H∞ CONTROL AND SLIDING MODE CONTROL OF MAGNETIC
LEVITATION SYSTEM

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ABSTRACT

In this paper, H∞ disturbance attenuation control and sliding mode disturbance estimation and compensation control of a magnetic levitation system are studied. A magnetic levitation apparatus is established, and its model is measured. Then the system model is feedback linearized. A H∞ controller is then designed. For comparison, a sliding mode controller and a PID controller also were designed. Some experiments were performed to compare the performance of the H∞ controller, the sliding mode controller and the PID controller.

KeyWords: Magnetic levitation system, H∞ control, sliding mode control, feedback linearization.

I. INTRODUCTION

Magnetic levitation systems have received increasing attention recently, due to their practical importance in many engineering systems. They are becoming popular in many applications, such as high-speed trains, magnetic bearings, and vibration isolation tables [1-4]. Magnetic levitation systems are nonlinear and open loop unstable. This has led to a significant demand for control technologies for use in magnetic levitation systems.

Feedback linearizing control is an approach to the design of nonlinear controllers [5,6]. Recently, many researchers have applied this approach to magnetic suspension systems [7-9], especially in cases where magnetic suspension systems are required to operate under large variation of the air gap. Feedback linearization utilizes the complete nonlinear description and, hence, yields consistent performance largely independent of the operation point. Usually, feedback-linearizing control does not guarantee robustness in the presence of modeling error and disturbances.

It is known that H∞ control is an effective method for attenuating such disturbances [10,11] and that the sliding mode control scheme can be used to estimate and then effectively compensate for the disturbances [12,13]. The purposes of this paper are to examine the feedback linearizing technique and to compare the H∞ disturbance attenuation control and sliding mode disturbance estimation scheme for magnetic levitation systems. To fulfill the purposes, a magnetic levitation system is constructed, and its model is measured. Then this model is feedback linearized. The H∞ controller, the sliding mode controller and the PID controller are then designed, and their performance in set-point regulation, disturbance attenuation and trajectory following are experimentally evaluated. We also present experimental data which verify the ability of the feedback linearization technique to mitigate the effect of nonlinearities.

The experimental apparatus and the model are described in Section 2. Section 3 describes the design of the H∞ controller, sliding mode controller and PID controller. Experimental results are presented in Section 4. Finally, conclusions are given in Section 5.

II. EXPERIMENTAL APPARATUS AND
CONTROL MODEL

Figure 1 shows the single-axis magnetic levitation system used in the experiment. The levitation object is a ping-pong ball with a permanent magnet attached inside it to provide an attractive force. The attraction force is controlled by means of a computer-controlled electromagnet mounted directly above the ball. A light source and a linear image sensor (LIS, Hamamatsu S5462-512Q) are used to determine the displacement of the ball. There are 512 photo diode cells in LIS, and the length of each cell is 0.05mm. The light source and the sensor are tuned such that the outputs of the photocells are saturated when the
ball does not cover the cells. A comparator is used to
compare the outputs of each cell with a preset voltage to
judge whether the cell is saturated or not. We then obtain
the levitation displacement of the ball by utilizing a counter
to count the numbers of saturated cells. The sampling rate
used is 200 Hz. This low sampling rate is used due to the
bandwidth limitation of LIS. The control computer is an
industrial personal computer with a Pentium processor
and an Advantech PCL818H analog I/O and counter
board.

A calibration experiment was performed, and the
data were least squares fitted as follows:

\[ x = 0.045n + 20.49, \]  

where \( n \) is the number of saturated cells and \( x \) is the
displacement of the ball from the bottom of the electro-
magnet in millimeters. Note that (1) is valid in the range
20.75mm \( \leq x \leq 42.95 \)mm. The accuracy in this range is
better than \( \pm 0.075 \)mm.

A force balance analysis yields the following equa-
tion for this levitation system:

\[ mx = mg - F, \]  

where \( m \) is the mass of the levitation ball in grams, \( g \) is
gravity, and \( F \) is the magnetic control force in millinewtons.
The mass of the ball is 1.71g.

The force/current/displacement relationship of this
magnetic levitation apparatus is extremely difficult to
determine using an analytic method. Therefore, the mag-
netic force characteristics were experimentally calibrated
as a function of the applied current and the displacement.
We rested the ball on a xyz-stage capable of 1mm incre-
mental positioning and determined the minimum current
required to pick up the ball at various heights. The model
of the force distance relationship can be determined by
means of least square fitting as follows:

\[ F = \frac{i}{a_2x^2 + a_1x + a_0} , \]  

where \( i \) is the current of the electromagnet in amperes, \( a_2 = 5.4e-5, a_1 = -1.022e-2 \) and \( a_0 = 1.34e-2 \). The curve-
fit calibration is valid for \( x \) in the range from 20.7 to
42.7mm. The characteristics of the solenoid change with
the temperature. Therefore, the coefficients of (3) change
significantly when the temperature of the solenoid
increases. This drift combined with other nonlinear ef-
fects result in deviation of the calibration data from the
curve-fit in (3) of up to \( \pm 10\% \).

Define the state variables as \( x_1 = x \) and \( x_2 = \dot{x} \). The
equations of motion for the magnetic levitation system can
be written as

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = g - bi + d, \]  

where \( d \) represents unknown disturbances; \( b \) denotes the
actual force-distance relationship and can be expressed as

\[ b = b_n + \Delta b, \]  

where

\[ b_n = (m(a_2x_1^2 + a_1x_1 + a_0))^{-1} \]  

denotes the nominal model of the force-distance relation-
ship and \( \Delta b \) represents the modeling error, the upper bound
which is approximately \( \pm 10\% \) of the nominal model.

In order to cancel out the nonlinearity, we define a
new control input:

\[ u = g - b_i. \]  

Substituting (7) into (4), the system state equations be-
come

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = u - \delta(b - g) + d. \]  

From (7), we obtain the control current from \( u \) as follows:

\[ i = (g - u)/b_n. \]  

Substituting (9) into (8), we get

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = u + \delta(u - g) + d, \]  

where \( \delta = \Delta b/b_n \) is uncertain. The modeling error is about
10%; therefore, \( \delta \in [-0.1, 0.1] \). Note that (10) is a
nominally linear uncertain system.

The control objective is to keep the levitation dis-
placement \( x \) at some desired value \( x_d \), which may be time
varying. Define

$$e = x - x_d = x_1 - x_d$$  \hspace{1cm} (11)

as the levitation displacement error.

### III. THE CONTROLLER

In this section, the development of the $H^{-}$ controller, the sliding mode controller and the PID controller for the magnetic levitation system is presented.

#### 3.1 $H^{-}$ controller

In order to get rid of steady-state error in set-point regulation when uncertainties and disturbances exist, a new state ($e_i$) such that

$$\dot{e}_i = e$$  \hspace{1cm} (12)

is introduced. Denote

$$e = [e, e, \dot{e}]^T.$$  \hspace{1cm} (13)

From (10), (11), (12) and (13), we obtain

$$\dot{e} = \begin{bmatrix} e \\ \dot{e} \\ u + \delta(u - g) + d - \ddot{x}_d \end{bmatrix}.$$  \hspace{1cm} (14)

Let the control law be

$$u = \ddot{x}_d - K_1 e_i - K_2 \dot{e} - K_3 \ddot{e} + u_h$$

$$= \ddot{x}_d - K_1 \int_0^t e \, dt - K_2 \dot{e} - K_3 \ddot{e} + u_h, \hspace{1cm} (15)$$

where $K_1$, $K_2$ and $K_3$ are positive constants to be designed and $u_h$ is a disturbance compensation component yet to be specified by means of $H^{-}$ control theory. Substituting (15) into (14) leads to

$$\dot{e} = A e + B u_h + B z,$$  \hspace{1cm} (16)

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 - K_2 - K_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad z = \delta(u - g) + d.$$  \hspace{1cm} (17)

**Remarks.**

1) $K_1$, $K_2$ and $K_3$ are specified so that $A$ has the desired eigenvalues and the error dynamics in (16) have the desired response while the magnetic levitation system is free of uncertainties and disturbances.

2) Note that $z$ includes modeling error and external disturbances, and depends on control input $u$.

It is known that the minimum $H^{-}$ control [10,11] is the most efficient method for eliminating the worst-case effect of the uncertain $z$ on $e$ in (16). Therefore, it is employed here to attenuate the effect of $z$.

The control signal $u_h$ is specified such that the worst-case effect of $z$ on $e$ is attenuated as much as possible and kept below a prescribed level $\rho$; that is, the following minimax performance must be satisfied for the error dynamics in (16):

$$\min_{u_h \in L_2[0, t_f]} \max_{z \in L_2[0, t_f]} \int_0^{t_f} (e^T Q e + \gamma u_h^2) dt \leq e^T(0) P e(0) + \int_0^{t_f} \rho^2 \dot{z}^2 dt,$$  \hspace{1cm} (18)

where $\gamma > 0$ is a weighting factor, $Q = Q^T > 0$ and $P = P^T > 0$ are some positive definite matrices, and the final time $t_f > 0$.

For the error dynamics in (16), if we choose

$$u_h = -\gamma^{-1} B^T P e,$$  \hspace{1cm} (19)

and if $P$ is the solution of the following Riccati-like equation

$$A^T P + PA + Q - \gamma^{-1} P B B^T P + \frac{1}{\rho^2} P B B^T P = 0,$$  \hspace{1cm} (20)

then the performance of (18) is achieved with a prescribed attenuation level, $\rho$ [10,11].

**Remarks.**

1) Equation (20) has a solution $P^T = P \geq 0$ if and only if $\rho^2 \geq \gamma$.

2) Note that $z$ depends on control input $u$. Due to the trade-off between the attenuation level and the control input, the desired performance robustness may not be achievable, especially in the case without external disturbances and with very small $\rho$ [14]. In this magnetic levitation system, external disturbances always exist, hence the $H^{-}$ controller still can work well.

In the experiment, we selected $K_1 = 10125$, $K_2 = 1575$ and $K_3 = 75$, in which the eigenvalues of the desired error dynamics were $-15$ with a multiplicity of $2$ and $-45$. For convenience, the parameter $\gamma$ was selected as $\gamma = \rho^2$, and the prescribed attenuation level was chosen as $\rho = 0.05$. When the matrix $Q$ was chosen, significant attention was given to minimizing the overshoot of set-point response.
and to optimizing the disturbance response. The final $Q$ obtained is then

$$Q = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$  

The solution of equation (20) can be obtained as

$$P = \begin{bmatrix} 9300 & 825.7 & 0.4938 \\ 825.7 & 185.3 & 0.8417 \\ 0.4938 & 0.8417 & 0.0779 \end{bmatrix}.$$  

Thus, we have $u_0 = [-197.5 \ 336.7 \ 31.16] e$ and $u = \tilde{x}_d - [10323 \ 1912 \ 106] e$. Applying this control law to (16), the eigenvalues of the error dynamics become $-10.5423 \pm 3.2287 i$ and $-84.92$. While the system is free of uncertainties and disturbances, the error dynamics will have the desired response.

To determine closed-loop stability, we denote the switched action assigned as

$$S = \tilde{x}_2 - x_2.$$  

Thus, the sliding condition is satisfied. Note that $\tilde{x}_2(0) = x_2(0)$; therefore,

$$S = 0 \text{ for } t = 0.$$  

From (21) and (14), we obtain

$$\dot{e} + (1 + \delta) C_1 e + (1 + \delta) C_2 \dot{e} + (1 + \delta) C_\psi \dot{e} = \delta(\tilde{x}_d - g) + d.$$  

Therefore, for stability, $C_1 > 0$, $C_2 > 0$, $C_\psi > 0$ and $C_\psi C_\delta (1 + \delta) > C_1$ are required. For the parameter values derived above, the stability condition requires that $\delta$ satisfy the inequality $\delta > -0.95$. This requirement with respect to the accuracy of the model parameters for the sake of stability can be satisfied easily.

### 3.2 Sliding mode controller

In this subsection, we will develop the sliding mode controller. In this study, the sliding mode disturbance estimation and compensation scheme [12,13] is applied to design the sliding mode controller for this magnetic levitation system.

From (10) and (11), we have

$$\ddot{e} = \dot{\delta}(u - g) + d - \tilde{x}_d.$$  

Let the control law be

$$u = \dot{x}_d - k_1 e - k_\psi \dot{e} + u_d,$$  

where $k_1$ and $k_\psi$ are the feedback gains to be designed so that the error dynamics will have the desired response while the system is free of uncertainties and disturbances, and $u_d$ is the uncertainty and the disturbance compensation component yet to be determined by the sliding mode estimator.

To establish the sliding mode disturbance estimator, we define the switching function as

$$S = \tilde{x}_2 - x_2.$$  

With

$$\dot{\tilde{x}}_2 = \tilde{x}_d - k_1 e - k_\psi \dot{e} + u_d + \psi = u + \psi,$$  

$$\tilde{x}_2(0) = x_2(0),$$  

where $\tilde{x}_d$ is the state variable of this auxiliary process. $\psi$ is the switched action assigned as

$$\psi = -\eta \text{ sign } (S), \text{ sign } (S) = \begin{cases} 1 & \text{if } S > 0 \\ -1 & \text{if } S < 0 \\ 0 & \text{if } S = 0 \end{cases}.$$  

and the positive constant $\eta$ satisfies

$$\eta > \left| \delta(u - g) + d \right|.$$  

Ensuring a sliding regime $S = 0$ requires consideration of the Lyapunov candidate $V = 0.5 S^2$. We differentiate $V$ with respect to time and substitute (10), (24) and (26) to obtain

$$\dot{V} = S \dot{x}_2 = S \{ \psi - [\delta(u - g) + d] \}.$$  

From (28) and (29), it is seen that

$$\dot{V} = SS < 0 \text{ if } S \neq 0.$$  

Thus, the sliding condition is satisfied. Note that $\tilde{x}_2(0) = x_2(0)$; therefore,

$$S = 0 \text{ for } t = 0.$$  

From (30) and (31), we can conclude that the sliding mode exists at all times, i.e.,

$$S = 0 \text{ for all } t \geq 0.$$  

Denote the equivalent value [13, 15] of the discontinuous function $\psi$ as $\psi_{eq}$. Since $S = 0$, $\psi_{eq}$ can be determined from (10), (25) and (26):

$$\psi_{eq} = \delta(u - g) + d.$$  

This means that the equivalent value of the discontinuous function $\psi$ equals the disturbances. By selecting $u_d = -\psi_{eq}$, the disturbances can be compensated. It was shown in [13, 15] that the equivalent value $\psi_{eq}$ is equal to the average value measured by a first-order linear filter, with the
switched action as its input. Therefore, we can write

\[ u_d = -\psi_{av} = -\psi_{eq} \]  

(34)

with

\[ \tau \psi_{eq} + \psi_{av} = \psi. \]  

(35)

The time constant \( \tau \) should be made small such that the plant and disturbance dynamics are allowed to pass through the filter without significant phase lag.

Substituting (34) and (24) into (23) yields

\[ e + k_2 \dot{e} + k_1 e = -\psi_{av} + \delta (u - g) + d, \]  

(36)

which is equivalent to

\[ \dot{e} + k_2 \dot{e} + k_1 e = 0. \]  

(37)

Equation (37) represents the desired error dynamics. This shows that the sliding mode controller can achieve the desired error dynamics by identifying and then compensating for the disturbances.

In our experiments, the feedback gains were selected as \( k_1 = 225 \) and \( k_2 = 30 \), in which the eigenvalues of the desired error dynamics were \(-15\) with a multiplicity of 2. To satisfy the inequality (28), the control parameter \( \eta \) was selected to be larger than \( |a|_{\text{max}} |u - g|_{\text{max}} + |d|_{\text{max}} \), where \( |a|_{\text{max}} \) denotes the maximum absolute value of \( a \). For the 10% upper bound on the modeling error and 4A maximum current, \( |a|_{\text{max}} |u - g|_{\text{max}} \) was chosen as 1330. In the experiments, \( \eta = 2000 \) was used. In [12], the authors indicated that the bandwidth of the first order filter in (35) should be roughly 5-10 times larger than the desired bandwidth of the closed-loop system. Figure 2 depicts the control results for \( \tau = 1/(5 \times 15) \) and \( \tau = 1/15 \). The desired levitation height is 30mm. The chattering in levitation height is caused by the switching action. As the figure demonstrates, a wide bandwidth (small \( \tau \)) will make the achieved steady-state accuracy unacceptable. In order to obtain acceptable steady-state accuracy, \( \tau = 1/15 \) must be selected. For convenience, the resulting controller is written as follows:

\[ u = \dot{x}_d - 225 \dot{e} - 30 e - \psi_{av}, \quad \frac{1}{15} \psi_{av} + \psi_{av} = -2000 \text{ sign } (S), \]

\[ S = \ddot{x}_2 - x_2, \quad \dot{x}_2 = u - 2000 \text{ sign } (S). \]

### 3.3 PID controller

From equation (10), the nominal transfer function (free of uncertainty and disturbances) of the linearized magnetic levitation system can be obtained as

\[ G(s) = \frac{X(s)}{U(s)} = \frac{1}{s^2}. \]

A PID controller can be designed for this system using the pole assignment method [16]. When the desired poles are selected, significant attention must be paid to optimizing the disturbance response and providing good damping to the closed-loop response. The final desired poles of the closed loop system are \(-13.5 \pm 6.538j \) and \(-45\). The final transfer function of the PID controller can be obtained as

\[ \frac{U(s)}{E(s)} = 1440 + \frac{10125}{s} + 72s. \]

In order to obtain better set-point response and avoid integrator wind-up, set-point weighting (W) and anti-windup based on back-calculation [16] are added. Moreover, the gain of the derivative term must be limited. The obtained PID controller is shown in Fig. 3. The actuator model shown in Fig. 3 is a function that simulates actuator saturation. The set-point weighting factor W is

![Fig. 2. Control results for the sliding mode controller with \( \tau = 1/15 \) (solid line) and \( \tau = 1/(5 \times 15) \) (dashed line).](image)

![Fig. 3. Block diagram of the PID controller.](image)
set to be 0.4. Typical values of N are 8 to 20. We selected N = 10 in the experiment. Following the suggestion made in [16], the tracking time constant T was selected as $\sqrt{72/10125} = 0.0843$.

IV. EXPERIMENTAL RESULTS

The performance of the H∞, the sliding mode and the PID controllers in set-point regulation, input disturbance attenuation and trajectory tracking were compared. In the experiments, all of the controller designs were discretized by means of backward approximation with a sampling rate of 200Hz.

4.1 Regulation control

The performance of the controllers designed as described in the previous section in terms of set-point regulation over the full operating range was examined by commanding 5mm steps in the reference position at various operating points. The results of these tests are shown in Figs. 4(a)-4(c). As the figures demonstrate, all three controllers could provide stable regulation over the full operation range. This proves that the feedback linearization technique can be used to effectively mitigate the effect of nonlinearity. The performance of the PID controller is excellent. It provides the fastest settling time. Figure 5 depicts the steady state of regulation control. The PID controller provides perfect regulation up to the sensor accuracy. Due to the large feedback gains of $e$ and $\dot{e}$, the steady state accuracy of the H∞ controller in regulation is worse than that of the PID controller. Due to the switching action, the sliding mode controller has the worst steady state accuracy. Figure 6 shows the performance in disturbance attenuation of these controllers. As the figure demonstrates, the H∞ controller achieves the best performance in disturbance attenuation.

4.2 Tracking control

Figure 7 shows the results for trajectory following, where the desired trajectory is a sine wave of 1Hz with 2.5mm amplitude. As the figure demonstrates, the H∞ controller achieves slightly better tracking performance than the sliding mode controller in this case. There exists phase lag between the output of the system controlled by the PID controller and the reference trajectory.

Both controllers were also evaluated for 2, 3, 4 and 5 Hz sine wave trajectories. We found that the performance of the PID controller degraded rapidly in the higher-frequency trajectory following cases. For a 3Hz trajectory, the PID controller could not achieve satisfactory performance. For the same case, the H∞ controller and sliding mode controller could still achieve acceptable performance, as depicted in Fig. 8. Note that the controlled result of the sliding mode controller exhibits sig-

![Graphs of experimental results for each controller type.](image)
Fig. 5. Steady state accuracy of set-point regulation for (a) the $H^\infty$ controller, (b) the sliding mode controller and (c) the PID controller.

Fig. 6. Experimental results of regulation with an input disturbance $-0.1A$ introduced at $t = 1$ second for (a) the $H^\infty$ controller, (b) sliding mode controller and (c) the PID controller.

Fig. 7. Experimental results for a 1Hz trajectory for (a) the $H^\infty$ controller, (b) the sliding mode controller and (c) the PID controller.

Fig. 8. Experimental results for a 3Hz trajectory for (a) the $H^\infty$ controller and (b) the sliding mode controller.

Significant phase lag. This is caused by the large time constant $\tau$ of the filter. The phase lag can be decreased if a smaller value of $\tau$ is chosen. However, a smaller value of $\tau$ will cause the steady state accuracy of regulation control to worsen. Clearly, the $H^\infty$ controller also achieves better tracking performance in this case. Under a 4Hz trajectory, the $H^\infty$ controller becomes unstable. In the identical case, the sliding mode controller still achieves
stable performance. The reason is that the bandwidth of the system with the $H^\infty$ controller is narrower than that with the sliding mode controller. Under trajectories higher than 4Hz, the sliding mode controller also becomes unstable.

V. CONCLUSIONS

The $H^\infty$ controller, sliding mode controller and PID controller have been experimentally compared for set-point regulation and trajectory tracking with a magnetic levitation system. The performance of the PID controller is quite good in set-point regulation and disturbance attenuation. The performance of the $H^\infty$ controller and sliding mode controller in trajectory tracking is superior to that of the PID controller. In general, the performance of the $H^\infty$ controller is superior to that of the sliding mode controller. From the experimental results, we can also conclude that the feedback linearization technique is useful for designing nonlinear control systems with large variation in operating points. We can also conclude that the performance of the $H^\infty$ controller in disturbance attenuation is good.

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