d} \) are known matrices.

Let \( A_c(t) + B_c(t) = A_0(t) + E_c(t)S_c(t)F_c(t) \) and \( A_{dk}(t) + B_{dk}(t) = A_{00}(t) + E_{dk}(t)S_{dk}(t)F_{dk}(t), \) \( S_{dk} \in \Sigma^*, \)

\( k \in \mathbb{N}. \)

**Corollary 4.** Assume that there exist scalar functions \( \lambda_k(t) > 0, \sigma(t) > 0, k \in \mathbb{N}, \) and a uniformly positive definite matrix function bounded from above \( X(t) \)
such that

(i) \( X(t) \) is differentiable at \( t \neq t_k \) and for all \( t \in (t_k-1, t_k), k \in N \), the Riccati inequality holds:

\[
\dot{X} + X(A_c + K_c) + (A_c + D_c K_c)^T X + 2 \lambda_k^2 X E_c E_c^T X + \frac{1}{2 \lambda_k^2} F_c^T F_c < -\alpha X,
\]

where \( X = X(t) \), \( A_c = A_c(t) \), \( E_c = E_c(t) \), \( F_c = F_c(t) \), \( K_c = K_c(t) = -S_c^{-1}(t) D_c^T(t) X(t) \), \( \lambda_k = \dot{\lambda}_k(t) \), and \( \alpha = \alpha(t) \); and the following Riccati equation holds:

\[
R_c + K_c^T S_c K_c + \dot{X} + X(A_c + D_c K_c)
+ (A_c + D_c \cdot K_c)^T X + 2 \lambda_k^2 X E_c E_c^T X + \frac{1}{2 \lambda_k^2} F_c^T F_c = 0
\]

(ii) there exist some \( r_k \geq 0 \) with \( \sum_{k=1}^{n} r_k = +\infty \), and \( \varepsilon_k > 0, k \in N \) such that

\[
- \int_{t_k}^{t_{k+1}} \alpha(s) ds + \ln \beta_k \leq -r_k,
\]

for all \( k \in N \),

(iii) for all \( k \in N \),

\[
R_d + K_d^T S_d K_d - X_k + (I + A_{d0} + D_{d0} K_{d0})^T \]

\[
[X_k + \varepsilon_k^{-1} X_k E_d E_d^T X_k] (I + A_{d0} + D_{d0} K_{d0}) + [e_k + \lambda_{\max}(E_d^T E_d)] X_k E_d E_d^T F_d K_{d0} = 0
\]

Then, with the state feedback control law

\[
(u_c, u_d) = (\phi_c(x), \phi_d(x)) = (K_c x, K_d x),
\]

the zero solution \( x(t) = 0 \) of the closed-loop system (86) and (92) is robust asymptotic stable for all \((B_c, B_{d0}) \in \Sigma = \Sigma^* \times \Sigma^*_d\), and

\[
\sup_{(\Sigma, \Sigma_{d0}) \in \Sigma} J^{\text{eff}}_{(\Sigma, \Sigma_{d0})}(x_0, u_c, u_d)
\leq I^{\text{eff}}_{(A_c, A_{d0})}(x_0, \phi_c(x), \phi_d(x))
\leq x_0^T X_0 x_0.
\]

Specially,

\[
\sup_{(\Sigma, \Sigma_{d0}) \in \Sigma} J^{\infty}_{(\Sigma, \Sigma_{d0})}(x_0, u_c, u_d)
\leq I^{\infty}_{(A_c, A_{d0})}(x_0, \phi_c(x), \phi_d(x))
= x_0^T X_0 x_0,
\]

where \( X_0 = X(t_0) \), and,

\[
I^{\text{eff}}_{(A_c, A_{d0})}(x_0, u_c, u_d)
= \int_{t_0}^{t_f} \left( X_c^T R_c + 2 \lambda_k^2 X_c E_c E_c^T X_c + \frac{1}{2 \lambda_k^2} F_c^T F_c \right) x + u_c^T S_c u_c \right) dt
+ \sum_{t_0 < t_k < t_f} \{ x_k [R_d + \varepsilon_k^{-1} (I + A_{d0})^T X_k E_d E_d^T X_k (I + A_{d0})] \}
\times X_k E_d E_d^T X_k (I + A_{d0})
+ [e_k + \lambda_{\max}(E_d^T E_d X_k)] F_d^T F_d X_k
+ u_d^T (t_k) S_d (t_k) u_d (t_k) \}
\]

where \( t_f < \infty \) or \( t_f = \infty \), and in (95), \( x(t) \), \( t \neq t_0 \), is a solution to system (86) with \((B_c, B_{d0}) = (0, 0), k \in N \).

Moreover,

\[
I^{\infty}_{(A_c, A_{d0})}(x_0, \phi_c(x), \phi_d(x))
= \min_{(u_c, u_d) \in (U_c, U_d)} I^{\infty}_{(A_c, A_{d0})}(x_0, u_c, u_d).
\]

**Proof.** Let \( V(t, x) = x^T X(t, x) \). Then, the results of this theorem are the direct consequence of Theorem 4 and Remark 3b.

**Remark 4.** In order to guarantee the uniqueness of solution to equation (89), by [17], we need the following terminal condition:
(ii*) for some positive constant $\lambda_k$, (89) has the only stable equilibrium $X_{\infty}$, which is a unique, maximal, positive definite solution of the Riccati equation:

$$R_c + K_c^T S_c K_c + X_{\infty} (A_{c0} + D_c K_c) + (A_{c0} + D_c K_c)^T \cdot X_{\infty} + 2\lambda_k^2 X_{\infty} E_c E_c^T X_{\infty} + \frac{1}{2\lambda_k} F_c^T F_c = 0.$$ 

If system (86) is time-invariant, then (ii*) is not needed. \(\square\)

VI. EXAMPLE

As the application, we give an example in this section.

Example 1. Consider the system in form of (86), where the matrices of system (86) are given as follows:

$$A_{c0} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -2 & -1 \end{pmatrix}, \quad D_c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$A_{dk} = A_{d0k} = \begin{pmatrix} -0.9 & 0 & 0.1 \\ 0 & -0.9 & 0 \\ 0.1 & -0.2 & -0.9 \end{pmatrix},$$

$$D_{dk} = (0, 1, -1)^T,$$

$$B_{dk} = 0,$$ and $E_c$, $F_c$ satisfying: $E_c E_c^T = 0.1I$, and $F_c^T F_c = \text{diag}(0.1, 0.1, 0.03)$. The matrices in the quadratic hybrid performance functional (87) are given by:

$$R_c = \begin{pmatrix} 25.9500 & 11.1999 & 11.0000 \\ 11.1999 & 7.5500 & 4.9999 \\ 11.0000 & 4.9999 & 5.7850 \end{pmatrix},$$

$$R_d = \begin{pmatrix} 2.9600 & 1.0300 & 0.0199 \\ 1.0300 & 0.8000 & 0.0100 \\ 0.0199 & -0.0500 & 0.9600 \end{pmatrix},$$

and $S_c = 1$, $S_d = 1$.

Setting

$$X = \begin{pmatrix} 3.0000 & 1.0000 & 0.0000 \\ 1.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix},$$

and $\alpha = 0.7070$, $\beta_k = 0.1107$, then $X$ satisfies Corollary 4. Thus, for any impulsive instances $t_k$, $k \in N$, by Corollary 4, the hybrid feedback control law $(u_c, u_d) = (K_c x, K_d x)$ can stabilize this closed-loop system, where $K_c = (-4 - 2 - 2)$, $K_d = (0 - 0.3000 0)$. Moreover, (93) holds under the control law. For example, if $x_0 = (0.2 - 0.4 3)^T$, then

$$\sup_{(\Sigma_j, \Sigma_{d_j}) \in \Sigma} J^{tf}_{(A_{c0}, \tilde{A}_{dk})} (x_0, u_c, u_d) \leq J^{tf}_{(A_{c0}, A_{d0k})} (x_0, \phi_c(x), \phi_d(x)) \leq x_0^T X \cdot x_0 = 9.1200.$$

Remark 5. Note $\alpha = 0.7070 > 0$, hence the conditions in [5] are not satisfied.

REFERENCES

Department of Applied Mathematics at the Hong Kong Polytechnic University, China, from 1999 to 2004. He is currently Professor of Applied Mathematics and Head of Department of Mathematics and Statistics at Curtin University of Technology. He has published 5 books and over 300 journal papers. He has a software package, MISER3.3, for solving general constrained optimal control problems. He is Editor-in-Chief of the *Journal of Industrial and Management Optimization*, and *Dynamics of Continuous, Discrete and Impulsive Systems, Series B*. He also serves as an associate editor of a number of international journals, including *Automatica, Nonlinear Dynamics and Systems Theory, Journal of Global Optimization, Optimization and Engineering, Discrete and Continuous Dynamic Systems (Series A and Series B), Dynamics of Continuous, Discrete and Impulsive Systems, Series A, Optimization Letters*, and *Journal of Inequality and Applications*. His research interests include both the theoretical and practical aspects of optimal control and optimization, and their practical applications such as in signal processing in telecommunications, and financial portfolio optimization.

**Xinzhi Liu** received the B.Sc. degree in mathematics from Shandong Normal University, Jinan, China, in 1982, and the M.Sc. and Ph.D. degrees, all in applied mathematics, from University of Texas, Arlington, in 1987 and 1988, respectively. He was a Post-Doctoral Fellow at the University of Alberta, Edmonton, Alberta, Canada, from 1988 to 1990. He joined the Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada, in 1990, where he became an Associate Professor in 1994, and a Professor in 1997.

His research areas include systems analysis, stability theory, hybrid dynamical systems, impulsive control, chaos synchronization, nonlinear oscillations, artificial neural networks, and communication security. He is the author or coauthor of over 200 research articles and two research monographs and five other books. He is the Chief Editor of the journal, *DCDIS Series A: Mathematical Analysis*, and the Co-Chief Editor of the Journal, *DCDIS Series B: Applications and Algorithms*, and Associate Editor of several other journals. He is an Honorary Professor at six universities outside Canada. Dr. Liu served as General Chair for several international conferences on dynamical systems and engineering applications.