CHATTERING REDUCTION OF SLIDING MODE CONTROL BY LOW-PASS FILTERING THE CONTROL SIGNAL

Ming-Lei Tseng and Min-Shin Chen

ABSTRACT

The conventional approach to reducing control signal chattering in sliding mode control is to use the boundary layer design. However, when there is high-level measurement noise, the boundary layer design becomes ineffective in chattering reduction. This paper, therefore, proposes a new design for chattering reduction by low-pass filtering the control signal. The new design is non-trivial since it requires estimation of the sliding variable via a disturbance estimator. The new sliding mode control has the same performance as the boundary layer design in noise-free environments, and outperforms the boundary layer design in noisy environments.

Key Words: Sliding mode control, control chattering, boundary layer design, disturbance estimator.

I. INTRODUCTION

The sliding mode control is a widely studied control scheme that provides robustness to certain disturbances and system uncertainties [1, 2]. However, its control signal exhibits high frequency oscillations called chattering after the system state reaches the sliding surface. Such chattering has many negative effects in real world applications since it may damage the control actuator and excite the undesirable unmodeled dynamics. Among the various solutions to reducing chattering, the boundary layer design [3, 4] is probably the most common approach. In the boundary layer design, a smooth continuous function is used to approximate the discontinuous sign function in a region called the boundary layer around the sliding surface. However, the boundary layer design has two disadvantages. First, chattering reduction of the control signal is achieved at the sacrifice of control accuracy. To obtain smoother control signals, the boundary layer width must become larger. But a larger boundary layer width results in larger errors in control accuracy. Second, when there is high-level measurement noise, the boundary layer design becomes ineffective in chattering reduction.

There is a new design in the literature [5, 6] that can avoid the disadvantages of conventional boundary layer design while effectively reduce chattering. The new design adopts a control structure as shown in Fig. 1, where an integrator is placed in front of the system to be controlled. A sliding mode control \( w \) is then constructed for the extended system (the original system plus the integrator). The control signal \( w \) hence has chattering, but the true control signal \( u \) going into the system is smooth since the high frequency chattering in \( w \) will be filtered out by the integrator, which acts as a low pass filter. With such a design, the chattering reduction is achieved by low pass filtering, and at the same time the control accuracy can be maintained. Another advantage of the new design is that in noisy environments, measurement noise causes even more severe chattering in the signal \( w \), but the integrator can still effectively filter out the chattering. Hence, the new design has better noise immunity than the conventional boundary layer design. Previous literature [5, 6] does not carry out
stability analysis and performance analysis and does not address noise-induced chattering. In [7], two first-order filters are employed but again noise-induced chattering is not addressed.

The new low-pass-filtering design for chattering reduction is nevertheless non-trivial. As is known, the sliding variable in sliding mode control design must be chosen such that the control input shows up in the time derivative of the sliding variable. In this way, the control input can influence how the sliding variable evolves. Such a design guideline must also be observed in the new design in Fig. 1. In other words, the time derivative of the new sliding variable for the extended system should contain the sliding mode control input. This in turn suggests that the new sliding variable itself for the extended system contains the integration of the control signal which is the true control signal. Since in Fig. 1 the unknown disturbance enters the system in the same place as the control signal (the so-called matching condition), the new sliding variable will inevitably contain the unknown disturbance, and this makes evaluation of the sliding variable difficult. This is a problem that is unique to the low-pass-filtering design. Previous literature [9, 10] has attempted to solve this problem with only partial success. In [9], a variable structure estimator is proposed to estimate the sliding variable, but it must assume a priori that the system state is uniformly bounded before proving the system stability. In [10], a one-dimensional observer is proposed to estimate the sliding variable, but stability is guaranteed only if a differential inequality with bounded coefficients is satisfied. This paper will propose a complete solution by using the disturbance estimator proposed in [11] for sliding variable estimation. A rigorous stability proof of the new sliding mode control will also be provided.

This paper is organized as follows. Section II reviews the boundary layer design for the sliding mode control of a linear uncertain system. A simulation example is given to reveal the weakness of boundary layer design. Section III introduces the new chattering reduction control design. A second simulation example is given to confirm the advantage of the new design. Finally, Section IV gives the conclusions.

II. BOUNDARY LAYER DESIGN

Consider the sliding mode control design for a linear system with uncertainty:

$$\dot{x} = Ax + Bu + d,$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^1$ is the scalar control input, and $d$ is an external disturbance satisfying the following upper bounds, $|d| \leq D_0$, $|d| \leq D_1$. The system state $x$ is accessible for measurement, and $(A, B)$ is controllable. The case when the system uncertainty $d$ contains state dependent parametric uncertainty can also be treated. But for simplicity it is assumed in this paper that $d$ is a bounded external disturbance. The purpose of this section is to review the boundary layer design in sliding mode control.

A typical sliding mode control with boundary layer design can be as follows.

$$u = -Kx + u_1,$$ \hspace{1cm} (2)

where $K$ is chosen to place all eigenvalues of $A - BK$ to desired locations under the controllability assumption of $(A, B)$ pair, and $u_1$ is

$$u_1 = -\rho_1 \frac{s_1}{s_1 + \varepsilon_1}, \quad \rho_1 > |d|,$$ \hspace{1cm} (3)

in which $\varepsilon_1$ is a small positive parameter which specifies the boundary layer width, and the sliding variable $s_1$ is

$$s_1 = 2B^T P x,$$

with $P$ a positive definite matrix satisfying the Lyapunov equation

$$(A - BK)^T P + P(A - BK) = -I.$$  

The following theorem, whose proof has now become standard and hence is omitted, states the stability property achieved by the above boundary layer control (3).

**Theorem 1** ([12]). The boundary layer control (3) practically stabilizes the system (1) in the sense that the system state is driven to a small residual set containing the state space origin, with the size of residual set proportional to the boundary layer width $\varepsilon_1$.

Theoretically, any small and non-zero boundary layer width $\varepsilon_1$ ensures that the boundary layer control (3) is a continuous function of state. The resultant control signal is therefore smooth and chattering free. However, when the state measurements are corrupted with noise whose level is significant with respect to the boundary layer width, the boundary layer design...
can no longer avoid chattering. This situation is best demonstrated by the following simulation example.

**Example 1** (Boundary layer control). Consider the system (1) with

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-0.12 & 1.06 & -2.84 & 2.9 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
-15.10 \\
\end{bmatrix},
\]

and a disturbance uncertainty \(d = \cos(2t)\). One tests the boundary layer control (2) with stabilizing state feedback gain \(K = [83.88, 112.91, 52.71, 15.10]\), and a small boundary layer width \(\epsilon_1 = 0.5\) and \(\rho_1 = 1.2\) in (3). When the state measurement is noise-free, the control signal will be smooth because of the use of a boundary layer. The noise-free simulation is presented in Fig. 2. The upper plot shows that the state converges to almost zero, and the lower plot shows the control signal is indeed smooth. But when the state measurement is contaminated with a uniform noise with zero-mean and standard deviation 0.05, the results are quite different. The noise corrupted case is simulated and depicted in Fig. 3. The upper plot shows the system state (with noise removed) is still convergent to almost zero, but the lower plot shows the control signal has severe chattering as a result of the measurement noise.

**III. FILTERED SLIDING MODE CONTROL**

### 3.1 Sliding variable design

As is demonstrated in the simulation example 1, sliding mode control with the boundary layer design still exhibits the chattering phenomenon when there is a high level of measurement noise. Hence, a solution better than the boundary layer design is required to reduce the chattering in sliding mode control. To this end, one will introduce the Filtered Sliding Mode Control in this section, whose control structure is depicted in Fig. 1. In Fig. 1, an integrator is intentionally placed in front of the system, and \(w = \dot{u}\) is treated as the control variable for the extended system. A switching sliding mode control law is chosen for \(w\) to suppress the effects of disturbance \(d\). Even though \(w\) is chattering, the control input \(u\) to the system will be smooth because the high-frequency chattering will be filtered out by the integrator, which acts as a low-pass filter. In other words, the new control design removes chattering by filtering the control signal, hence, the control structure in Fig. 1 is called the filtered sliding mode control (FSMC).

For the design of filtered sliding mode control, one chooses the sliding variable as follows.

\[
s_2 = \dot{z} + \lambda z, \quad z = Cx,
\]

where \(\lambda\) is a positive constant, and the row vector \(C \in \mathbb{R}^{1 \times n}\) is chosen such that \((A, B, C)\) is of relative degree one, and the \((n - 1)\) zeros of the system \((A, B, C)\) are in the stable locations. It will be shown in the proof of Theorem 2 below that when \(s_2\) is driven to zero, the system state \(x\) will also be convergent to zero.

Using (4) and (1), one finds

\[
s_2 = CAx + CB(u + d) + \lambda Cx,
\]
and, by taking the time derivative of $s_2$,

$$
\dot{s}_2 = (CA^2 + \lambda CA)x + (CAB + \lambda CB)u + CB w
+ (CAB + \lambda CB)d + CB \dot{d}.
$$

Note that the control variable $w = \dot{u}$ appears in the time derivative of the sliding variable $s_2$, suggesting that one can control the evolution of $s_2$ by properly choosing the control variable $w$. However, there is a problem that according to (5), the expression of $s_2$ contains the unknown disturbance term $d$. Therefore, it is difficult to evaluate the sliding variable $s_2$.

To solve this problem, one will use the Disturbance Estimator proposed in [11] to estimate the disturbance $d$. With an estimate of $d$, one can obtain an estimate of the sliding variable $s_2$ via (5). In the sequel, an estimator for the unknown disturbance $d$ will be constructed based on the scalar variable $z$ defined in (4). Note that $z$ satisfies the following differential equation,

$$
\dot{z} = CAx + CB(u + d), \tag{6}
$$

Call $\hat{z}$ an estimate of $z$, and denote the estimation error as

$$
\dot{e} = z - \hat{e}.
$$

Construct the governing equation of $\hat{e}$ as follows.

$$
\dot{\hat{e}} = CAx + \beta e + CB(u + v), \quad v = \rho \frac{e}{|e| + \epsilon}, \tag{7}
$$

where $\beta$ is a positive constant, $\rho$ an estimator gain larger than the disturbance upper bound $D_0$, and $\epsilon$ is a positive constant close to zero. With the above estimator (7), an estimate of the disturbance $d$ will be provided by

$$
\dot{\hat{d}} = \frac{\beta}{CB}e + v = \frac{\beta e}{CB} + \rho e \frac{\epsilon}{|\epsilon| + \epsilon}. \tag{8}
$$

Once one has obtained an estimate of $d$, one can approximate $s_2$ in (5) by

$$
\dot{s}_2 = CAx + CB(u + \hat{d} + \lambda Cx). \tag{9}
$$

The following theorem proves the effectiveness of the above disturbance estimator (7) and (8).

**Theorem 1.** The disturbance estimation error $d - \hat{d}$, where $\hat{d}$ is given by (8), will become arbitrarily small if the estimator gain $\rho$ in (7) is sufficiently large.

**Proof.** One can refer to the original paper on disturbance estimator [11]. For completeness of this paper, a simple proof will be given below. From (6)–(7), one can easily obtain

$$
\dot{e} = -\beta e - CB(v - d)
= -\beta e - CB\left(\rho \frac{e}{|e| + \epsilon} - d\right). \tag{10}
$$

It will be shown that both $e$ and $\dot{e}$ will become arbitrarily small if $\rho$ is sufficiently large. Notice from (10) and (8) that $\dot{e} = CB(d - \hat{d})$. Therefore, the smallness of $\dot{e}$ implies the smallness of $d - \hat{d}$ and hence, the success of disturbance estimation.

Let Lyapunov function $V_1 = \frac{1}{2} e^2$, and take its time derivative,

$$
\dot{V}_1 = e\dot{e} = e(-\beta e - CB(v - d))
\leq -\beta e^2 - CB\left(\rho \frac{|e|}{|e| + \epsilon} - D_0\right)
\leq -\beta n_1^2, \quad \text{for all } e \notin N_1,
$$

where $N_1 = \{e : |e| < n_1 = (\epsilon D_0) / (\rho - D_0)\}$. With the last inequality, one can prove [see [11]] that $|e(t)| < n_1$ for all $t > T_1$ for some finite time $T_1$. Since $n_1 = (\epsilon D_0) / (\rho - D_0)$ becomes arbitrarily small as the disturbance estimator gain $\rho$ becomes sufficiently large, one concludes that $e$ becomes arbitrarily small within a finite time if $\rho$ is sufficiently large.

To check the behavior of $\dot{e}$, one chooses $V_2 = \frac{1}{2} \dot{e}^2$, and take its time derivative,

$$
\dot{V}_2 = \ddot{e}e = e\left(-\beta \dot{e} - CB\left(\frac{\rho \dot{e}}{|\dot{e}| + \epsilon} - \hat{d}\right)\right)
\leq -\beta \dot{e}^2 - \rho \dot{e}|CB||\dot{e}| \left(\frac{\rho \dot{e}}{|\dot{e}| + \epsilon} - D_1\right), \quad t \geq T_1,
\leq -\beta \dot{e}^2 - \rho |CB|\left(\frac{\dot{e}}{|\epsilon| + \epsilon} - \frac{D_1(n_1 + \epsilon)^2}{\rho \epsilon}\right),
\leq -\beta n_2^2, \quad \text{for all } \dot{e} \notin N_2,
$$

where $N_2 = \{\dot{e} : |\dot{e}| < n_2 = D_1(n_1 + \epsilon)^2 / (\rho \epsilon)\}$. From the last inequality, one can prove [11] that $|\dot{e}(t)| < n_2$ for all $t > T_2$ for some finite time $T_2$. Since $n_2 = D_1(n_1 + \epsilon)^2 / (\rho \epsilon)$ becomes arbitrarily small as the disturbance estimator gain $\rho$ becomes sufficiently large, one concludes that $\dot{e} = CB(d - \hat{d})$ becomes arbitrarily small within a finite time if $\rho$ is sufficiently large. End of proof.

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3.2 Control variable design

In the filtered sliding mode control, the objective of the control variable \( w = \dot{u} \) is to drive the sliding variable \( s_2 \) to (almost) zero in the face of system uncertainties. For this purpose, one chooses

\[
\dot{u} = w = -(CA^2 + CA)x - (CAB + CB)u - \sigma s_2 - \delta \text{sgn}(s_2),
\]

where \( \sigma > 0 \), \( \text{sgn}(\cdot) \) is the sign function, and \( \delta \) is an upper bound of the uncertainty \( |\Delta p| \) with

\[
\Delta p = (CAB + CB)d + \dot{d}.
\]

As explained in the previous section, it is impossible to evaluate the sliding variable \( s_2 \) due to the uncertainty \( d \) involved. Hence, to implement the proposed control, one uses the estimate \( \hat{s}_2 \) in place of \( s_2 \),

\[
\dot{\hat{s}}_2 = C B (d - \hat{d}).
\]

To study the evolution of \( s_2 \), choose Lyapunov function \( V = \frac{1}{2} s_2^2 \) and check its time derivative under the proposed control \( w \) in (12),

\[
\dot{V} = s_2^2 [(CA^2 + CA)x + (CAB + CB)u + CBw + (CAB + CB)d + CB\dot{d}]
\]

\[
= s_2 [-\sigma \hat{s}_2 - \delta \text{sgn}(\hat{s}_2) + \Delta p]
\]

\[
= -s_2^2 + s_2 \hat{s}_2 + s_2 [-\delta \text{sgn}(\hat{s}_2) + \Delta p],
\]

where \( \Delta p \) is as given in (11), and one has used \( \hat{s}_2 = s_2 - \hat{s}_2 \) to obtain the third equality. There are two possible cases for the square brackets in the above equation.

**Case 1.** \( |s_2| > |\hat{s}_2| \): In this case, \( \text{sgn}(\hat{s}_2) = \text{sgn}(s_2 - \hat{s}_2) = \text{sgn}(s_2) \). Equation (14) then becomes

\[
\dot{V} \leq -s_2^2 + s_2 \hat{s}_2 - |s_2|(\delta - |\Delta p|)
\]

\[
\leq -s_2^2 + s_2 \hat{s}_2
\]

\[
\leq -|s_2|^2 + s_2 |\mu|,
\]

where the second inequality results from the design choice \( \delta > |\Delta p| \), and the third inequality (with \( \mu \) an arbitrarily small number) comes from Theorem 1 that \( \hat{s}_2 = CB(d - \hat{d}) \) becomes arbitrarily small asymptotically. From the last inequality, it is not difficult to show that asymptotically one has \( \lim_{t \to \infty} |s_2| \leq \mu \); that is, \( s_2 \) becomes arbitrarily small asymptotically.

**Case 2.** \( |s_2| \leq |\hat{s}_2| \): Since \( \hat{s}_2 = CB(d - \hat{d}) \), it follows from Theorem 2 that \( |s_2| \) becomes arbitrarily small asymptotically.

Judging from conclusions of both Case 1 and 2, one can say that the sliding variable \( s_2 \) becomes arbitrarily small asymptotically. One next shows that the system state \( x \) will also become arbitrarily small as \( s_2 \) does. To this end, introduce a state transformation [13],

\[
x = T \begin{bmatrix} z \\ \eta \end{bmatrix}, \quad T \in \mathbb{R}^{n \times n}.
\]

(14)

where the external state \( z \in \mathbb{R}^1 \) is as defined in (4), and the internal state \( \eta \in \mathbb{R}^{n-1} \) satisfies

\[
\dot{\eta} = Q \eta + P z,
\]

(15)

for some matrices \( Q, P \), in which \( Q \) is a square matrix whose eigenvalues are open-loop zeros of the triple \( (A, B, C) \) [13]. Since, in the design of sliding variable in (4), \( z = Cx \) is chosen such that \( (A, B, C) \) has only stable zeros, \( Q \) is stable.

When \( s_2 \) becomes arbitrarily small, it follows from (4) that the external state \( z \) also becomes arbitrarily small since \( \dot{z} + Az = s_2 \) can be regarded as a stable system \( z \) subject to small input signal \( s_2 \). Similarly,
Fig. 4. Filtered sliding mode control with noise.

(15) can be regarded as a stable system \( \eta \) subject to small input signal \( z \). Hence, its state \( \eta \) will also become arbitrarily small asymptotically. Finally, since both \( z \) and \( \eta \) become arbitrarily small, so does the original system state \( x \) according to the state transformation (14). End of proof.

To show the efficacy of the proposed filtered sliding mode control in noisy environments, a simulation example is presented below.

**Example 2 (Filtered sliding mode control).** The same system as in Example 1 is tested again for the proposed filtered sliding mode control (12). Here one has chosen

\[
C = [10.12, 14.06, 6.50, 1].
\]

The disturbance uncertainty \( d = \cos(2t) \) and the state measurement is contaminated with a uniform noise with zero-mean and standard deviation 0.05. The parameters are chosen such that \( \lambda = 2 \) in (4) and \( \beta = 100, \rho = 1.2, \varepsilon = 0.5 \) in (7). Other design parameters are \( \sigma = 30 \) and \( \delta = 16.08 \) in (12). The upper plot of Fig. 4 shows the time history of system state, which achieves almost the same performance as that with the boundary layer control in Example 1. However, note from the lower plot of Fig. 4 that the filtered sliding mode design has successfully removed chattering in the control input \( u \) even in this noisy environment.

**IV. CONCLUSIONS**

In this paper, a new design is proposed to reduce control chattering in sliding mode control by low-pass filtering the control signal. The new design requires estimation of the sliding variable, and this is achieved by the use of a disturbance estimator. The unique feature of this new design is that chattering reduction is achieved by low-pass filtering the control signal, and control accuracy can be maintained by a sufficiently large disturbance estimator gain. This is contrary to the conventional boundary layer design, where chattering reduction is achieved at the price of sacrificing the control accuracy. This paper further shows via simulation examples that when there is high-level measurement noise, the boundary layer design can no longer reduce chattering, but the new design in this paper can effectively reduce chattering even in noisy environments.

**REFERENCES**


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