OUTPUT FEEDBACK SLIDING MODE CONTROLLER
DESIGN VIA $H_\infty$ THEORY

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ABSTRACT

For a linear system with mismatched disturbances, a sliding mode controller using only output feedback is developed in this paper. Through application of the $H_\infty$ control theory, the designed switching surface can achieve robust stabilization and guarantee a level of disturbance rejection during sliding mode. Although the system exhibits disturbances, a state estimator is used which, using only measured outputs, can asymptotically estimate the system states. The control law is designed with respect to the estimated signals. Finally, a numerical example is presented to demonstrate the proposed control scheme.

KeyWords: Sliding mode control, mismatched disturbance, state estimator, $H_\infty$ approach.

I. INTRODUCTION

The sliding mode control (SMC) is a useful tool for dealing with system uncertainties and external disturbances [1]. Earlier results for the SMC were based on the assumption that all the system states are available [2-5]. However, controller designers often encounter systems for which only output information is obtainable. To deal with this situation, several output feedback sliding mode control (OFSMC) algorithms have recently been proposed [6-9]. In these papers [6-9], the uncertain term or disturbance must satisfy the so-called matching condition [10]. In fact, however, mismatched uncertainty or disturbance exists in many practical systems. There currently exist effective control technologies designed to handle systems with mismatched disturbances, for example, $H_\infty$ control [11-12], adaptive control [13], and SMC [4-5]. When an SMC system suffers from mismatched disturbances, a component of the disturbance will affect system performance while in sliding mode [4]. In [4-5], the researchers considered only the state feedback SMC for systems with mismatched uncertain terms. In this paper, we will study the OFSMC problem and replace the matching condition.

Since the system states are not available and unknown disturbances exist in the system, robust state estimation is needed. This problem is known in the literature as that of robust fault detection or unknown input observers [17-19]. In this paper, we will apply the design technique in OFSMC [6-9] to find a practical transformation matrix to effectively estimate the true states. In addition, a switching surface design method will be proposed, which combines $H_\infty$ control theory and our prior work [15]. The main feature is that the switching surface is determined by solving the so-called algebraic Riccati equation arising in $H_\infty$ control theory [11,12]. This method can guarantee robust stability and disturbance attenuation for the system in sliding mode. A similar strategy can be found in the work by Nonami and Nishimura [15-16]. They applied the dynamic sliding mode and used $H_\infty$ control theory to obtain the frequency-shaped switching surface. However, in our method, the system dynamics are not extended. Based on the estimated signals, the total control law is designed. Note that the estimator and controller design can be taken as two separate steps. This concept is similar to that of an indirect adaptive control system [13].

In the next section, a class of systems to be controlled is introduced along with some important assumptions related to the system matrices. Section 3 presents the switching surface design and Section 4 describes the state estimator a long with its novel transformation matrix. After showing that the system’s states are well estimated, we propose a controller design in Section 5. To
demonstrate the developed controller, a numerical example is given in Section 6. Finally, Section 7 gives concluding remarks.

II. PROBLEM STATEMENT

Consider an MIMO linear system expressed by the following state-space equations:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Gd(t), \\
y(t) &= Cx(t),
\end{align*}
\]

where \( x \in \mathbb{R}^n \) are the states of the system, \( u \in \mathbb{R}^n \) are control inputs, \( y \in \mathbb{R}^m \) are measured outputs, and \( d(t) \in \mathbb{R}^m \) are disturbances. The system matrices satisfy \( A \in \mathbb{R}^{nm} \), \( B \in \mathbb{R}^{nm} \), \( C \in \mathbb{R}^{m} \), and \( G \in \mathbb{R}^{mp} \), with \( \text{rank}(B) = m \), \( \text{rank}(C) = l \), and \( \text{rank}(G) = p \). For the plant, we assume the following hold with respect to the system:

(A1) The pair \((C, A, B)\) is controllable and observable.

(A2) The disturbances \( d(t) \) are \( L_2 \) bounded [11, 11], and they satisfy

\[
\|d(t)\| \leq \beta,
\]

where \( \beta \) is a known constant.

(A3) \( \text{rank}(CG) = \text{rank}(G) = p \).

(A4) The invariant zeros of the pair \((C, A, G)\) must lie in the left-half complex plane.

Assumptions (A1) and (A2) are standard in SMC design [1-9], and assumptions (A3) and (A4) are important for the state estimator design [17-19].

Since the sliding mode controls only cancel out the matched disturbance [1-10], how to estimate the effect of the mismatched disturbance on the system (1) is the main problem. In the next section, a transformation matrix method that decomposes the disturbance term \( Gd \) into its matched part and mismatched part is established. Based on this transformation, we then propose a method for obtaining the switching surface design by using \( H_\infty \) control theory to achieve disturbance attenuation. The problem of disturbance attenuation involves designing a controller such that the closed-loop system is internally stable, and such that there exists \( 0 \leq \gamma < \infty \) for the following inequality [11, 12]:

\[
\|y(t)\| \leq \gamma \|d(t)\| \leq \gamma \beta.
\]

Moreover, since the system (1) exhibits disturbance, a practical state estimator method is also presented such that \( \hat{x}(t) \to x(t) \) for \( t \to \infty \), where \( \hat{x}(t) \) is the estimated state.

III. SWITCHING SURFACE DESIGN VIA \( H_\infty \) THEORY

Since the system (1) is controllable, according to linear control theory, a gain matrix \( K \in \mathbb{R}^{m \times n} \) can be obtained by assigning \( n \) desired eigenvalues. We decompose these \( n \) eigenvalues into two sets, \( \{\lambda_1, \ldots, \lambda_{m-1}\} \) and \( \{\omega_1, \ldots, \omega_n\} \). This implies that \( \{\lambda_1, \ldots, \lambda_{m-1}\} \) and \( \{\omega_1, \ldots, \omega_n\} \) is the set of desired eigenvalues for the matrix \( A - BK \). Moreover, these \( n \) eigenvalues must satisfy the following conditions:

(C1) Any eigenvalue in \( \{\omega_1, \ldots, \omega_n\} \) is not in the spectrum of \( A \).

(C2) If a complex eigenvalue is in \( \{\lambda_1, \ldots, \lambda_{m-1}\} \), so is its conjugate.

(C3) The number of repeated eigenvalues in \( \{\lambda_1, \ldots, \lambda_{m-1}\} \) is not greater than \( m \).

Note that the eigenvalues \( \{\omega_1, \ldots, \omega_n\} \) are not necessarily located in the left-half complex plane. Sinswatt and Fallside [20] have shown that if the number of repeated eigenvalues of \( A - BK \) is not greater than the rank of \( B \), then \( A - BK \) is diagonalizable. Hence, based on their results and condition (C3), \( A - BK \) can be diagonalized as

\[
(A - BK)W_r = W_r J_r ,
\]

(4)

\[
A - BK = W_r J_r ,
\]

(5)

where \( W_r \in \mathbb{C}^{m \times m} \) and \( W_r \in \mathbb{C}^{n \times n} \) are right eigenvector matrices which correspond to \( \{\lambda_1, \ldots, \lambda_{m-1}\} \) and \( \{\omega_1, \ldots, \omega_n\} \). In addition, \( J_r \) and \( J_s \) are diagonal matrices with elements \( \{\lambda_1, \ldots, \lambda_{m-1}\} \) and \( \{\omega_1, \ldots, \omega_n\} \), i.e.,

\[
J_r = \text{diag} \{\lambda_1, \ldots, \lambda_{m-1}\} \quad \text{and} \quad J_s = \text{diag} \{\omega_1, \ldots, \omega_n\}. \]

Let

\[
W = \begin{bmatrix} W_r & W_s \end{bmatrix} \quad \text{and} \quad V = W^{-1} = \begin{bmatrix} V_r & V_s \end{bmatrix},
\]

where \( V_r \in \mathbb{C}^{m \times m} \) and \( V_s \in \mathbb{C}^{n \times n} \). In fact, \( V \) is the left eigenvector matrix for \( A - BK \). Similar to (4) and (5), we obtain

\[
V_r (A - BK) = J_r V_r ,
\]

(6)

\[
V_s (A - BK) = J_s V_s .
\]

(7)

Rearranging (7) yields

\[
V_s A - J_s V_r = (V_s B) K .
\]

(8)

Lemma 1 [14]: The matrix \( V_s B \in \mathbb{C}^{n \times n} \) is nonsingular under condition (C1).

Lemma 2 [14]: The matrix \( W_r \) can be decomposed as

\[
W_r = U_r \Gamma_r .
\]

(9)
where $U_i \in \mathbb{R}^{m \times n}$ is full rank and $\Gamma_i \in \mathbb{C}^{(s-n) \times (s-n)}$ is invertible.

From $VV = I_i$, we know that $V, W_i = 0$. Since $V W_i = \theta$ and $W_i = U_i \Gamma_i$, it follows that $V, U_i, \Gamma_i = \theta$. Hence, $U_i$ is known to span the null space for $V_i$. From Lemma 1, we know that $B$ is independent of the null space of $V_i$. Hence, $B$ is independent of $U_i$, i.e.,

$$R(U_i) \cap R(B) = \emptyset,$$

(10)

where $R(U_i)$ and $R(B)$ represent the range spaces of $U_i$ and $B$, respectively. Define

$$M = \begin{bmatrix} B & U_i \end{bmatrix},$$

(11)

where $M \in \mathbb{R}^{s \times n}$. Since $B$ and $U_i$ are both full rank, we can conclude from (10) and (11) that $M$ is invertible [14]. Moreover, its inverse matrix $M^{-1}$ can be defined as

$$M^{-1} = \begin{bmatrix} B^+ \cr U_i^+ \end{bmatrix},$$

(12)

where $B^+ \in \mathbb{R}^{n \times s}$ and $U_i^+ \in \mathbb{R}^{n \times s}$ can be obtained as the generalized inverses of $B$ and $U_i$. From $M^{-1} M = I_s$, the following equations can be obtained:

$$B^+ B = I_n, \quad U_i U_i = I_{s-n},$$

$$B^+ U_i = \theta, \quad U_i^+ B = \theta.$$

(13)

Now, the switching surface is chosen as

$$\sigma = S x = B^+ x + EU_i U_i x,$$

(14)

where $\sigma \in \mathbb{R}^s$, $S = B^+ \in \mathbb{R}^{s \times n}$, and $E \in \mathbb{R}^{s \times (s-n)}$ is a gain matrix to be determined later. Significantly, from (13) and (14) we have

$$SB = B^+ B + EU_i U_i B = I_n,$$

(15)

and

$$\begin{bmatrix} S \
U_i^+ \end{bmatrix} \begin{bmatrix} B & U_i - BE \end{bmatrix} = \begin{bmatrix} B^+ + EU_i^+ \
U_i^+ \end{bmatrix} \begin{bmatrix} B & U_i - BE \end{bmatrix} = \begin{bmatrix} I_n & 0 \
0 & I_{s-n} \end{bmatrix} = I_s.$$

(16)

Next, we perform the following system transformation:

$$\begin{bmatrix} \sigma \
\eta \end{bmatrix} = \begin{bmatrix} S \
U_i^+ \end{bmatrix} x \quad \text{and} \quad x = B \sigma + (U_i - BE) \eta,$$

(17)

where $\eta = U_i^+ x \in \mathbb{R}^{n \times s}$. System (1) can then be rewritten as

$$\begin{align*}
\dot{\sigma}(t) &= SAB \sigma(t) + SA(U_i - BE) \eta(t) + u(t) + SGd(t), \\
\eta(t) &= U_i^+ AB \eta(t) + U_i^+ A(U_i - BE) \eta(t) + U_i^+ Gd(t), \\
y(t) &= CB \sigma(t) + C(U_i - BE) \eta(t).
\end{align*}$$

(18-20)

If the system is in sliding mode, $\sigma = 0$, and $\eta = 0$, from (18)-(20), then the reduced-order system can be obtained as

$$\begin{align*}
\eta(t) &= A \eta(t) + B d(t) - B \eta \eta(t), \\
y(t) &= C \eta(t) - C BE \eta(t),
\end{align*}$$

(21-22)

where $A = U_i^+ AB \in \mathbb{R}^{n \times n}$, $B = U_i^+ G \in \mathbb{R}^{n \times m}$, $A AB \in \mathbb{R}^{s \times (n-s)}$, $C_1 = CU_i \in \mathbb{R}^{r \times n}$, and $D_2 = CB \in \mathbb{R}^{r \times r}$. Now, we will construct the matrix $E$ such that $A_2 - B_2 E$ is stable and the disturbance attenuation $\|y\| \leq \gamma \|v\|_2$ is guaranteed. This is a standard $H_\infty$ control problem [11,12], and the following theorem gives the entire solution.

**Theorem 1** [11,12]. Consider the system (21)-(22) and assume that the following hold:

1. The pair $(A_2, B_2)$ is controllable, and the pair $(C_1, A_1)$ is observable.
2. If for a given value of $0 \leq \gamma < \infty$, there exists a real, symmetric solution $P \geq 0$ to the algebraic Riccati equation (ARE), that is,

$$P \begin{bmatrix} A_1 - j\omega I & B_2 \\
C^*_1 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_2 \\
C^*_1 & D_2 \end{bmatrix}^* P$$

(23)

and

$$E = (D_2^* D_2)^{-1} D_2^* C_1 + (B_2^* D_2^*)^{-1} B_2^* P,$$

(24)

then $A_2 - B_2 E$ is stable and the condition $\|y\|_2 \leq \gamma \|v\|_2$ can be satisfied. In addition, only matched disturbances exist in

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system (1); the switching surface can be described by \( \sigma = Sx = B^T x \). For this case, the stability analysis in sliding mode can be found in our earlier work [14].

### IV. STATE ESTIMATOR

Consider the plant given in (1). The standard Lu- enberger observer is not suitable for this plant due to the unknown input term \( d(t) \). Robust estimation of states in the presence of unknown input has been discussed in many papers [17-19]. In this section, we shall apply the design technique in output sliding mode control [6-9] to find a suitable transformation matrix which can be used to appropriately transform the original system into a new model containing two sub-systems; one is related to the disturbances, and the other is not. Based on this new model, the reduced-order state-estimator will then be designed.

**Lemma 3** [6]. If assumptions (A3) and (A4) hold, then a matrix \( F \in \mathbb{R}^{n \times m} \) can be found such that the \( n - p \) non-zero eigenvalues for \( A(GFCG)^{-1}FCA \) are located in the left-half complex plane. Moreover, [6] provided a method for selecting the matrix \( F \). Note that the matrix \( F \) must be chosen such that \( FCG \) is nonsingular.

**Lemma 4.** The eigenvalues for two matrices \( A - G(FCG)^{-1}FCA \) and \( A - AG(FCG)^{-1}FC \) are the same.

**Proof:** By means of calculations, we have

\[
\begin{bmatrix}
I_n & 0 \\
\frac{1}{\lambda}FC & I_p
\end{bmatrix}
\begin{bmatrix}
\lambda I_n - A & G \\
-FC & 0
\end{bmatrix}
\begin{bmatrix}
I_n & 0 \\
\frac{1}{\lambda}FC & I_p
\end{bmatrix}
\begin{bmatrix}
\lambda I_n - A + G(FCG)^{-1}FCA & \lambda G \\
0 & FCG
\end{bmatrix}
\]

(25)

and

\[
\begin{bmatrix}
I_n & 0 \\
\frac{1}{\lambda}FC & I_p
\end{bmatrix}
\begin{bmatrix}
\lambda I_n - A - G(FCG)^{-1}FC & 0 \\
-FC & 0
\end{bmatrix}
\begin{bmatrix}
I_n & 0 \\
\frac{1}{\lambda}FC & I_p
\end{bmatrix}
\begin{bmatrix}
\lambda I_n - A - AG(FCG)^{-1}FC & 0 \\
-FC & 0
\end{bmatrix}
\]

(26)

Interestingly, if the determinants on both sides from the above equations (25) and (26) are calculated, then it follows that

\[
\lambda^n \det \left( \lambda I_n - A + G(FCG)^{-1}FCA \right) = \det \left( \lambda I_n - A + AG(FCG)^{-1}FC \right) \det(FCG).
\]

From (27) and (28), we can conclude that the eigenvalues for the two matrices \( A - G(FCG)^{-1}FCA \) and \( A - AG(FCG)^{-1}FC \) are the same.

Hence, from Lemma 4, \( A - AG(FCG)^{-1}FC \) has \( n - p \) non-zero eigenvalues \( \lambda^{\ell}, \ell = 1, 2, \ldots, n - p \), satisfying \( \text{Re} \{ \lambda^{\ell} \} < 0 \), and there exists a full rank matrix \( W_o \in C^{(s-p)(s-p)} \) such that

\[
(A - AG(FCG)^{-1}FC)W_o = W_oJ_o,
\]

(29)

where \( J_o \in C^{(s-p)(s-p)} \) is the Jordan form for the eigenvalues \( \{\lambda^{\ell}, \ldots, \lambda^{\ell+p}\} \). Since \( G \) is full rank and

\[
(A - AG(FCG)^{-1}FC)G = 0,
\]

(30)

\( G \) can be treated as the right eigenvector matrix of \( A - AG(FCG)^{-1}FC \) corresponding to \( p \) zero eigenvalues. Combining (29) and (30), we have

\[
[A - AG(FCG)^{-1}FC] [W_o \ G] = [W_o \ G] \begin{bmatrix} J_o & 0 \\ 0 & 0 \end{bmatrix}
\]

(31)

From (31), \( [W_o \ G] \) is the right eigenvector matrix and is invertible. Similar to Lemma 2, the matrix \( W_o \) can be decomposed into

\[
W_o = U_o \Gamma_o,
\]

(32)

where \( U_o \in \mathbb{R}^{n \times (s-p)} \) is full rank and \( \Gamma_o \in C^{(s-p)(s-p)} \) is invertible. Note that \( [U_o \ G] \) is a nonsingular matrix because \( [W_o \ G] \) is invertible. Let

\[
U_o^\dagger = [U_o \ G]^{-1},
\]

(33)

where \( G^\dagger \in \mathbb{R}^{(s-p) \times p} \) and \( U_o^\dagger \in \mathbb{R}^{(n-p) \times (n-p)} \). Clearly,

\[
U_o U_o^\dagger + GG^\dagger = I_o,
\]

(34)

\[
G^\dagger G = I_p,
\]

(35)

From (29), (32), and (35), it is easy to verify that

\[
U_o^\dagger (A - AG(FCG)^{-1}FC) = \Gamma_o J_o \Gamma_o^{-1} U_o^\dagger = R U_o^\dagger,
\]

(36)
I FCGF U z
UB u
I FCGF U z
UB u
2
UB u
I FCGF U z
UB u
2
UB u
I FCGF U z
UB u
2
UB u
I FCGF U z
UB u
2
UB u
where \( R = \Gamma, J, \Gamma \) \( R \) is a stable matrix with eigenvalues \( \lambda_{\alpha}, \ldots, \lambda_{\alpha} \). From (34) and (35), it is easy to check that

\[
\begin{bmatrix}
(FCG)^{-1} FC \\
U_1^T
\end{bmatrix} G \left(I - (FGFC)^{-1} FC\right) U_1 = I. \tag{37}
\]

Now, we define the system transformation as

\[
z = \begin{bmatrix} z_1 \\
z_2 \end{bmatrix} = \begin{bmatrix}
(FCG)^{-1} FC \\
U_1^T
\end{bmatrix} x = \begin{bmatrix}
U_1^T Fy \\
U_1^T x
\end{bmatrix} \tag{38}
\]

and

\[x = Gz_1 + (I - (FGFC)^{-1} FC) U_1 z_2;
\]

\[
= G(FCG)^{-1} Fy + (I - (FGFC)^{-1} FC) U_1 z_2, \tag{39}
\]

where \( z_1 = (FCG)^{-1} FCx \in \mathbb{R}^p \) and \( z_2 = U_1^T x \in \mathbb{R}^{n-p} \). Evidently, in order to estimate \( x \), it is necessary to set up a reduced-order observer for \( z_2 \). Based on (39) only. In the following, a state-estimator for \( z_2 \) is proposed.

First, pre-multiplying (1) by \( U_1^T \) and using (36), we have

\[
\begin{aligned}
\dot{z}_2 &= U_1^T \dot{x} + U_1^T Bu + U_1^T Gd \\
&= U_1^T (A - AG(FCG)^{-1} FC)x + U_1^T AG(FCG)^{-1} Fy + U_1^T Bu \\
&= Rz_2 + U_1^T AG(FCG)^{-1} Fy + U_1^T Bu. \tag{40}
\end{aligned}
\]

From (40), it is clear that the disturbance term \( d \) does not exist in the dynamics of \( z_2 \). Hence, our novel reduced-order observer is constructed as

\[
\dot{z}_2 = Rz_2 + U_1^T AG(FCG)^{-1} Fy + U_1^T Bu, \tag{41}
\]

where \( z_2 \) is the estimate of \( z_2 \). We next define the estimation error as \( e = z_2 - \tilde{z}_2 \), and from (40) and (41), we have

\[
\dot{e} = Re. \tag{42}
\]

Since the matrix \( R \) is stable, it follows that \( e(t) = z_2(t) - \tilde{z}_2(t) \rightarrow 0 \) for \( t \rightarrow \infty \). This means that \( \tilde{z}_2 \) successfully estimates \( z_2 \). Finally, we define the estimate of \( x \) as

\[
\hat{x} = G(FCG)^{-1} Fy + (I_1 - (FGFC)^{-1} FC) U_1^T \tilde{z}_2, \tag{43}
\]

where \( \hat{x} \in \mathbb{R}^n \). Let the state error be \( e = x - \hat{x} \). From (39) and (43), we have

\[
e = (I_1 - (FGFC)^{-1} FC) U_1^T e = Me, \tag{44}
\]

where \( M = (I_1 - (FGFC)^{-1} FC) U_1^T \in \mathbb{R}^{n-x} \). Since the condition \( e(t) \rightarrow 0 \) for \( t \rightarrow \infty \) is guaranteed, it follows from (44) that \( \hat{x}(t) \rightarrow x(t) \) for \( t \rightarrow \infty \).

\section{V. CONTROLLER DESIGN}

Since the true states \( x \) cannot be measured directly, the estimated states \( \hat{x} \) should be used in the control algorithm. Hence, the overall control scheme becomes

\[
\hat{\phi} = S \hat{x} = B^T \hat{x} + E U_1^T \hat{x}, \tag{45}
\]

\[
u = -SA \hat{x} - \left(\beta \left[SG\right] + \rho + k\right) \frac{\tilde{\phi}}{\left[\tilde{\phi}\right]}, \tag{46}
\]

\[
\dot{x} = G(FCG)^{-1} Fy + (I_1 - (FGFC)^{-1} FC) U_1^T \tilde{z}_2, \tag{47}
\]

\[
\dot{\tilde{z}}_2 = Rz_2 + U_1^T AG(FCG)^{-1} Fy + U_1^T Bu \tag{48}
\]

where \( k > 0 \), given below, is a constant. From (44) and (45), we have \( \sigma = \hat{\phi} + SM e \). Since the system (42) is stable, there exist real positive numbers \( \sigma \) such that

\[
\left\|e(t)\right\| \leq \gamma e^{-\omega t} \left\|e(0)\right\|, \tag{49}
\]

Theorem 2. Consider the plant (1) and the control algorithm described in (45)-(48). If the control inputs are designed as in (46) with

\[
k \geq \gamma \left[\left\|SA - SM R\right\|\right] \left\|e(0)\right\|, \tag{50}
\]

then we have \( \sigma(t) \rightarrow 0 \) for \( t \rightarrow \infty \).

Proof. Using (1) and (44), we can yield the derivative of \( \hat{\phi} \) as

\[
\dot{\hat{\phi}} = Ax + Bu + Gd - MRe \tag{51}
\]

\[
\dot{\hat{\phi}} = \dot{\hat{x}} = AKx + Bu + Gd + \left\|AM - MR\right\| e, \tag{51}
\]

From (45) and (51), the dynamics for \( \dot{\phi} \) can be obtained as

\[
\dot{\phi} = S \hat{\phi} + u + SGd + (SA - SMR) e \tag{52}
\]

Pre-multiplying (52) by \( \tilde{\phi} \) obtains

\[
\tilde{\phi} \dot{\tilde{\phi}} = -\left\|\left[SG\right] + \rho + k\right\| \|e\| + \tilde{\phi} \tilde{\phi} SGd
\]

\[
\leq -\rho \tilde{\phi} \|e\|. \tag{52}
\]
Hence, the system will reach $\dot{\sigma} = 0$ in finite time [2]. Since $\sigma = \sigma + SM \varepsilon$, and $e(t) \to 0$ for $t \to \infty$, we can conclude $\sigma(t) \to 0$ for $t \to \infty$. 

For alleviating unwanted chattering, the term $2\dot{\varepsilon}$ can be modified by using the saturation function $\dot{\varepsilon}(\varepsilon)$, where $\varepsilon > 0$ [3]. It is easy to conclude that once the system gets into the sliding layer $2\varepsilon < \sigma$, its trajectory will be completely restricted in this layer [22].

VI. SIMULATION RESULTS

Consider the following plant introduced as:

$$\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} -0.277 & 1 & -0.0002 \\ -1.71 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 6.67 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$d = 0.5 \times (\sin(\pi t) + \cos(\pi t/4))$$

The disturbance term $d(t)$ is $L_2$-bounded, and it satisfies $\|d\|_2 \leq 1$. Moreover, it is easy to check that the system (54) satisfies the four assumptions (A1)-(A4). First, we choose $\lambda_1 = -3 + i$, $\lambda_2 = -3 - i$, and $\delta_0 = 1$. Since these eigenvalues for $A$ are $(-0.67, -0.2275 \pm 4.1349)$, the three important conditions (C1)-(C3), introduced in section 3, are satisfied. With the help of the MATLAB software package, we found $B^\dagger$ and $U^\dagger$ to be

$$B^\dagger = \begin{bmatrix} 0.1067 & -0.0681 & 0.1499 \end{bmatrix},$$

$$U^\dagger = \begin{bmatrix} -1.0394 & -0.3440 & 0 \\ 0.6528 & 10432 & 0 \end{bmatrix}.$$  

Setting $\gamma = 0.1$ and solving the ARE equation (23) yields the symmetric, positive-definite matrix $P$ and the matrix $E$ as

$$P = \begin{bmatrix} 1.3099 & 0.3944 \\ 0.3944 & 0.2411 \end{bmatrix} > 0 \quad \text{and} \quad E = \begin{bmatrix} 0.1227 & -0.1299 \end{bmatrix}.$$  

Hence, the design goal is to find the control inputs $u$, using the system outputs $y$ only, such that

$$\|y\|_2 \leq \gamma \|u\|_2 = 0.1.$$  

By means of calculations, the matrix $S = B^\dagger + EU^\dagger_1$ can be obtained as

$$S = \begin{bmatrix} 0.0640 & -0.2458 & 0.1499 \end{bmatrix}.$$  

Since none of the system states $x$ are measurable, we can begin to design the state estimator. We choose

$$F = \begin{bmatrix} -1 & 2 \end{bmatrix}$$

and verify that the eigenvalues for $A - AG(FCG)^{-1} FC$ are $\{0, -3.0815 \pm 5.1314i\}$. After performing the steps proposed in Section 4, we found that the corresponding matrices for the state-estimator are

$$U^\dagger_2 = \begin{bmatrix} -1.0028 & 0 \\ -0.0249 & 0.7454 \\ 6.67 \end{bmatrix},$$

$$R = \begin{bmatrix} -0.2102 & -2.6905 \\ 12.8511 & -5.9528 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.9972 & 0 \\ -0.0666 & 2.6833 \\ -0.0333 & 1.3416 \end{bmatrix},$$

Hence, the reduced-order observer is constructed as

$$\dot{x} = (FCG)^{-1} FY + U^\dagger_2 Bu,$$

and the estimated states $\hat{x}$ are

$$\hat{x} = G(FCG)^{-1} FY + \hat{M}\hat{z}.$$  

The control algorithms are designed as

$$\dot{\sigma} = \hat{S}\hat{x},$$

$$u = -SA\hat{x} - (k + \beta) \|SG\| + \rho \hat{s}(\sigma, e),$$

where $\rho = 0.1, k = 3$, and $\epsilon = 0.05$. Figures 1-5 show simulation results obtained using the initial state $x(0) = [0 \ 2 \ 0]'$. The time responses of $y$ and $\|y\|_2$ are shown in Fig. 1 and Fig. 2, respectively. The control input $u$ is shown in Fig. 3. Figure 4 shows the trajectory of $\sigma$. From Fig. 2, one can see that $\|y\|_2 \leq 0.1$ and, hence, that the control goal (59) is achieved. The trajectories of true states and estimated states are shown in Fig. 5, which clearly shows that the estimated states $\hat{x}(t)$ approach the true states $x(t)$ as $t \to \infty$. 

\[\text{Equation} (53)\]
Fig. 1. Outputs $y_1$ and $y_2$.

Fig. 2. The response of $y_2$.

Fig. 3. Control input $u$.

Fig. 4. The trajectory of $\sigma$.

Fig. 5. (a) State $x_1$ and estimated state $\hat{x}_1$, (b) State $x_2$ and estimated state $\hat{x}_2$, (c) State $x_3$ and estimated state $\hat{x}_3$.

VII. CONCLUSIONS

In this paper, we have presented an output feedback sliding mode control for linear systems with mismatched disturbances via $H_\infty$ theory. Applying the $H_\infty$ approach and solving the algebraic Riccati equation, we have found that the desired switching surface can guarantee some level of robust stability and disturbance rejection for the system once it is in sliding mode. In addition, a state observer has been developed to effectively estimate the system states. Hence, the overall sliding mode control design can be divided into two separate
steps. First, the observer is used to estimate the states, and then the control law is designed with respect to the estimated states.

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