STABILIZATION OF A CLASS OF NONLINEAR NON-MINIMUM PHASE SYSTEMS

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Abstract

This paper tackles the problem of stabilization of a class of non-minimum phase nonlinear systems which have zero dynamics with an eigenvalue zero of multiplicity 2. By adding some new terms, called cross terms, we are able to generalize the concept of the Lyapunov function with a homogeneous derivative along the trajectory, which was introduced in [4], to produce a suitable Lyapunov function. The Lyapunov function assures that the stability of an approximate system, which consists of some lower order terms of a nonlinear system with an eigenvalue zero of multiplicity 2, implies the stability of the whole system. Applying these to non-minimum phase zero dynamics of nonlinear systems with such a center, a sufficient condition and a design method of state feedback control are obtained for stabilizing the systems.

Key Words: Zero dynamics, non-minimum phase, center manifold, stabilization, Lyapunov function with homogeneous derivative.

I. INTRODUCTION

Use of the center manifold method to stabilize control systems was proposed in [7] and elsewhere. Then it was developed into a systematic design method, called minimum phase zero dynamics [1-2]. In [4], a systematic design method was proposed for non-minimum phase zero dynamics of a single zero eigenvalue at the origin by combining the center manifold theory with the Lyapunov function method. When the Jacobian of the dynamics on the center manifold has an eigenvalue zero of multiplicity k at the origin, we simply call it the k-zero case, and we call it the multi-zero case when the multiplicity k is not specified. Stabilization of a 2-zero center has already been discussed for some special classes in [7-8] and elsewhere. In this paper, we generalize the method developed in [4] to provide a design method for the canonical form of affine nonlinear control systems.

Even though some concepts employed here are inherited from [4], the design technique is completely new, and the discussion in this paper does not depend very much on [4]. The background of homogeneous stability can be found in [3]. The basic theory of the center manifold used in this paper is mostly taken from [5].

The stabilization of a nonlinear system with zero dynamics of multi-zero is theoretically a natural development based on extensive discussion of the single-fold zero case.

In practice, it is also important. Assume a nonlinear system has a linearized canonical form as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \\
\dot{x}_n &= f(x) + g(x)u
\end{align*}
\]

(1)

but with \(f(0) = g(0) = 0\); then the problem of stabilization of multi-zero systems emerges. A similar case results with parabola center zero dynamics.

The stabilization of a system with zero dynamics of 2-zero has been discussed in [8]. The systems discussed in [8] have a linearized part of dimension 1. The tool used there is the theory of the normal form of ordinary differential equations. In this paper, general affine control systems in the canonical form [2] are considered. The tool used is a combination of the center manifold theory with the Lyapunov function method. The main result obtained here is a generalization of that obtained in [8].

Most of the existing stabilization techniques are designed to stabilize nonlinear systems with minimum phase zero dynamics [2,9,10]. For instance, in [2], a
systematic design process was proposed to stabilize canonical nonlinear systems. But the minimum phase condition is assumed. The systems considered in [10] are linearizable or cascaded. As for general nonlinear control systems with non-minimum phase zero dynamics, to the best of the author’s knowledge, there is no systematic method for stabilizing nonlinear systems with non-minimum phase zero dynamics.

The paper is organized as follows: Section 2 describes the basic concepts and results obtained using the Lyapunov function with the homogeneous derivative along trajectory (LFHD). Section 3 provides a method used to generalize the LFHD by adding some cross terms. The main results of stabilizing of systems with 2-zero center zero dynamics are presented in section 4. Section 5 describes via an example the design technique used to stabilize controls. Section 6 is a concluding summary.

II. PRELIMINARY

Consider an affine nonlinear system with the following form:

\[
\begin{align*}
\dot{x} &= A(\xi) + B(\xi)u, \quad x \in \mathbb{R}^n \\
w &= Sw + p(x, w, z), \quad w \in \mathbb{R}^m, \quad \mathcal{R}(\sigma(S)) < 0 \\
\dot{z} &= Cz + q(x, w, z), \quad z \in \mathbb{R}^l, \quad \mathcal{R}(\sigma(C)) = 0 \\
y &= x_1,
\end{align*}
\]

where $\xi = \text{col}(x, w, z)$ and

\[
\begin{align*}
A(\xi) &= \text{col}(x_2, \ldots, x_n, F(\xi)); \quad F(0) = 0 \\
B(\xi) &= \cos(0, \ldots, 0, G(\xi)); \quad G(0) \neq 0.
\end{align*}
\]

$\mathcal{R}(\sigma(S)) < 0 (= 0)$ means that all the eigenvalues of $S$ have negative (zero) real parts. Moreover, the functions $p, q$ vanish at $(0, 0, 0)$ together with their first-order partial derivatives with respect to $w, z$. The zero dynamics of system (2.1) are [1]

\[
\begin{align*}
\dot{w} &= Sw + p(0, w, z) \\
\dot{z} &= Cz + q(0, w, z).
\end{align*}
\]

If (2.2) is asymptotically stable, then the system (2.1) is said to have minimum phase zero dynamics. Then a control,

\[
u = -\frac{F(\xi)}{G(\xi)} + \frac{1}{G(\xi)} \sum_{j=1}^{n} a_j x_j
\]

can be used to stabilize the overall system as long as $\lambda^n - \sum_{j=1}^{n} a_j \lambda^{j-1}$ is Hurwitz [1-2]. For the non-minimum phase case, a method called the Lyapunov function with homogeneous derivative (along the trajectory), or, LFHD, was proposed in [4]. We cite a few related results here: Given a dynamic system

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n.
\]

Applying Taylor expansion to each component of $f(x)$ and letting $d_i, i = 1, \ldots, n$, be the lowest degree of the non-vanishing terms of $f$, then the system

\[
\dot{x}_i = g_i(x) = \sum_{|s| = d_i} \frac{1}{S!} \frac{\partial |f(x)|}{\partial x_s} x^s, \quad i = 1, \ldots, n
\]

is called the approximate system of (2.3), where $S = (S_1, \ldots, S_n) \in \mathbb{Z}_+^n$ is a multi-index and

\[
[S] = \sum_{i=1}^{n} S_i, \quad S! = \prod_{i=1}^{n} S_i!, \quad x^S = \prod_{i=1}^{n} (x_i)^{S_i}.
\]

Express the vector $g$ as $g = \text{col}(g_1, \ldots, g_r)$, with $\text{dim}(g_i) = n_r$, and the degree of $g_i$ equals $k_i, j = 1, \ldots, n_r, i = 1, \ldots, r$.

Assume that $k_i, i = 1, \ldots, r$, are all odd numbers. Let $m$ be a given integer satisfying $2m \geq \max\{k_1, \ldots, k_r\} + 1$, and set $2m_1 = 2m - k_i + 1$. Then we define the LFHD as

\[
V = \sum_{i=1}^{r} ([x_i]^{m_i}, \ldots, [x_r]^{m_r}) P_i ([x_1]^{m_1}, \ldots, [x_r]^{m_r})^T,
\]

where $P_i, i = 1, \ldots, r$, are $n_i \times n_i$ positive definite matrices. It is easy to see that the derivative of $V$ along the trajectory of (2.4), $V|_{t=0}$, is a homogeneous polynomial with degree $2m$. Then, we have the following theorem.

**Theorem 2.1.** [4] System (2.1) is asymptotically stable at the origin if there exists an LFHD such that the derivative of the LFHD along its approximate system (2.4) is negative definite.

For the 1-zero case, the theorem provides a convenient way to stabilize certain non-minimum phase systems via a designed center manifold. The idea for the design is: If the original lowest degree of the non-vanishing terms is odd, then choose a suitable state feedback such that $V|_{t=0}$ becomes negative definite. Otherwise, use the feedback to delete the lowest degree terms and make $V|_{t=0}$ negative for the remaining (odd degree) approximate system. This won’t work for the 2-zero case because, as shown in the next section, $V|_{t=0}$ can never be negative definite. Further discussion on this issue depends on irreducible variety factorization [6]. Therefore we will try to generalize the LFHD in such a way that Theorem 2.1
remains true for a generalized LFHD.

**III. GENERALIZED LFHD**

For convenience, we will now consider the following system:

\[
\begin{aligned}
  x_1 &= x_2 \\
  &\vdots \\
  x_{n-1} &= x_n \\
  x_n &= F(\xi) + G(\xi) u,
\end{aligned}
\]

(3.1)

\[
\begin{pmatrix}
  \dot{z}_1 \\
  \dot{z}_2
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix} +
\begin{pmatrix}
  q_1(\xi) \\
  q_2(\xi)
\end{pmatrix},
\]

where \(\xi = (x, z)\) and \(q_1(\xi)\) and \(q_2(\xi)\) vanish at zero with their first order partial derivatives. Following the standard procedure described in [4], we assume that the center manifold is

\[
x_i = h_i(z) = \psi(z) + 0(\|z\|^3), \quad i = 1, \ldots, n.
\]

(3.2)

That is, \(x_i = h_i(z), i = 1, \ldots, n\), determine the center manifold, and they are approximated by \(x_i = \psi(z), i = 1, \ldots, n\), up to \(0(\|z\|^3)\). Moreover, if we use state feedback control,

\[
u = -\frac{F(\xi)}{G(\xi)} + \frac{1}{G(\xi)} \left( \sum_{j=1}^{n} a_j x_j - a_i(\psi_i) \right),
\]

(3.3)

then the dynamics of the center manifold of the closed loop system will be

\[
\begin{aligned}
  \dot{z}_1 &= z_2 + q_1(\psi(z), z) + 0(\|z\|^3) \\
  \dot{z}_2 &= q_2(\psi(z), z) + 0(\|z\|^3),
\end{aligned}
\]

(3.4)

Where \(\psi = (\psi_1, \ldots, \psi_n)^T\).

We may assume that the degree of the lowest degree terms of \(q_1(\psi(z), z)\) is 3, which is the only possible odd number \(k\) for \(1 < k < 4\). Then an LFHD may be constructed as

\[
V = az_1^3 + bz_2^3.
\]

(3.5)

Now however, the derivative of \(V\) along the trajectories of the approximate system of (3.4) can never be negative definite. Therefore, we have to change (3.5).

Of course, the choice of LFHD is not unique. By changing the approximation accuracy (approximation degree), we can alter (3.5). However, no matter how we change it, \(x_3\) is always a factor of \(V\), so it can never be negative definite.

We will try to modify (3.5) by adding some new terms to preserve the positivity of \(V\) and provide negativity of \(V\) via suitable feedback.

**Definition 3.1.** Given an LFHD as (2.5) and a monomial \(h(x)\)

\[
h(x) = \prod_{i=1}^{n} \prod_{j=1}^{p_i} (x_j)^{\alpha_i}.
\]

Let \(I\) be such an index that \(m_i = \max\{m_i : \exists x_i \neq 0\}\), where \(m_i\) is the degree of \(x_i\) in LFHD (2.5). \(h(x)\) is called a cross term with respect to (2.5) if

\[
2m_j = \sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} > \sum_{i=1}^{k} t_{ij}.
\]

(3.6)

**Example 3.2.** Assume that \(x = (x_1^4, x_2^3, x_3^2) \in R^3\). Consider an LFHD as follows:

\[
V = ((x_1^4)^2, ((x_2^3)^2)P((x_1^4)^3, ((x_2^3)^3)^2 + q(x_3)^6),
\]

where \(P > 0\) is a \(2 \times 2\) positive definite matrix and \(0 < q \in R\).

(a) Let \(h = (x_1^4)^2(x_3^2)\). Then \(m_1 = 2, m_2 = 3\). Since \(t_{12} = 3 \neq 0\) and \(t_{23} = 1 \neq 0\), \(m_i = \max\{m_1, m_2\} = 3\) and \(I = 2\). Now,

\[
\begin{aligned}
  2m_1 &= 2m_2 = 6 \\
  \sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} &= 3 + 1 = 4. \\
  \sum_{j=1}^{p_i} t_{ij} &= 1
\end{aligned}
\]

Since the equality part of (3.6) is not satisfied, \(h\) is not a cross term of \(V\).

(b) Let \(h = (x_2^3)^2(x_3^2)^4\). A similar argument provides

\[
\begin{aligned}
  2m_1 &= 2m_2 = 6 \\
  \sum_{i=1}^{k} \sum_{j=1}^{p_i} t_{ij} &= 2 + 4 = 6. \\
  \sum_{j=1}^{p_i} t_{ij} &= 4
\end{aligned}
\]

Hence, this \(h\) is a cross term of \(V\).

**Proposition 3.3.** Let \(K\) be a finite set and

\[
H(x) = \sum_{k \in K} c_k h_k(x),
\]

where \(h_k(x), k \in K\), are cross terms, \(c_k \in R\). \(V(x)\) is as in (2.5). Then, \(V(x) + H(x)\) is locally positive definite around zero.
**Proof.** We will first prove this for monomial and monic \( H(x) = h(x) \). That is, we will show that \( V(x) + h(x) \) is locally positive definite. According to the inequality part of (3.6), there exists \( t_0 \neq 0 \) for some \( f_0 \) and \( m_0 < m_2 \).

Using the inequality [4]

\[
\prod_{i=1}^{n} (x_i)^{t_i} \leq \frac{1}{t} \left( \sum_{i=1}^{n} t_i \right) \left( t \right)
\]

where \( t = \sum_{i=1}^{n} t_i \),

we can easily get

\[
\prod_{i=1}^{n} (x_i)^{t_i} \leq \sum_{ij} \frac{\varepsilon^{2m_i t_j} (x_i)^{2m_i t_j}}{m_i} + \frac{\delta^{2m_i t_j} (x_i)^{2m_i t_j}}{m_i}
\]

where \( \varepsilon^{2m_i t_j} \), \( \delta^{2m_i t_j} = 1 \). We choose \( \varepsilon > 0 \) small enough such that

\[
\frac{\varepsilon^{2m_i t_j}}{m_i} < \min P^{t_j}_{2m_i}
\]

where \( P^{t_j}_{2m_i} \) is the diagonal element of the positive definite matrix \( P_i \) in \( V(x) \).

Moreover, since \( m_0 < m_2 \), we choose \( \left| x \right| \) small enough such that, when \( x_0 \neq 0 \),

\[
\frac{\delta^{2m_i t_j} (x_i)^{2m_i t_j}}{m_i} < P^{t_j}_{2m_i} (x_i)^{2m_i t_j}
\]

The conclusion follows immediately.

When \( K \) is a finite set, a similar argument remains true while the parameters \( \varepsilon \) and \( \delta \) can be modified by the number of terms and corresponding coefficients.

We will call \( V(x) + h(x) \) a generalized LFHD because it is obtained from an LFHD by adding some cross terms. In the next section, we will show how it can be used to construct a feedback control to stabilize (3.1).

Proposition 3.3 itself is interesting because it provides a way to produce various locally positive definite functions with a number of freely chosen coefficients. In this way, a set of adjustable Lyapunov candidates can be produced. It can also be applied to some other stabilization problems.

**Example 3.4.** Recall (3.5). Then, \((z_1)^3(z_2)\) is a cross term because \( l = 1, m_1 = 2, \) and (3.6) becomes

\[
2m_i = 4 = \sum_{i=1}^{n} t_i + 3 = 1 \sum_{j=1}^{n} t_j = 3.
\]

Similarly, \((z_2)^3(z_1)\) is also a cross term. According to Proposition 3.3,

\[
V(z) = (z_1)^i + (z_2)^j + c(z_1)^i(z_2) + d(z_1)(z_2)^j, \quad c, d \in R
\]

is a locally positive definite polynomial.

**IV. STABILIZATION OF A 2-ZERO CENTER MANIFOLD**

Consider system (3.1) again. Before we can state the main result, some notations are required.

Let \( \alpha, \beta \in R \), \( P_i(z) \) be the set of the \( k \)-th degree homogeneous polynomials. We define a set of quadratic forms and use them to approximate the functions which determine the center manifold.

Define \( \psi_i(z) \in P_i(z), t = 1, \ldots, n, \) as

\[
\begin{align*}
\psi_i(z) &= \alpha z_i^2 + \beta z_i z_2 \\
\psi_j(z) &= 2 \alpha z_i z_j + \beta z_j^2 \\
\psi_i(z) &= 2 \alpha z_i^2 \quad \psi_i(z) = 0, \quad i \geq 4
\end{align*}
\]

Denote

\[
\begin{align*}
a_{jk} &= \frac{1}{j!} \frac{\partial^j q_{ij}(0)}{\partial z_i^j}, \quad j + k = 3 \\
b'_{jk} &= \frac{\partial^j q_{ij}(0)}{\partial z_i^j}, \quad j + k = 3
\end{align*}
\]

Set \( q_j \in P_i(z), j = 1, 2, \) as

\[
q_j = \sum_{i=0}^{\infty} a_{i+3} z_i^3 z_j + (b_{i+j} z_1 + b_{i+j} z_2) \psi_1 + (b_{i+j} z_1 + b_{i+j} z_2) \psi_2 + (b_{i+j} z_1 + b_{i+j} z_2) \psi_3.
\]

In fact, \( q_j \) is the third degree terms of \( q_i(\psi(z), z), \) \( j = 1, 2. \) Note that when \( n < 3 \) for ease of notation, we render \( \psi_i = 0 \) for \( i > n \).

Our main result is the following theorem, which provides a sufficient condition for (3.1) to be state feedback stabilizable. Meanwhile, it provides a precise procedure for constructing a state feedback stabilizing control.

**Theorem 4.1.** Consider system (3.1). Assume that

\[
\begin{align*}
A_1. \frac{\partial [r^j + |S|]}{\partial \psi_i(0)} &= 0, \quad i = 1, 2, \quad 2 |T| + |S| \leq 2; \\
A_2. \frac{\partial^2 q_{ij}}{\partial x_i \partial z_j} &\neq 0.
\end{align*}
\]

Then system (3.1) is state feedback stabilizable. Moreover, a stabilizing control is
where \( \lambda^n - \sum_{j=1}^n a_j \lambda^{n-j} \) is Hurwitz. \( \alpha \) and \( \beta \) are two proper real numbers (which are determined later by (4.10) and (4.12), respectively).

**Proof.** We intend to use \( u = \psi(z) = (\psi(z), \ldots, \psi(z)) \) to approximate the center manifold. Using the above notations and assumption A1, we may express

\[
q_j(\psi(z), z) = \bar{q}_j + \sum_{|\Gamma| = 2} \frac{\partial^2 q_j}{\partial z^2} (0) z^2 + \sum_{|\Gamma| = 3} \frac{\partial^3 q_j}{\partial z^3} (0) z^3 + 0(\|z\|^4), \quad j = 1, 2.
\]

Therefore, the approximated dynamics on the center manifold can be expressed as

\[
\begin{aligned}
z_1 &= \dot{z}_1 + \frac{\partial^2 q_j}{\partial z^2} (0) z^2 + \sum_{|\Gamma| = 3} \frac{\partial^3 q_j}{\partial z^3} (0) z^3 + 0(\|z\|^4) \\
\dot{z}_2 &= \dot{z}_2 + \frac{\partial^2 q_j}{\partial z^2} (0) z^2 + \sum_{|\Gamma| = 3} \frac{\partial^3 q_j}{\partial z^3} (0) z^3 + 0(\|z\|^4).
\end{aligned}
\]

(4.6)

First, we claim that the approximation of the center manifold \( x = h(z) \) by \( x = \psi(z) \) has precision \( 0(\|z\|^4) \). According to the approximation theorem of the center manifold, it is sufficient to show that

\[
\frac{\partial \psi}{\partial z} = \begin{pmatrix}
\psi_1(z) \\
\vdots \\
\psi_n(z)
\end{pmatrix}
\begin{pmatrix}
z_1 \\
\vdots \\
z_n
\end{pmatrix}
= 0(\|z\|^4).
\]

(4.7)

We will prove (4.7) for \( n \geq 3 \). When \( n = 2 \) or \( n = 1 \), using the corresponding controls, the proof is similar. When \( n \geq 3 \), using (4.1) and (4.3), the left hand side of (4.7) is

\[
\left( \begin{array}{c}
2\alpha_1 + \beta_2 \\
2\alpha_2 \\
\vdots \\
4\alpha_2
\end{array} \right) \left( \begin{array}{c}
z_1 \\
0 \\
\psi(z) + g(\psi(z), z)u(z)
\end{array} \right) = 0(\|z\|^4).
\]

which is, obviously, of \( 0(\|z\|^4) \). The claim follows.

Next, we claim that the dynamics on the center manifold constitute a particular case of (4.6). Assume that the center manifold is determined by \( x = h(z) \). From the previous claim, we know that \( \|h(z) - \psi(z)\| = 0(\|z\|^4) \).

Then, we can replace \( \psi(z) \) in (4.6) with \( h(x) + 0(\|z\|^4) \). Straightforward computation shows that (4.6) remains unchanged, which proves the claim.

The last claim is fundamental because now, if we can stabilize (4.6), then we have stabilized the dynamics on the center manifold of the closed-loop system. Then according to the equivalence theorem of the center manifold, the overall closed-loop system is stabilized.

The last thing we have to do is to use the method of the generalized LFHD to determine the parameters \( \alpha, \beta \) in the control (4.4) and show that under the designed state feedback control (4.4) system, (4.6) is asymptotically stable.

Recall Example 3.4; the LFHD \( V(z) \) in (3.5) can be modified by setting \( a > 0 \), \( b > 0 \) and adding two cross terms as follows:

\[
V = a\zeta_1^2 + b\zeta_2^2 + c\zeta_1\zeta_2 + d\zeta_1^2\zeta_2.
\]

(4.8)

Now consider the derivative of \( V(z) \) along (4.6); we have

\[
V_{|\zeta_1\neq 0} = 4\alpha_1\zeta_1^2 + 4\beta_1\zeta_1\zeta_2 + 3c\zeta_1^2\zeta_2 + 4\beta_1\zeta_1^2 + d\zeta_1^2 + 2b\zeta_1\zeta_2 + a\zeta_1\zeta_2 + a\zeta_1^2\zeta_2 + a\zeta_1\zeta_2 + a\zeta_1^2\zeta_2
\]

\[
+ ab\zeta_1\zeta_2 + 2ab\zeta_1\zeta_2 + a\zeta_1\zeta_2 + a\zeta_1^2\zeta_2 + ab\zeta_1\zeta_2 + ab\zeta_1^2\zeta_2
\]

\[
+ 2ab\zeta_1\zeta_2 + 2ab\zeta_1\zeta_2 + 2ab\zeta_1\zeta_2 + 2ab\zeta_1\zeta_2 + ab\zeta_1\zeta_2 + ab\zeta_1\zeta_2
\]

\[
+ 2ab\zeta_1\zeta_2 + 2ab\zeta_1\zeta_2 + R_1 + R_2,
\]

where

\[
R_1 = 0(\|z\|^4) \quad l + r = 6 \quad \lambda \geq 0 \quad r > 0 \quad R_2 = 0(\|z\|^4).
\]

Note that assumption A2 means that \( h_{11} \neq 0 \). After collecting the terms of (4.9), we can choose the parameters
in the state feedback control (4.4) and adjust the parameters in (4.8) simultaneously as follows:

First, \( \alpha \) can be chosen as

\[
\alpha = -\frac{2a + ba_z^2}{bb_z^2}, \tag{4.10}
\]

in which the term \( z_1^2z_2 \) is deleted.

Then, we choose \( c \) such that the coefficient of \( z_1^6 \), denoted by \( \mu \), becomes negative:

\[
\mu = c(a_z^6 + b_z^6 \alpha) + 4a(a_z^6 + b_z^6 \alpha) < 0 . \tag{4.11}
\]

Note that we may have to adjust \( a \) or \( b \) in (4.10) such that the coefficient of \( c \) in (4.11) isn’t zero. Since \( b_z^2 \neq 0 \), this can always be done.

Thirdly, we choose \( \beta \) such that the coefficient of the term \( z_1^3z_2^2 \), denoted by \( \lambda \), is negative:

\[
\lambda = 2b(a_z^3 + \beta b_z^3 + ab_z^2 + 2ab_z^3 + 3c < 0 .
\]

We can first choose any negative number \( \lambda < 0 \). Then, we solve \( \beta \) by means of

\[
\beta = \frac{\lambda - 2b(a_z^3 + \beta b_z^3 + 2ab_z^3) - 3c}{b_z^3} . \tag{4.12}
\]

Fourthly, consider the term \( z_1z_2^3 \). Its coefficient is

\[
w = 2b(a_z^3 + \beta b_z^3 + \beta b_z^3 + 2ab_z^3 + 2ab_z^3) .
\]

Note that

\[
|z_1z_2^3| \leq \frac{1}{2}(\varepsilon z_1^3 + \varepsilon^2 z_2^4) .
\]

We choose \( \varepsilon \) small enough such that

\[
\eta = \lambda + \frac{\varepsilon}{2}w < 0 . \tag{4.13}
\]

Finally, we choose \( d \) such that

\[
v = d + \left| 2ba_z^3 \right| + \left| 2b \beta b_z^3 \right| + \left| 4b \beta b_z^3 \right| + \left| 2 \frac{w}{2 \varepsilon} \right| < 0 . \tag{4.14}
\]

Summarizing (4.10) - (4.14), we can see that

\[
V \leq \mu z_1^6 + \eta z_1^3z_2^2 + \varepsilon z_1^3 + R_1 + R_2 . \tag{4.15}
\]

To see that (4.15) is locally negative definite, we have only to consider \( R_1 \). Because \( R_1 = 0(\varepsilon \varepsilon^3) \), a case by case discussion over different powers can easily show that when \( R_1 = 0 \), the right hand side of (4.15) is locally negative definite.

Recall the form of \( R_1 \). It is obvious that if \( r > 1 \), then such terms will be the higher order terms of either \( z_1^2 \) or \( z_1^3z_2^2 \). Therefore, we have only to handle the term \( z_1^3z_2 \). Using an old trick, when \( \varepsilon \) is small enough and \( \varepsilon \) is small enough, we have

\[
\left| z_1^3z_2 \right| \leq \frac{1}{2}(\varepsilon z_1^3 + \frac{1}{4}z_2^4) \times \frac{\mu}{2}z_1^6 + \frac{\eta}{2}z_1^3z_2^2 .
\]

Therefore, when \( \varepsilon \) is small enough, the derivative of \( \varepsilon \) the trajectories of the system depicted in equation (4.6) is negative definite. This completes the proof. ■

**Remark.** As pointed out in [8], for the dynamics of the center manifold to be stable, the second degree terms should be zero. Hence, assumption A1 is necessary in the normal form. Without loss of generality, we can assume that the dynamics on the center manifold are already in their normal form. Otherwise, we can convert them first.

**V. AN EXAMPLE**

In this section, an example will be presented to illustrate the control design. Consider

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -5x_1 + u \\
z_1 &= z_2 + z_1^3 - 2z_2x_2 + z_1z_2^2 \\
z_2 &= z_1^3 + z_1x_1 \\
y &= x_1 .
\end{align*}
\]

The zero dynamics are obtained by setting \( x_1 = x_2 = 0 \) as follows:

\[
\begin{align*}
\dot{z}_1 &= z_2 - z_1^3 + z_1z_2^2 \\
z_2 &= z_1^3 . \tag{5.2}
\end{align*}
\]

Using the Lyapunov function \( V = z_1^3 + z_2^2 \), we can see easily that (5.2) is unstable because

\[
\frac{d}{dt} V_{z_1z_2} = 2(z_2z_2 - z_1^3 + z_1z_2^3 + z_1^2z_2) .
\]

Now, along the line \( z_1 = z_2 \), we have

\[
\dot{V}_{z_1z_2} = 2z_1^3 + z_1^4 .
\]

Hence, in any neighborhood of origin, there exists at least one point where \( V_{z_1z_2} > 0 \). Therefore, (5.1) has non-minimum phase zero dynamics [11].

It is easy to check that both A1 and A2 in Theorem 4.1 hold; hence the system is stabilizable.

Next, we will follow the procedure proposed in the proof of Theorem 4.1 to construct the stabilizing control.
Meanwhile, we will also verify our result by constructing the generalized LFHD as also proposed in the previous section.

First, we need to figure out the parameters in (4.2) as follows:

\[ a_{10} = -1, \quad a_{11} = 0, \quad a_{12} = 1, \quad a_{13} = 0; \]
\[ b_{11} = 0, \quad b_{12} = 0, \quad b_{13} = 0, \quad b_{14} = 1; \]
\[ a_{20} = 1, \quad a_{21} = 0, \quad a_{22} = 0, \quad a_{23} = 0; \]
\[ b_{21} = 1, \quad b_{22} = 0, \quad b_{23} = 0, \quad b_{24} = 1. \]

Then, as in (4.8), we assume that the generalized LFHD is

\[ V = a z_1^4 + b z_2^4 + c z_1^2 z_2 + d z_1 z_2^3. \]

Since \( a, b \) can be chosen arbitrarily, we set \( a = \frac{1}{2}, \quad b = 1. \)

Using (4.10), \( \alpha = -\frac{1 + 1}{2} = -2. \)

According to (4.11), to make \( \mu < 0 \), we can simply choose \( c = 0. \) Say, we choose \( \lambda = -2 < 0. \) Then, using (4.12), we have \( \beta = \lambda = -2. \)

It is easy to see that \( w = 0. \) (4.13) yields \( \eta = \lambda = -2 < 0. \)

Finally, we can choose \( d = -1. \) Then, from (4.14), we have \( V = d = -1 < 0. \) According to Theorem 4.1, we know that the center manifold is

\[ x_i = h_i(z) = \psi_i(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4, \quad i = 1, 2, \]

where

\[ \begin{cases} \psi_1 = -2 z_1^3 - 2 z_1 z_2 \\ \psi_2 = -4 z_2 z_1 - 2 z_2^3. \end{cases} \]

Then, the dynamics on the center manifold are

\[ \begin{align*} \dot{z}_1 &= z_2 - z_1^3 + z_1 z_2^2 - 2 z_2 \psi_2(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4 \\ \dot{z}_2 &= z_1^3 + z_1 \psi_1(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4. \end{align*} \]

From (4.4), the proposed control is

\[ u = -x_1 - x_2 - (2 z_1^3 + z_1 z_2) - (4 z_1 z_2 + 2 z_2^3) - 2 z_2^3 = -x_1 - x_2 - 2 z_1^3 - 6 z_1 z_2 - 6 z_2^3. \]

The LFHE is

\[ V = \frac{1}{2} z_1^2 + z_2^2 - z_1 z_2^2, \]

which is locally positive definite. Then it follows that there exists a neighborhood of zero: \( 0 \in U \in \mathbb{R}^2 \). Over \( U \), we have

\[ V = 2 z_1^4 (z_2^2 - z_1^2 + z_4^2 - 2 z_2 \psi_2(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4) \]
\[ + 2 z_1 (z_3^3 + z_1 \psi_1(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4) \]
\[ - z_4^3 (z_2^2 - z_1^2 + z_4^2 - 2 z_2 \psi_2(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4) \]
\[ - 3 z_4 z_3^2 (z_1 + z_4 \psi_1(z) + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4) \]
\[ \leq -2 z_1^4 - 4 z_2^3 z_1^2 - z_2^4 + 0\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)^4, \]

\( k + l = 6, \quad l \geq 1, \quad z \in U \in \mathbb{R}^2, \)

which obviously is locally negative definite.

VI. CONCLUSIONS

In this paper, we have considered stabilization of a class of nonlinear non-minimum phase systems. The dynamics on the center manifold of this class of systems have an eigenvalue zero of multiplicity 2 at the origin. First of all, the concept of a cross term was introduced. On this basis, a generalized Lyapunov function with a homogeneous derivative along the trajectory was proposed which is a development of the results presented in [4]. Combining this Lyapunov approach with the center manifold theory, a set of easily verifiable sufficient conditions for stabilizability of this class of systems was obtained. Correspondingly, a new design technique for the state feedback control which stabilizes the systems was also presented.

However, the case where systems have zero dynamics with an eigenvalue zero of multiplicity \( k > 2 \) at the origin is much more complex and difficult. The method proposed in the paper can only be applied to some particular cases. The general case remains to be considered in further studies.

REFERENCES