DISCRETE OUTPUT FUZZY CONTROLLER DESIGN FOR ACHIEVING COMMON CONTROLLABILITY GRAMIAN

Wen-Jer Chang, Chein-Chung Sun and Chyun-Chau Fuh

ABSTRACT

In control theory, the robust properties of linear systems can be related directly to properties of the controllability or observability Gramian. In this paper, a discrete fuzzy controller for a class of nonlinear systems is developed to achieve a common controllability Gramian. We assume that the nonlinear system is represented by the Takagi-Sugeno fuzzy model. The purpose of this paper is to find the output feedback gains for the T-S fuzzy controller after assigning a certain common controllability Gramian. Finally, we provide a numerical example to verify the effects of the proposed method.

KeyWords: Controllability Gramian, discrete Takagi-Sugeno fuzzy model, theory of generalized inverses.

I. INTRODUCTION

In recent years, many research efforts have focused on fuzzy control issues based on the Takagi-Sugeno (T-S) fuzzy model [7], which is described by fuzzy IF-THEN rules. For a nonlinear system that is transformed successfully into a T-S fuzzy model, the stability of the overall nonlinear system still cannot be guaranteed even if each subsystem of the T-S fuzzy model is stable. To overcome this problem, Wang and Tanaka proposed the concept of parallel distributed compensation (PDC) [11], which is employed in the design of state feedback gains for each rule. Based on the PDC concept, some works in the literature [6,8] have studied stability analysis of T-S fuzzy models. This method requires finding a common positive definite matrix \( P \) such that the proposed sufficient stability conditions are satisfied for every IF-THEN rule [12]. LMI is a powerful tool for finding a common positive definite matrix \( P \). Thus, many approaches have been developed to find \( P \) based on the LMI method [10].

We will propose a new approach to designing the output feedback gains of a fuzzy controller for discrete T-S fuzzy models. The proposed method is based on the theory of generalized inverse and the controllability Gramian constraint. When we consider the controllability Gramian assignment problem, the common positive definite matrix \( P \) becomes \( G_c \). \( G_c \) is the so-called common controllability Gramian. In control theory, the controllability Gramian has been extensively applied in analysis and synthesis of linear dynamic systems. Almost all the robustness properties of linear systems can be related directly to the properties of the controllability Gramian [13-14]. For this reason, we will attempt to assign the controllability Gramian for a T-S discrete fuzzy model. To obtain the output feedback gains of a T-S type fuzzy system, we will derive some theorems to find the feedback gains, which guarantee that each rule is asymptotically stable. If the remaining influencing Lyapunov inequality is satisfied by these output feedback gains, then the overall system is global asymptotically stable. For similar purposes, the fuzzy controller design problem with common observability Gramian assignment has been successfully solved by the authors [2-3]. Different from the derivations in [2-3], this paper will give a new methodology to obtain the solutions of output feedback gains for discrete T-S type fuzzy controllers as well as the specified common controllability Gramian for all rules.

II. DESCRIPTIONS OF THE DISCRETE TAKAGI-SUGENO FUZZY MODEL

A nonlinear system can be approximated by a T-S fuzzy model [7]. The T-S fuzzy model consists of a set of IF-THEN rules which represent the local linear input-output relations of a nonlinear system and has the following form:
Rule:\n
IF \( x_1(k) \) is \( M_{11} \) \ldots and \( x_{n_2}(k) \) is \( M_{n_2} \),
THEN \( x(k + 1) = A_x(k) + B_xu(k) \) and \( y(k) = C_x(k) \),

\[(1)\]

where \( x(k) \in \mathbb{R}^{n_x} \) is the state vector; \( u(k) \in \mathbb{R}^{n_u} \) is the control input vector; \( y(k) \in \mathbb{R}^{n_y} \) is the output vector; \( i = 1, 2, \ldots r \); and \( r \) is the number of IF-THEN rules. \( M_i \) represents the fuzzy sets, \( A_i \in \mathbb{R}^{n_x \times n_x} \) and \( B_i \in \mathbb{R}^{n_x \times n_u} \). It is assumed that \((A_i, B_i)\) is a controllable pair, and that \( B_i \) is of full-column rank.

For the overall system, the state equation and output equation can be represented as

\[ x(k + 1) = \frac{\sum_{i=1}^{r} \omega_i(k)(A_i x(k) + B_i u(k))}{\sum_{i=1}^{r} \omega_i(k)}, \quad \text{(2)} \]

\[ y(k) = \frac{\sum_{i=1}^{r} \omega_i(k)C_i x(k)}{\sum_{i=1}^{r} \omega_i(k)}, \quad \text{(3)} \]

where

\[ \omega_i(k) = \prod_{j=1}^{n_2} M_{ij}(x_j(k)); \quad \text{(4)} \]

\( M_{ij}(x_j(k)) \) is the grade of membership of \( x_j(k) \) in \( M_{ij}; \) and \( \omega_i(k) \) is the weight of the \( i \)-th rule.

Using the concept of Parallel Distributed Compensation (PDC) \([11]\), we design the output feedback gains so as to compensate each rule in the T-S fuzzy model. We consider a discrete fuzzy controller, which is represented as a T-S type fuzzy controller as follows:

Rule:\n
IF \( x_1(k) \) is \( M_{11} \) \ldots and \( x_{n_2}(k) \) is \( M_{n_2} \),
THEN \( u(k) = F_i y(k) \).

\[(5)\]

From (5), it can be found that the feedback gains \( F_i \) can be switched by the membership functions of system states via the IF-THEN fuzzy rule. By substituting (3) into (5), the overall output feedback fuzzy controller \( u(k) \) can be rewritten as

\[ u(k) = \frac{\sum_{i=1}^{r} \sum_{j=1}^{r} \omega_i(k) \omega_j(k) [F_i C_j x(k)]}{W}, \quad \text{(6)} \]

where

\[ W = \sum_{i=1}^{r} \sum_{j=1}^{r} \omega_i(k) \omega_j(k). \quad \text{(7)} \]

Substituting (6) into (2), we obtain

\[ x(k + 1) = \frac{1}{W} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} \omega_i(k) \omega_j(k) [A_i + B_i F_i C_j] x(k) \right\}, \quad \text{(8)} \]

or

\[ x(k + 1) = \frac{1}{W} \left\{ \sum_{i=1}^{r} \omega_i(k) \omega_i(k) [A_i + B_i F_i C_i] x(k) + 2 \sum_{i=1}^{r} \omega_i(k) \omega_i(k) H_i x(k) \right\}, \quad \text{(9)} \]

where

\[ H_i = \frac{(A_i + B_i F_i C_i) + (A_i + B_i F_i C_i)^T}{2}, \quad i \neq j. \quad \text{(10)} \]

**Lemma 1.** \([9]\) The discrete fuzzy system (9) is asymptotically stable in the large if there exists a common positive definite matrix \( P \) such that

\[(A_i + B_i F_i C_i)^T P (A_i + B_i F_i C_i) - P < 0, \quad i = 1, 2, \ldots, r. \quad \text{(11)} \]

\[ H_i^T P H_i - P < 0, \quad i \neq j, \quad \text{(12)} \]

To satisfy the stability conditions in Lemma 1, it is necessary to determine \( F_i \) \((i = 1, 2, \ldots, r)\) with a common positive definite matrix \( P \) for (11) and (12). Note that (11) and (12) are not linear matrix inequalities for \( F_i \) and \( P \). It is difficult to use the LMI approach to directly solve for \( F_i \) and \( P \) from (11) and (12). However, if we reconstruct (11) and (12) in the linear matrix form using the method developed in \([4]\), then the solutions of (11) and (12) can be obtained using the LMI method. In this paper, we will attempt to use the theory of generalized inverse to develop a methodology to solve this problem. In order to consider the system performance, we will utilize the concept of controllability Gramian assignment.

For each rule, we first define the controllability Gramian as follows:

\[ G_{ii} = \sum_{i=1}^{r} (A_i + B_i F_i C_i)^T B_i B_i^T (A_i + B_i F_i C_i)^T \quad \text{(13)} \]

Here, \( G_{ii} \) satisfies the following Lyapunov equation for each rule:
Based on Lemma 1, if there exists a common positive definite matrix $P$ satisfying (11) and (12), then the T-S fuzzy system is asymptotically stable. Applying the concept of controllability Gramian assignment to achieve the stability conditions of Lemma 1, we have to assign the controllability Gramian of each rule so as to make it equal to a common positive definite matrix. Hence, we define $G_i = G$, $i = 1, 2, \ldots, r$, where $G$ is called the common controllability Gramian for all rules. In other words, we define a common controllability Gramian $G_c$ to replace $P$ matrix in Lemma 1. In the next lemma, we will modify Lemma 1 so as to provide the new stability conditions subject to the common controllability Gramian $G_c$.

**Lemma 2.** Given a positive definite common controllability Gramian $G_c > 0$ for the discrete T-S fuzzy system (9). If the common controllability Gramian $G_c$ satisfies the following two conditions, then the system (9) is asymptotically stable in the large:

\[
(A_i + B_i F_i C_i) G_c (A_i + B_i F_i C_i)^T - G_c + B_i B_i^T = 0,
\]

\[
i = 1, 2, \ldots, r,
\]

\[
H_i G_i H_i^T - G_c < 0, \quad i < j \leq r.
\]

From (11), it can be found that the closed-loop state matrix $(A_i + B_i F_i C_i)$ will be stable if there exists a positive definite matrix $P$ satisfying (11) for each rule. That is, satisfaction of (11) will guarantee the stability of each linear subsystem for each rule. Applying the linear control theory to consider the Lyapunov equation (15), we can find that if $G_c > 0$, $(A_i, B_i)$ is controllable and $B_i$ is full-column rank, then the closed-loop state matrix $(A_i + B_i F_i C_i)$ is stable (Corollary 5.5 of [5]). Hence, we can use the Lyapunov equation (15) to replace the stability condition (11) in order to guarantee the stability of each rule. Moreover, (12) and (16) have the same structures, besides replacing replacement of the common matrix $P$ by the common controllability Gramian $G_c$. Therefore, the stability conditions of Lemma 1 can be replaced with those of Lemma 2 subject to the common controllability Gramian $G_c$.

In order to achieve the stability conditions of Lemma 2, we must find the common controllability Gramian $G_c$ and the output feedback gains $F_i$ in order to satisfy (15) and (16). In the next section, we will provide a way to assign the common controllability Gramian $G_c$, and to then find the output feedback gains $F_i$ for each rule. This method is based on condition (15) of Lemma 2. It will be shown that this problem can be solved using the theory of generalized inverse.

**III. DESIGN OUTPUT FEEDBACK GAINS FOR DISCRETE FUZZY CONTROLLERS**

To design the stable output feedback fuzzy controller with the common controllability Gramian for all rules, it is necessary to find suitable $F_i$ and $G_i$ in order to satisfy stability conditions (15) and (16). If we assign the common controllability Gramian $G_c$, then the first problem is to solve the output feedback gain $F_i$ referring to equation (15). After designing the output feedback gains $F_i$ of the fuzzy controllers, the remaining problem is to check whether (16) is satisfied. In this section, we will derive the following theorem, which can be used to solve for the output feedback controller for each rule.

**Theorem 1.** Given a common controllability Gramian $G_c > 0$ for the discrete fuzzy system (9), there exist output feedback gains $F_i$ for each rule, which satisfies (15) for some $\tilde{Z}_i$, if and only if

\[
G_c = A_i G_i A_i^T + B_i B_i^T - (A_i G_i C_i^T)(C_i G_i C_i^T)^{-1} C_i G_i A_i^T
\]

\[
+ [(I - B_i B_i^T) (A_i G_i C_i^T) + B_i B_i^T \tilde{Z}_i] (C_i G_i C_i^T)^{-1}
\]

\[
[(I - B_i B_i^T) (A_i G_i C_i^T) + B_i B_i^T \tilde{Z}_i]^T,
\]

where $\tilde{Z}_i$ is used to satisfy (17).

If condition (17) holds, we can obtain the output feedback gains of each rule as follows:

\[
F_i = B_i^T (\tilde{Z}_i - A_i G_i C_i^T) (C_i G_i C_i^T)^{-1} + (I - B_i B_i^T) \tilde{Z}_i,
\]

where $Z_i$ is an arbitrary matrix.

**Proof.** Suppose there exists $G_c$ satisfying the controllability Gramian equation

\[
(A_i + B_i F_i C_i) G_c (A_i + B_i F_i C_i)^T - G_c + B_i B_i^T = 0
\]

given that $G_c > 0$; then, the above equation can be rewritten as

\[
G_c - A_i G_i A_i^T + S R_i S_i^T - B_i B_i^T = L_i L_i^T,
\]

where

\[
L_i = (B_i F_i - S R_i^{-1}) \Gamma_i,
\]

\[
R_i = C_i G_i C_i^T = \Gamma_i \Gamma_i^T,
\]

\[
S_i = A_i G_i C_i^T.
\]

**Necessity.** By supposition, the output feedback gains $F_i$ of each rule satisfying (21) exist, and we can expand (21) as follows:
\[ B_F \Gamma_i = L_i + S R_i \Gamma_i^\top. \]  

(24)

We apply the generalized inverse technique [1] to solve for \( F \), based on (24), which is guaranteed if and only if

\[ (I - BB^\top) (L_i - SR_i \Gamma_i^\top) = 0. \]  

(25)

All the solutions, \( F \), are given by

\[ F_i = B_i^\top (L_i \Gamma_i^\top - S R_i \Gamma_i^\top) + (I - B_i B_i^\top) Z_i, \]  

(26)

where \( Z_i \) is arbitrary. Solving (25) for \( L_i \) yields

\[ L_i = (I - B_i B_i^\top) S \Gamma_i^\top + B_i B_i^\top Z_i, \]  

(27)

where \( Z_i \) is arbitrary. Substituting (27) into (20) gives

\[ G_i = A_i G_i A_i^\top + B_i B_i^\top - S R_i \Gamma_i^{-1} S_i \]

\[ + (I - B_i B_i^\top) S_i \Gamma_i^{-1} + (I - B_i B_i^\top) Z_i, \]  

(28)

where \( Z_i \equiv \tilde{Z}_i \Gamma_i^{-1} \) is arbitrary.

Substituting (27) into (26) yields

\[ F_i = B_i^\top (\tilde{Z}_i - S_i) \Gamma_i^{-1} + (I - B_i B_i^\top) Z_i. \]  

(29)

This completes the necessity.

**Sufficiency.** For sufficiency, we assume that condition (17) is satisfied. We will show that \( F \), given by (29), solves (20) or, equivalently, (19). Putting (29) into (21), we have

\[ L_i = (I - B_i B_i^\top) S_i \Gamma_i^{-1} + B_i B_i^\top Z_i. \]  

(30)

Substituting (30) into (20) yields (28), which is equivalent to (17). Since (17) holds, equation (20) can be solved using \( F \), given in (29). The sufficiency of this proof is completed.

From the above proof, we can find that if condition (17) holds, then the forms of the output feedback gains \( F_i \) have been derived in (29). By substituting (22) and (23) into (29), the output feedback gains \( F_i \) can be constructed in a form such as that in (18).

In Theorem 1, it is necessary for us to assign a common controllability Gramian \( G_c \) and arbitrary matrix \( \tilde{Z} \), so as to satisfy (17). If (17) is satisfied, we can find the output feedback gains \( F_i \) of each rule using (18), and the subsystems will be guaranteed to be asymptotically stable. Hence, Theorem 1 gives the condition and solution for the output feedback gains \( F_i \). In the following section, we will give a simple example to verify the effects of the proposed method.

**IV. A NUMERICAL EXAMPLE**

Let us consider a nonlinear system, which can be transformed successfully into a discrete T-S fuzzy model with two rules as follows:

**Rule 1:**

\[ I \times \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \]

\[ y(k) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \]

**Rule 2:**

\[ I \times \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(k), \]

\[ y(k) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \]

The membership functions of \( x_1(k) \) are shown in Fig. 1, where the universe of discourse of \( x_1(k) \) is the closed interval \([-40, 40]\). Our goal is to choose a common controllability Gramian \( G_c \) and to determine the feedback gains for each rule. Before assigning \( G_c \), we have to find the relationship between \( G_c \) and \( \tilde{Z} \), for satisfying (17). Let us define \( G_c \) and \( \tilde{Z} \), as follows:

\[ G_c = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}, \quad \tilde{Z}_i = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \tilde{Z} \quad \tilde{Z}_i = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}. \]

Substituting the above matrices into (17), we have

![Fig. 1. Membership function of \( x_1(k) \) in the example.](image-url)
\[ g_{22} = 4g_{11}, \quad \text{(31a)} \]
\[ z_{22} = 2g_{12}, \quad \text{(31b)} \]
\[ z_{12} = -4g_{12}, \quad \text{(31c)} \]
\[ 4g_{11} \times z_{12}^2 + 12g_{12}^2 \times z_{12} - (144g_{11}^3 - 72g_{11}g_{12}^2 \]
\[ - 36g_{11}^2 + 9g_{12}^2) = 0, \quad \text{(31d)} \]
\[ g_{11} \times z_{22}^2 - 2g_{12}^2 \times z_{22} - (16g_{11}^3 - 8g_{11}g_{12}^2 \]
\[ - 4g_{11}^2 + g_{12}^2) = 0. \quad \text{(31e)} \]

Now, we assign the common controllability Gramian matrix as follows:

\[
G_c = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 4 \end{bmatrix}.
\quad \text{(32)}

Substituting the above \( G_c \) into (31), \( \hat{Z}_1 \) and \( \hat{Z}_2 \) can, respectively, be solved as follows:

\[
\hat{Z}_1 = \begin{bmatrix} 3 & 5.00766 \\ 5.00766 & -1 \end{bmatrix}, \quad \hat{Z}_2 = \begin{bmatrix} -5 & 3.33844 \\ 3.33844 & 0.5 \end{bmatrix}.
\quad \text{(33)}

Putting \( G_c, \hat{Z}_1 \) and \( \hat{Z}_2 \) into (18), we get the output feedback gains as follows:

\[
F_1 = \begin{bmatrix} 0.242487 & -0.866483 \end{bmatrix},
\quad \text{(34)}
\]
\[
F_2 = \begin{bmatrix} 0.136269 & -1.23297 \end{bmatrix}.
\]

According to the stability conditions of Lemma 2, all the subsystems of the T-S type fuzzy system can be guaranteed to be asymptotically stable by using the output feedback gains \( F_1 \) and \( F_2 \). Substituting the output feedback gains into (16), we have

\[
H_{12} G_c H_{12}^T - G_c = \begin{bmatrix} -1 & -0.25 \\ -0.25 & -2.77379 \end{bmatrix}.
\quad \text{(35)}

It is easy to verify that (35) is negative definite; hence, condition (16) is satisfied. By Lemma 2, we can conclude that the fuzzy control system is stable in the large with the specified common controllability Gramian matrix (32). Figure 2 and Fig. 3 show the responses of \( x_1(k) \) and \( x_2(k) \), where the sampling time is 0.01 sec and the initial conditions are \( x_1(0) = 8 \) and \( x_2(0) = -25 \), respectively.

**V. CONCLUSIONS**

Based on finding output feedback fuzzy controllers, this paper has developed a method for assigning the common controllability Gramian matrix for the discrete Takagi-Sugeno fuzzy model. In order to assign a specified common controllability Gramian for the discrete Takagi-Sugeno fuzzy systems, we use the theory of generalized inverses to find the output feedback gains of the discrete fuzzy controllers. Based on common controllability Gramian assignment, the proposed approach provided a concept opposite to the LMI method.

**REFERENCES**

2. Chang, W.J. and C.C. Sun, “Fuzzy Control with Common Observability Gramian Assignment for Con-


