DESIGN OF NONLINEAR TERMINAL GUIDANCE/AUTOPILOT CONTROLLER FOR MISSILES WITH PULSE TYPE INPUT DEVICES

Fu-Kuang Yeh

ABSTRACT

This investigation addresses a nonlinear terminal guidance/autopilot controller with pulse-type control inputs for intercepting a theater ballistic missile in the exoatmospheric region. Appropriate initial conditions on the terminal phase are assumed to apply after the end of the midcourse operation. Accordingly, the terminal controller seeks to minimize the distance between the commanded missile and the target missile to ensure a hit-to-kill interception. In particular, a 3D terminal guidance law is initially developed to eliminate the so-called “sliding velocity,” thus, constraining the relative motion between the missile and the target along the line of sight. Sliding mode control is adopted to design stable pulse-type control systems. Then, a quaternion-based attitude controller is used to orient appropriately the commanded missile, taking into account the fact that the missile is a rigid body, to realize interceptability. The stability of the overall integrated terminal guidance/autopilot system is then analyzed thoroughly, based on Lyapunov stability theory. Finally, extensive simulations are conducted to verify the validity and effectiveness of the integrated controller with the pulse type inputs developed herein.

Key Words: Pulse type inputs, divert control system, attitude control, sliding mode, target interception.

I. INTRODUCTION

Medium- and long-range ballistic missiles have been developed and utilized to attack distant targets. Since a low-altitude interception point will probably cause damage as scraps fall after the missiles hit, theater high altitude area defense (THAAD) [1] was designed to intercept a theater ballistic missile as it coasts at an approximate altitude of 150 km. Pulse width modulation (PWM) and pulse width pulse frequency (PWPF) thrusters are adopted to satisfy the need for input discontinuity, thus realizing pulse-type inputs. PWM actuators have been used to make the error state of the spacecraft system converge [2], while PWPF-type actuators have been used to reduce the vibration of the flexible spacecraft system, thereby helping to control the attitude [3]. Hablani [4] proposed the pulsed proportional navigation guidance of a missile equipped with divert thrusters to intercept non-maneuvering reentry vehicles. Since constant bearing [5, 6] is critical to guidance law design, a sliding mode [6, 7] controller is proposed herein to drive robustly to the constant bearing condition. Additionally, quaternion is a global representation [8], which has been applied to specify the attitude of a spacecraft [9–11] or to describe the orientation of general mechanical systems [12–14].
The sliding mode control (SMC) technique, which is imposed to design controllers with robustness [15] against external disturbances and parameter perturbations, has been researched since the early 1960s, and has been applied to missile interception by Yeh et al. [6], Das et al. [16], Idan et al. [17], and Shim et al. [18]. Despite the popularity of such a control technique, it is, however, well known that the chattering problem is worthy of more attention for the sake of practical deployment. Taking into consideration the aforementioned reason, a guide to sliding mode control for practical implementation has been proposed by Young et al. [19]. Levant [20] proposed a super-twisting algorithm, which is continuously completely robust to disturbances, using second order sliding mode control. In this work, the terminal phase operation, at the end of the midcourse [21, 22], determines the initial condition of the present stage. The terminal guidance system, developed in this approach, with the zero-sliding guidance law integrated with the quaternion-based attitude controller, is presented. Extensive simulations confirm the overall controller design.

II. PRELIMINARIES

2.1 Equations of motion for missiles

Given the location coordinate of the missile’s center of gravity, \( r_M = [ r_{Mx} \ r_{My} \ r_{Mz} ]^T \), the translational motion can be described by Newton’s law as follows:

\[
\begin{align*}
\dot{r}_{Mx} &= v_{Mx}, & \dot{v}_{Mx} &= \frac{1}{m} F_{Mx} \\
\dot{r}_{My} &= v_{My}, & \dot{v}_{My} &= \frac{1}{m} F_{My} \\
\dot{r}_{Mz} &= v_{Mz}, & \dot{v}_{Mz} &= \frac{1}{m} F_{Mz},
\end{align*}
\]

(1)

where \( v_M = [v_{Mx} \ v_{My} \ v_{Mz}]^T \) and \( F_M = [F_{Mx} \ F_{My} \ F_{Mz}]^T \) denote the velocity of and the exerting force on the body of the missile, respectively, and \( m \) is the mass of the missile. Clearly, (1) is formulated in an inertial coordinate frame. The variation of \( J \) is normally caused by vibration of the body, the inner gas, or the fuel flow. In this work, the missile body is assumed to be rigid, and the mass loss and the vibration of the center of gravity caused by the consumption of fuel are assumed to be negligible. \( J = 0 \), so that Euler’s equation of rotation is adopted with the following general form, describing the attitude and the revolving rate of a rigid body:

\[
J \ddot{\omega} = - (\omega \times) J \omega + T_b.
\]

(2)

where \( \omega \) is the angular velocity of the rigid body with inertia matrix \( J \), both in the body coordinate frame; \( T_b \) is the torque on the missile also in the body coordinate frame, and \( (\times) \) is an operator defined for any vector \( v_M \in \mathbb{R}^3 \), as

\[
(v_M \times) = \begin{bmatrix} 0 & -v_{Mz} & v_{My} \\ v_{Mz} & 0 & -v_{Mx} \\ -v_{My} & v_{Mx} & 0 \end{bmatrix}.
\]

For a symmetrical rigid body, the inertia matrix \( J \) will be of the simple diagonal form or will have negligible off-diagonal elements:

\[
J = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix}.
\]

(3)

A vector in an inertial coordinate frame to a body coordinate frame can be transformed by multiplying the original vector by the rotation matrix of the body coordinate system, \( B_b \) as indicated below:

\[
V_i = B_b V_b,
\]

(4)

where the subscript \( i \) denotes a vector in the inertial coordinate frame, and the subscript \( b \) refers to a vector in body coordinates.

2.2 Quaternion

According to Euler’s rotation theory [8], for two arbitrary coordinate frames with coincident origins, there exists a unit vector \( n \) and an angle \( \phi \) in one of the coordinate frames, such that if the other coordinate frame rotates through an angle \( \phi \) with respect to \( n \), then the two frames will coincide. The quaternion can be defined as four parameters \( q = [q_1 \ q_2 \ q_3 \ q_4]^T = [\bar{q}^T \ q_4]^T \), obtained from \( n \) and \( \phi \):

\[
\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} n_1 \sin(\phi/2) \\ n_2 \sin(\phi/2) \\ n_3 \sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix}
\]

(5)
Fig. 1. Pulse width pulse frequency modulator.

Thrust \(b_{zb} + b_{ix}b_{yb} + b_{iz}\)

Missile

Thrust \(b_{xb} - b_{xb} + b_{yb} + b_{by} + \) Center of Gravity

Fig. 2. Missile divert control system.

Therefore, the time derivative of a quaternion can be found to be a function of the corresponding angular velocity and the quaternion itself [6, 10]:

\[
\dot{\bar{q}} = \frac{1}{2} (\bar{q} \times \omega) + \frac{1}{2} q_{4}\omega \\
\dot{q}_{4} = -\frac{1}{2} \omega^{T} \bar{q}
\] (6)

Notably, (6) is a globally valid expression. Therefore, the quaternion is more suitable when the controlled object may exhibit large-angle maneuverability.

2.3 Pulse-type input

Traditionally, the missile control system reaches the desired acceleration by applying aerodynamic forces. For instance, a skid-to-turn (STT) missile [24, 25] controls the yaw and pitch angles to enlarge the air surface, thereby generating the necessary lift and turn forces, whereas a bank-to-turn (BTT) missile [26, 27] initially controls its roll angle such that the largest air surface is facing the commanded direction of acceleration. A survey [28] has demonstrated that the combination of the traditional aerodynamic controller and the pulse-type input devices might improve the controllability and maneuverability of aircraft. Via an integrator-type feedback loop, as shown in Fig. 1, the PWPF-type actuators try to output a signal that has almost the same frequency as the original input signal.

2.4 Problem description

Soon after the intercepting missile has been launched, the midcourse guidance [6] phase with thrust vector control (TVC) [6, 29] is initiated. When the missile approaches its pre-defined interception point at an altitude of around 150 km, the air density is low and the aerodynamic forces are difficult to exploit. Specifically, the divert control system (DCS) with thrust, as shown in Fig. 2, is assumed to be located near the missile’s center of gravity and aligned with the two axes, \(b_{bx}\) and \(b_{bzc}\), perpendicular to the longitudinal axis, \(b_{by}\), of the kill vehicle, to generate motion. In contrast, the attitude control system (ACS) with thrusts, shown in Fig. 3, is located and aligned such that only three pure rotational moments about the principal axes are generated.

III. ZERO-SLIDING GUIDANCE LAW

A guidance system is generally designed to generate suitable motion commands for a missile to adjust its velocity and trajectory to fulfill the tactical goal. Therefore, the guidance law to be investigated in this work is proposed.

In Fig. 4, \(B_{i} = [b_{ix} b_{iy} b_{iz}]\) is the inertial coordinate frame whose origin coincides with the missile’s center of gravity, and \(B_{b} = [b_{bx} b_{by} b_{bzc}]\) is the missile’s body coordinate frame. Let \(v\) be the relative velocity; \(x\) and \(\theta\) are the yaw and the pitch angles of the LOS in the inertial coordinate frame whose origin coincides with the intercepting missile. For convenience, a new LOS coordinate frame, \(B_{L}\), is defined as follows: \(b_{Ly}\) is the unit vector along the LOS, \(b_{Lx}\) is the unit vector perpendicular to \(b_{Ly}\) and parallel to the \(x-y\) plane in the inertial coordinate frame, and \(b_{Lz} = b_{Lx} \times b_{Ly}\) is the third unit vector. Notably, the rotation matrix that corresponds to the coordinate frame \(B_{L}\) can be described in
terms of \( \alpha \) and \( \theta \) as:

\[
B_L = \begin{bmatrix} b_{Lx} & b_{Ly} & b_{Lz} \end{bmatrix} =
\begin{bmatrix}
\sin \alpha & \cos \alpha \cos \theta & -\cos \alpha \sin \theta \\
-\cos \alpha & \sin \alpha \cos \theta & -\sin \alpha \sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\] (7)

The following lemmas are introduced to enable the guidance law to be designed.

**Lemma 1 (Constant Bearing Condition).** If the relative direction between the missile and the target can be fixed, then the missile will hit the target as long as the approaching velocity is positive and bounded away from zero.

The proof of this lemma is trivial and is omitted here.

**Lemma 2 (Zero-Sliding Guidance Law).** Define the relative velocity, \( v = v_T - v_M \), where \( v_T \) and \( v_M \) are the velocities of the target and the missile, respectively, and define \( r = r_T - r_M \) as the displacement vector from the missile to the target. If \( v \) has no component normal to the direction \( r \), and \( v^T r < 0 \), with \( v \) bounded away from zero, then the intercepting missile will eventually hit the target.

**Proof.** Define \( \hat{r} \) as the unit vector in the direction \( r \), then \( \hat{r} \) can be expressed as:

\[
\hat{r} = B_L \hat{r}_L = B_L \begin{bmatrix} 0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
\cos \alpha \cos \theta \\
\sin \alpha \cos \theta \\
\sin \theta
\end{bmatrix},
\]

where \( \hat{r}_L \) is the unit vector of \( r \) expressed in the LOS coordinate frame. Let \( v^T = [v_{Lx} \ v_{Ly} \ v_{Lz}]^T = B_L^T v \) be the expression for \( v \) with respect to the LOS coordinates. Since \( v_{Ly} \) is defined to be along the LOS, such that \( v_{Ly} \) is parallel to \( \hat{r} \), both \( v_{Lx} \) and \( v_{Lz} \) are the normal components of \( v \) with respect to \( \hat{r} \). The relationship between the variation of \( \alpha, \theta \) and \( v_{Lx}, v_{Lz} \) can be described, with reference to Fig. 4, by the following equations:

\[
\dot{\alpha} = -\frac{1}{|r| \cos \theta} v_{Lx}
\]

\[
\dot{\theta} = \frac{1}{|r|} v_{Lz}.
\]

When \( v_{Lx} \) and \( v_{Lz} \) are constrained to be zero, such that \( \dot{v}_{Ly} = v_{Ly} = 0 \), then, apparently \( \dot{\alpha} = \dot{\theta} = 0 \), thus, \( \hat{r} \) is constant. Moreover, since \( v^T r < 0 \) is assumed, not only does the direction of \( r \) remain fixed, but also the constant bearing condition in Lemma 1 is met. In summary, if the normal component of \( v \) with respect to \( r \) is zero, then, according to Lemma 1, the missile will eventually hit the target.

Define \( v_p \) as the normal components of \( v \) with respect to \( r \): \( v_p = v_{Lx} b_{Lx} + v_{Lz} b_{Lz} \), which will hereafter be called the sliding velocity for convenience. From Lemma 2, the necessary condition for successful interception is \( v_p = 0 \). Therefore, \( v_p \) can be selected as the control output of the guidance system. From vector algebra, \( v_p \) equals \( v \) minus its projection onto \( r \), with reference to Fig. 5.

\[
v_p = v - \frac{r^T v}{|r|^2} r,
\] (8)

and the time derivative of \( v_p \) is

\[
\dot{v}_p = \dot{v} - \left( \frac{1}{|r|^2} v - \frac{2 v^T r}{|r|^4} r \right) (r^T v) - \frac{r}{|r|^2} (v^T v + r^T u)
\]

\[
= u - \frac{\phi}{|r|^2} v + 2 \frac{\phi}{|r|^4} r - \frac{|v|^2}{|r|^2} - \frac{r^T u}{|r|^2} - \frac{r^T r}{|r|^2}
\]

\[
= A(r, v) + B(r) u,
\] (9)

where \( u = \dot{v} \) is the acceleration, which is treated as an input at this stage. \( \phi = r^T v = v^T r \) is the inner product of \( v \) and \( r \), and \( A(r, v) \) and \( B(r) \) are, respectively, defined as:

\[
A(r, v) = -\frac{\phi}{|r|^2} v + \frac{1}{|r|^4} (2 \phi^2 - |r|^2 |v|^2) r
\]

\[
B(r) = I_3 \times 3 - \frac{1}{|r|^2} [r_x r_y r_z],
\] (10)

where \( I_3 \times 3 \) is the identity matrix and \( r = [r_x \ r_y \ r_z]^T \).

The magnitude of the resulting acceleration generated by the divert control thrusters (DCT) is \( W_r \).
From (9), the pulsed control input can then be set to:

\[ u = -W_t \text{sgn}^v(v_p), \]  

(11)

where \( W_t \) is the constant magnitude of the acceleration generated by the thrusters, which will be specified later, and \( \text{sgn}^v(\cdot) \) is an operator defined as

\[ \text{sgn}^v(v_p) = \begin{cases} \frac{v_p}{|v_p|}, & \text{if } |v_p| \neq 0 \\ 0, & \text{if } v_p = 0 \end{cases} \]

for any \( v_p \in \mathbb{R}^3 \).

Given this controller design, the closed-loop dynamics of \( v_p \) from (9) are determined to be:

\[ \dot{v}_p = A(r, v) + B(r)u \]

\[ = -\frac{\varphi}{|r|^2} v + \frac{1}{|r|^4} (2\varphi^2 - |r|^2|v|^2)r + B(r)u \]

\[ = -\frac{\varphi}{|r|^2} v_p + \frac{1}{|r|^4} (2\varphi^2 - |r|^2|v|^2)r - B(r)W_t \text{sgn}^v(v_p) \]

\[ = -W_t \text{sgn}^v(v_p) - \frac{\varphi}{|r|^2} v_p \]

\[ + \frac{1}{|r|^4} (\varphi^2 - |r|^2|v|^2)r, \]  

(12)

where the fact that \( B(r)\text{sgn}^v(v_p) = \text{sgn}^v(v_p) \), since \( v_p \to 0 \), is exploited.

Define a Lyapunov function candidate as \( V_\delta(v_p) = \frac{1}{2} v_p^T v_p \); then, the time derivative of \( V_\delta \) under control law (11) is evaluated as:

\[ \dot{V}_\delta(v_p) = v_p^T \left( -W_t \text{sgn}^v(v_p) - \frac{\varphi}{|r|^2} v_p \right) \]

\[ + \frac{1}{|r|^4} (\varphi^2 - |r|^2|v|^2)r \]

\[ = -v_p^T W_t \text{sgn}^v(v_p) - v_p^T \frac{\varphi}{|r|^2} v_p \]

\[ \leq -\gamma_1 |v_p| < 0 \]

If \( W_t + \frac{\varphi}{|r|^2} |v_p| \geq \gamma_1 > 0 \) and \( |v_p| \neq 0 \),

(13)

**Remark 1.** From (11) and (14), the zero-sliding guidance law satisfies the reaching condition of the sliding-mode system; that is, the sliding surface is entered in finite time only if \( \gamma_1 \) is positive, at which time \( W_t \) is a positive and constant scalar that exceeds \( -\frac{\varphi}{|r|^2} |v_p| \) by a positive number throughout the course of the missile. Hence, the component \( v_p \) is driven to zero in finite time, and the distance vector \( r \) continues to decay but remains bounded away from zero because of the ultimate initiation of the zero-effort-miss (ZEM) phase that will be analyzed in Remark 2 (below). Here, the relative velocity \( v \) is assumed bounded away from zero. Therefore, the magnitude of the term \( \frac{\varphi}{|r|^2} |v_p| \) is finite, since the initial \( v_p \) is finite, so the finite scalar number \( W_t \) can be selected first as a finite scalar that is large enough to satisfy the requirement of pulse-type inputs.

To admit the constraint of the direction of the input \( u \) in (11) by the attitude of the missile and its consequent control by the autopilot system, the control input is expressed as:

\[ u = B_b u_b, \]

where \( B_b \) is the coordinate transformation from the body coordinate frame to the inertial coordinate frame, whereas \( u_b \) is the thrusting acceleration in the body coordinate frame. Then, the control law, (11), is modified as:

\[ u_b = -(W_t \text{sgn}^v(B_b^T v_p)). \]

(14)

Up to this point, the remaining problem is to adjust the attitude of the missile such that the DCT can be directed to provide the required acceleration, given by (11).

If the attitude of the missile is close to the desired attitude mentioned above when it enters the terminal phase, then the commanded acceleration described by (11) can be assumed to lie totally on the active plane of the missile and the acceleration can be divided into two components, of which one is in the \( b_{bx} \) direction whereas the other is in the \( b_{bz} \) direction. Therefore, the following algorithm is adopted for successful guidance to ensure the final interception, by driving \( v_p \) to zero.

1. Calculate the sliding velocity, \( v_p \).
2. Calculate the components of \( v_p \) on the \( b_{bx} \) and \( b_{bz} \) axes:

\[ v_p^{b_{bx}} = b_{bx} v_p \]

\[ v_p^{b_{bz}} = b_{bz} v_p. \]
3. If the zero-effort-miss (ZEM) exceeds some pre-defined threshold, which is typically set to the desired hit distance, then the applied thrust forces are:

\[ u_{bx} = -W_{tx} \text{sgn}(v_{px}^b) \]
\[ u_{bz} = -W_{tz} \text{sgn}(v_{pz}^b), \]

where \( W_{tx} \) and \( W_{tz} \) are the constant thrust magnitudes along \( b_{x} \) and \( b_{z} \), respectively, and \( \text{sgn}(\cdot) \) is a sign function. The resulting acceleration command can be expressed as follows.

\[ u = u_{bx} b_x + u_{bz} b_z \]
\[ = -W_{tx} \text{sgn}(b_x^T v_p) b_x \]
\[ -W_{tz} \text{sgn}(b_z^T v_p) b_z \]

(15)

The analysis of the stability of such a design is derived as in (13) and will be omitted here.

**Remark 2 (Zero-Effort-Miss (ZEM) Analysis).** The guidance law designed in this section tries to minimize the relative velocity component \( v_p \) normal to the relative displacement vector \( r \), or in the LOS direction, with reference to Fig. 5. The only singularity is at \( t=0 \), as evident from (8). Since our guidance law drives the output variable \( v \) to zero only when condition (13) holds, the fact that \( r \) approaches zero when a direct hit is to take place may eventually lead to a violation of (13), causing in turn, the failure of \( v \) to converge.

The missile and the target are both assumed to move with constant velocities for definition of ZEM:

\[ r_T = r_{T0} + v_T(t-t_0), \]
\[ r_M = r_{M0} + v_M(t-t_0), \]

where \( r_T \) and \( r_M \) are the position vector of the target and the missile, respectively; \( r_{T0} \) and \( r_{M0} \) are the position vectors at the starting time \( t_0 \); and \( v_T \) and \( v_M \) are the velocity vectors of the target and of the missile, respectively. Therefore, the relative position vector can be expressed as:

\[ r = (r_T - r_M) + (v_T - v_M)(t-t_0) = r_0 + v(t-t_0), \]

where \( r_0 = r_{T0} - r_{M0} \) and \( v = v_T - v_M \). By definition, ZEM is the minimum norm of \( r \) for \( t \geq t_0 \), and can be computed as:

\[ \text{ZEM} = \min_{t \geq t_0} |r| \]
\[ = \min_{t \geq t_0} \sqrt{r^T r} \]

\[ = \min_{t \geq t_0} \sqrt{{\begin{bmatrix} r^T & v^T \end{bmatrix} r(t-t_0) + v(t-t_0)^2} \]

\[ = \min_{t \geq t_0} \sqrt{|v|^2 (t-t_0)^2 + |r_0|^2 - \frac{|r_0|^2}{|v|^2}} \]

\[ = \sqrt{|r_0|^2 - \frac{|r_0|^2}{|v|^2}}. \]

From the above, the computation of ZEM is always valid, except when both the missile and the target have an identical velocity, resulting in parallel trajectories of respective objects. Such an exceptional case, however, will not be considered herein, since the condition defined in Lemma 2 that \( v \) is bounded away from zero is no longer valid.

**IV. AUTOPILOT SYSTEM DESIGN**

To facilitate the convergence analysis of the autopilot system, Assumption 1 may reasonably be given as follows.

**Assumption 1.** Throughout the course, the constant \( W_a \) satisfies:

\[ W_a > \frac{1}{2} p(\langle \bar{q}_e \times \rangle \omega_e + q_{e4} \omega_e - \dot{\omega}_d - J^{-1} \langle \omega \times \rangle J \omega) \]
\[ + \zeta, \quad \zeta > 0 \]

where \( \| \cdot \|_1 \) denotes the 1-norm,

\[ v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \in R^3, \quad \|v\|_1 = |v_x| + |v_y| + |v_z|. \]

Define the attitude error quaternion with respect to an inertial coordinate frame as:

\[ q_e = \left[ \begin{array}{c} q_{e1} \\ q_{e2} \\ q_{e3} \\ q_{e4} \end{array} \right]^T = [q_{e1} \ q_{e2} \ q_{e3} \ q_{e4}]^T. \]

Then, the dynamic equation that governs \( q_e \) can be derived from (6) as:

\[ \dot{q}_e = \frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \]
\[ \dot{q}_{e4} = -\frac{1}{2} \omega_e^T q_e, \]

(16)

where \( \omega_e = \omega - \omega_d \) is the error between the angular velocities at the present attitude and the desired attitude.
Define a sliding surface $S_a = P \bar{q}_e + \omega_e$ where $P = \text{diag}[p_1 \ p_2 \ p_3]$ is a positive definite matrix, and choose a Lyapunov function $V_a(S_a)$ as:

$$V_a(S_a) = \frac{1}{2} S_a^T S_a.$$  

Then, the time derivative of $V_a(S_a)$ can be derived from (2) and (16):

$$\dot{V}_a(S_a) = S_a^T \dot{S}_a = S_a^T (P \dot{\bar{q}}_e + \dot{\omega}_e + \frac{1}{2} q_e \omega_e)$$  

$$-K_a (P \bar{q}_e + \omega_e) + J^{-1} (T_b - \langle \omega \times J \omega \rangle - \ddot{\omega}_e)$$  

where $T_b$ is the control torque. Let $T_b$, according to the feedback linearization technique, be:

$$T_b = \langle \omega \times J \omega \rangle + J \dot{\omega}_e$$  

where $K_a = \text{diag}[k_{a1} \ k_{a2} \ k_{a3}]$ is also a positive definite diagonal matrix, so that $\dot{S}_a$ becomes

$$\dot{S}_a = P \left( \frac{1}{2} \langle \bar{q}_e \times \rangle \omega_e + \frac{1}{2} q_e \omega_e \right) + \dot{\omega}_e$$  

$$= -K_a (P \bar{q}_e + \omega_e)$$  

$$= -K_a S_a$$  

which, together with (17), implies

$$\dot{V}_a(S_a) = -S_a^T K_a S_a \leq 0$$  

According to Lyapunov stability theory, $S_a$ converges to zero exponentially fast, and the original system is driven to a sliding surface, $S_a = 0$, exponentially in time $t$. Herein, the controller is initially designed using the feedback linearization method, with reference to (18), to yield the torque input. Later, the sliding mode technique is incorporated, as in (26), into the design of the robust attitude controller within the autopilot system. The consequence of $q_e$ for a fixed sliding mode is studied below.

If the sliding mode is fixed, such that $S_a \equiv 0$, then the system dynamics are constrained by the following differential equations:

$$\dot{\bar{q}}_e = -\frac{1}{2} \langle \bar{q}_e \times \rangle P \bar{q}_e - \frac{1}{2} q_e P \bar{q}_e$$  

$$\dot{q}_e = \frac{1}{2} \bar{q}_e^T P \bar{q}_e.$$  

Define another Lyapunov function $V_e(q_e) = \bar{q}_e^T \bar{q}_e$. Then,

$$\dot{V}_e(q_e) = 2 \bar{q}_e^T \ddot{q}_e = -\bar{q}_e^T (\langle \bar{q}_e \times \rangle P \bar{q}_e + q_e P \bar{q}_e)$$  

$$= -q_e \bar{q}_e^T P \bar{q}_e,$$  

where $\bar{q}_e^T (\bar{q}_e \times) = 0$ is dropped in (22).

Given the quaternion definition, $q_e$ has two possible values that differ only in sign, which can be arbitrarily chosen to meet design convenience. Initially, $q_e$ is selected as $C = q_e(t = 0) > 0$, and since $q_e = \frac{1}{2} \bar{q}_e^T P \bar{q}_e \geq 0$, $q_e$ is a positive, growing variable.

Again, by Lyapunov stability theory, $\bar{q}_e$ is driven to zero when the system is constrained in sliding surface; the angular velocity is also driven to zero, whereby the system origin $(\bar{q}_e, \omega_e) = (0, 0)$ can be shown to be exponentially stable.

Now, with reference again to (18), given that $S_a$ converges to zero only exponentially, the actual differential equation that governs $q_e$ differs from (21) by some exponentially decaying term:

$$S_a = P \bar{q}_e + \omega_e = \epsilon(t),$$  

where $\epsilon(t) = e^{-K_a^T S_a}$ and $S_a = (P \bar{q}_e + \omega_e)|_{t=0}$ is an exponentially decaying function of time $t$.

From (16) and (23), the dynamics of $\bar{q}_e$ are:

$$\dot{\bar{q}}_e = \frac{1}{2} (\bar{q}_e \times) P \bar{q}_e + \frac{1}{2} q_e P \bar{q}_e$$  

$$-\frac{1}{2} (\bar{q}_e \times) (-P \bar{q}_e + \epsilon) + q_e (-P \bar{q}_e + \epsilon)$$  

$$= -\frac{1}{2} (\bar{q}_e \times) P \bar{q}_e + q_e P \bar{q}_e$$  

$$+ \frac{1}{2} (\bar{q}_e \times) + q_e I_3 \times 3 \epsilon$$  

$$= f_1(\bar{q}_e) + f_2(\bar{q}_e),$$  

where $f_1(\bar{q}_e) = -\frac{1}{2} (\bar{q}_e \times) P \bar{q}_e + q_e P \bar{q}_e$ and $f_2(\bar{q}_e) \equiv \frac{1}{2} (\bar{q}_e \times) + q_e I_3 \times 3 \epsilon$. As indicated by (21), $\dot{\bar{q}}_e = f_1(\bar{q}_e)$ is an exponentially stable system, implying that the stability and convergence of (24) are governed by $f_2(\bar{q}_e)$. If $f_2(\bar{q}_e) \to 0$ exponentially as $t \to 0$, then the
system is reduced to system (21), which is an exponentially stable system. From (23), \( \epsilon \) is an exponentially decaying function, and because \( \| \frac{1}{2}(\dot{\alpha}e \times) + q_{e4}I_{3x3} \| \) is bounded, \( \| \frac{1}{2}(\dot{\alpha}e \times) + q_{e4}I_{3x3} \| \leq \sigma \) for some constant \( \sigma > 0 \). \( \| f_3(\dot{q}_e) \| \leq \sigma e \) becomes also an exponentially decaying function, which now truly ensures the stability and convergence of (24).

The Lyapunov analysis of (17) is performed again:

\[
\dot{V}_a(S_a) = S^T_a \dot{S}_a = S^T_a \left[ P \left( \frac{1}{2}(\dot{q}_e \times)\omega_e + \frac{1}{2}q_{e4}\omega_e \right) + J^{-1}(T_b - \langle \omega_e \times J \omega \rangle - \dot{\omega}_d) \right] = S^T_a (D + J^{-1}T_b),
\]

where \( D \) is defined as,

\[
D = P \left( \frac{1}{2}(\dot{q}_e \times)\omega_e + \frac{1}{2}q_{e4}\omega_e \right) - J^{-1}(\omega_e \times J \omega - \dot{\omega}_d).
\]

Pulse-type torque input \( T_b \) generated by the attitude control thrusters (ACT) is chosen to promote system stability:

\[
T_b = -W_a J \text{sgn}(\epsilon)(S_a),
\]

where \( W_a > 0 \) is a constant design parameter, and \( \text{sgn}(\cdot) \) is an operator defined as

\[
\text{sgn}(S_a) = \begin{bmatrix} \text{sgn}(S_{a1}) \\ \text{sgn}(S_{a2}) \\ \text{sgn}(S_{a3}) \end{bmatrix}.
\]

for all \( S_a = [S_{a1} \ S_{a2} \ S_{a3}]^T \in R^3 \). The above torque command can be shown to be effective if the above Assumption 1 holds.

Given such a torque design, if Assumption 1 holds, then (25) becomes:

\[
\dot{V}_a(S_a) = S^T_a (D + J^{-1}T_b)
\]

\[
= S^T_a D - W_a S^T_a \text{sgn}(S_a)
\]

\[
\leq \| S_a \|_1 \| D \|_1 - W_a \| S_a \|_1
\]

\[
= \| S_a \|_1 (\| D \|_1 - W_a)
\]

\[
\leq -\| S_a \|_1.
\]

(27)

where \( \| D \|_1 - W_a \leq -\dot{\epsilon} \), as in Assumption 1. This result implies that \( S_a \) is bounded and converges to zero in finite time, which, in fact is consistent with the discussions of (16) to (27). The attitude control system thus determined to be an exponentially stable system.

**Remark 3.** Notably, the finite initial value of the angular velocity, \( \omega \), and the bounded time derivative of the desired angular velocity, \( \dot{\omega}_d \), throughout the guidance phase, are such that the former convergence analysis of the autopilot system can be carried out because Assumption 1 can indeed be assured to hold by choosing a sufficiently large constant \( W_a \) before the ZEM phase is initiated.

## V. SYSTEM INTEGRATION

The former subsystem, derived in Section III, receives the kinematic relationship between the missile and the target and, via the controller, according to (11) to (15), determines the acceleration command to be sent to the autopilot subsystem. The latter subsystem then converts the acceleration command to an attitude command and, via the controller described by (25) to (27), generates the torque command to the ACT, to alter the attitude of the missile so that the forces generated by the DCT can respond to the guidance command. Figure 6 presents the overall system, for which \( r \) is the relative position vector, \( v \) is the relative velocity vector, subscript \( M \) denotes the missile, subscript \( T \) denotes the target, \( u_d \) is the force commanded by the guidance law, and \( u, T_b \) are the divert force and the attitude control torque generated by the respective thrusters.

With reference to (5), the unit vector of rotation \( n = [n_1 \ n_2 \ n_3]^T \) of \( u_d \) can be easily seen to be the normal unit vector to the plane, which contains the y-axis of the inertial coordinate and LOS:

\[
n = \frac{[0 \ 1 \ 0]^T \times \hat{r}}{\| [0 \ 1 \ 0]^T \times \hat{r} \|}.
\]

As a continuation of this argument, the angle of rotation \( \phi \) is simply the angle between the y-axis of the inertial coordinate and the LOS direction:

\[
\phi = \cos^{-1}([0 \ 1 \ 0]^T \hat{r}).
\]

Fig. 6. Block diagram of terminal guidance.
 Accordingly, the desired angular velocity and its time derivative can be expressed as:

$$\omega_d = 2E^T(q_d) \dot{q}_d$$
$$\dot{\omega}_d = 2E^T(q_d) \ddot{q}_d,$$

as was proposed elsewhere [30, 31], where

$$E(q_d) = \begin{bmatrix} \langle \dot{q}_d \times \rangle + q_{dd}I_{3 \times 3} \\ -\dot{q}_d^T \end{bmatrix} \in R^{4 \times 3}.$$

If the desired quaternion $q_d, \dot{q}_d$, and $\ddot{q}_d$ are given with the unit-norm property of $q_d$, then the main goal of the attitude control is to let the quaternion $q$ approach $q_d$ and the angular velocity $\omega$ approach $\omega_d$ simultaneously. In this work, $q_{dd}$ must be of a constant sign, say positive, such that $q_{dd} > 0$, throughout the interception, so the Euler rotation angle is between $\pm \cos^{-1} q_{dd}$, to avoid the singularity $\langle \dot{q}_d \times \rangle + q_{dd}I_{3 \times 3}$.

To guarantee the convergence of $v_p$, the following assumption is introduced to specify clearly the requirement of the divert input device.

**Assumption 2.** The magnitudes of the thrusters, $W_{tx}$ and $W_{tz}$, are large enough so that throughout guidance:

$$W_{tx} > -\frac{\varphi}{|r|^2}|c_1|, W_{tz} > -\frac{\varphi}{|r|^2}|c_3|.$$

Now, we are ready to state the following theorem which will provide conditions under which the proposed overall guidance/autopilot system with 5 degree of freedom control inputs generated by the DCT and by the ACT altogether guarantees the stability of the overall system.

**Theorem 1.** If Assumptions 1 and 2 hold, then the terminal guidance of the missile with pulse-type inputs is asymptotically stable if the controller is designed according to

$$u = -W_{tx}\text{sgn}(b_{bx}^Tv_p)b_{bx} - W_{tz}\text{sgn}(b_{bz}^Tv_p)b_{bz}$$
$$T_p = -W_aJ\text{sgn}^e(S_u)$$

and the desired attitude of the missile is in the LOS direction.

**Proof.** Let the Lyapunov function candidate be $V_g(v_p) = \frac{1}{2}v_p^T v_p$. Then, the time derivative of $V_g$ with the time derivative of $v_p$ in (9) is:

$$\dot{V}_g(v_p) = v_p^T [A(r, v) + B(r)u]$$
$$= v_p^T A(r, v) + v_p^T B(r)u,$$

From (10) and (15), the vector $B(r)u$ can be derived as follows:

$$B(r)u = \left( I_{3 \times 3} - \frac{1}{|r|^2}(r_T r \ y y \ 0) \right) u$$
$$= u - \frac{1}{|r|^2}(r_T u)r$$
$$= -W_{tx}b_{bx}\text{sgn}(b_{bx}^Tv_p) - W_{tx}b_{bz}\text{sgn}(b_{bz}^Tv_p)$$
$$- \frac{r}{|r|^2}((r_T b_{bx})W_{tx}\text{sgn}(b_{bx}^Tv_p)$$
$$+ (r_T b_{bz})W_{tz}\text{sgn}(b_{bz}^Tv_p)).$$

Since, by definition $r_T v_p = 0$, $v_p^T A(r, v)$ and $v_p^T B(r)u$ can be determined from (30) and (31) to be

$$v_p^T A(r, v) = v_p^T \left( -\frac{\varphi}{|r|^2}|v_p| + \frac{1}{|r|^4}(\varphi^2 - |r|^2|v|^2)r \right)$$
$$= -\frac{\varphi}{|r|^2}|v_p|^2 v_p = -\frac{\varphi}{|r|^2}|v_p|^2 v_p^T B(r)u$$
$$= -W_{tx}v_p^T \text{sgn}(b_{bx}^Tv_p)b_{bx}$$
$$- W_{tz}v_p^T \text{sgn}(b_{bz}^Tv_p)b_{bz}$$
$$= -W_{tx}|b_{bx}^Tv_p| - W_{tz}|b_{bz}^Tv_p|,$$

the time derivative of $V_g(v_p)$ in (30) is derived as:

$$\dot{V}_g(v_p) = v_p^T A(r, v) + v_p^T B(r)u$$
$$= -\frac{\varphi}{|r|^2}|v_p|^2 - W_{tx}|b_{bx}^Tv_p| - W_{tz}|b_{bz}^Tv_p|. \quad (32)$$

Notably, Lemma 1 states that $\varphi = v_T r \leq 0$ is a basic requirement of successful interception. Hence, for $v_p$, the first term in (32) is unstable because it represents a positive feedback mechanism. Thus, the relationship between the body coordinate $B_b$ and the sliding velocity $v_p$ must be investigated.

The sliding velocity $v_p$ can be expressed in body coordinates $B_b$ as:

$$v_p = c_1 b_{bx} + c_2 b_{by} + c_3 b_{bz},$$

where $c_1 = b_{bx}^Tv_p$, $c_2 = b_{by}^Tv_p$, and $c_3 = b_{bz}^Tv_p$. Then, (32) becomes:

$$\dot{V}_g(c_1, c_2, c_3) = -\frac{\varphi}{|r|^2}(c_1^2 + c_2^2 + c_3^2) - W_{tx}|c_1|$$
$$- W_{tz}|c_3|$$
$$= -\left( \frac{\varphi}{|r|^2}c_1^2 + W_{tx}|c_1| \right) - \frac{\varphi}{|r|^2}c_2^2$$
$$- \left( \frac{\varphi}{|r|^2}c_3^2 + W_{tz}|c_3| \right). \quad (33)$$
Nevertheless, even if Assumption 2 is satisfied, the stability and convergence of \( v_p \) are not guaranteed, mainly because the missile is assumed to have no accelerating power in the \( b_{by} \) direction. If \( c_2 = b_{by}^T v_p \) is bounded away from zero, then the Lyapunov function candidate \( V_g(v_p) \) will not converge. The physical meaning is that, as long as the DCT are not aligned normal to \( r \), the missile will undergo acceleration in the direction of LOS, such that the convergence of \( v_p \) can be guaranteed only when the \( W_{tx} \) and \( W_{tz} \) are large enough to eliminate the term \( -\frac{\varphi}{|r|^2} c_2^2 \) in (33) throughout the course. Certainly, if the decay rate of \( |r| \) exceeds that of \( v_p \), then \( v_p \) will tend to infinity as \( |r| \) tends to zero.

Since the desired missile attitude \( b_{dy} \) is defined to be parallel to the LOS, where the \( b_{dy} \) axis is the unit vector in direction \( r \), and by definition,

\[
c_2 = b_{by}^T v_p,
\]

\[
c_2^2 = |b_{by}^T v_p|^2 = |(b_{by} - b_{dy})^T v_p + b_{dy}^T v_p|^2 = |(b_{by} - b_{dy})^T v_p|^2 \leq |b_{by} - b_{dy}|^2 |v_p|^2,
\]

when the attitude of the missile \( b_{by} \) is driven to the desired attitude \( b_{dy} \), the sliding velocity \( v_p \) will have no component along the \( b_{by} \) axis, resulting in \( b_{by}^T v_p = 0 \). Consequently, if the attitude controller of the missile can drive the attitude to converge to the desired attitude before \( r \) becomes too small, the system will be stabilized.

The Lyapunov function candidate \( V(S_a, S_g) \) of the overall system is defined as follows, to verify the stability of the overall system:

\[
V(S_a, S_g) = \frac{1}{2} S_a^T S_a + \frac{1}{2} S_g^T S_g,
\]

(34)

where \( S_g = v_p \) and \( S_a = P \dot{q}_a + \omega_a \). If Assumptions 1 and 2 are satisfied under the controller design given by (15) and (26), then the time derivative of the Lyapunov function \( V(S_a, S_g) \), with reference to (27) and (33), can be derived as:

\[
\dot{V} = S_g^T \dot{S}_g + S_a^T \dot{S}_a
\]

\[
= S_a^T D - W_a S_a^T \text{sgn}^r(S_a) - \left( \frac{\varphi}{|r|^2} c_1^2 + W_{tx} |c_1| \right)
\]

\[
- \frac{\varphi}{|r|^2} c_2^2 \left( \frac{\varphi}{|r|^2} c_3^2 + W_{tz} |c_3| \right)
\]

\[
\leq -\lambda_1 |c_1| - \lambda_2 |c_3| - \lambda_3 \|S_a\|_1 - \frac{\varphi}{|r|^2} c_2^2,
\]

(35)

where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are positive variables defined as

\[
\lambda_1 = W_{tx} + \frac{\varphi}{|r|^2} |c_1|,
\]

\[
\lambda_2 = W_{tz} + \frac{\varphi}{|r|^2} |c_3|,
\]

\[
\lambda_3 = W_a - \|D\|_1
\]

(36)

and the magnitude of the term \( \frac{\varphi}{|r|^2} c_2^2 \) decays to zero since the attitude of the missile is driven to the desired attitude before the ZEM phase is initiated. By Lyapunov stability theory, the system is asymptotically stable, and the interception of the target can succeed.

VI. SIMULATIONS

Performing various computer simulations validates the integrated terminal guidance and autopilot system that was proposed in Sec. V. In this section, the two scenarios defined below are considered. The initial conditions of the terminal phase for simulations are that the attitude of the missile body has an attitudinal error of around 10° and that the distance between the missile and the target is around 10 km. The missile has a sampling period of 1 ms, which is also the shortest switching period for both of the divert control thrust and the attitude control thrust. In the simulations, the switching distance ZEM from the terminal phase to the ZEM phase is set to ZEM = 700 m for both simulation scenarios, such that successful interception can be achieved with the minimal power consumption of the thrusting propulsion system. The two major factors that ensure successful interception are the acceleration generated by the thrusters and the servo rate of the missile controller. If the thrusting acceleration of

\[Fig. 7. The 3D engagement of Scenario 1.\]
the missile is too small, the missile cannot drive the sliding velocity \( v_p \) to zero before the distance between the missile and the target becomes too small. Hence, the sliding velocity continues to grow such that the constant bearing condition will not hold. A sufficiently high servo rate is also required for successful terminal guidance since the attitude command will change very fast during the terminal phase.

The simulations of the two scenarios indicate that the intercepting missile can intercept the target successfully. Figures 7 and 9 graphically simulate 3D engagements. These two scenarios represent two target trajectories: Scenario 1 shows that the target is flying down at an angle of around 70° at the final intercepting time, whereas in Scenario 2, the target travels along a much lower trajectory. Hence, the target flies nearly horizontal
Fig. 10. Various variables including position, relative velocity, and divert forces of Scenario 2.

throughout the flight over a long horizontal distance at the beginning of the terminal phase. The final velocity of the intercepting missile is around one third of the final velocity of the target at the time of interception, and is satisfactory in both scenarios.

With respect to the other controlled output effects and divert forces, and with reference to Figs 8 and 10, the hit-to-kill interception can be visualized from the intersections of the trajectories along the three axes of the inertial coordinate frame, and the velocity component perpendicular to LOS in LOS coordinates finally approach zero, verifying that the desired goal was reached. The output divert forces of DCS with respect to the inertial coordinate frame are also demonstrated. Eventually, the two interception periods of the terminal phase in the scenarios are 3.15 s and 4.09 s. Moreover, the switching rate of DCS in Scenario 2 exceeds that in Scenario 1, because the missile in the former case flows more horizontally, therefore, suffers more from the vertical gravitational field.

**Scenario 1**

<table>
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<th>Target:</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
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<td>1830.13</td>
<td>9659.26</td>
</tr>
<tr>
<td>Position (m)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Initial Velocity</td>
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<td>-507.582</td>
<td>-2490.83</td>
</tr>
<tr>
<td>(m/sec)</td>
<td></td>
<td></td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Missile:</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Position (m)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial Velocity</td>
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<td>108.447</td>
<td>730.778</td>
</tr>
<tr>
<td>(m/sec)</td>
<td></td>
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VII. CONCLUSIONS

In this work, the interception of a theater ballistic missile by a missile equipped with pulse-type actuators is investigated. The properties of pulse type actuators are taken into account and sliding mode control techniques adopted to design a stable controller following nonlinear Lyapunov stability analysis. The integrated controller, which performs guidance and attitude control, is developed to satisfy the requirements of the terminal phase. In particular, a new guidance law for the terminal phase is proposed. It is a zero-sliding guidance law, whose purpose is to eliminate the sliding velocity between the missile and the target in the normal direction of LOS. The designed quaternion-based attitude controller, however, guarantees exponential tracking ability. The validity of the proposed integrated system is thoroughly examined and established.

Various computer simulations were run to verify the feasibility of the presented integrated controller, which basically comprises a quaternion-based attitude control subsystem and a zero-sliding guidance subsystem. The results are satisfactory and the proposed design is very promising.

REFERENCES


