PID CONTROLLER DESIGN WITH GUARANTEED GAIN AND PHASE MARGINS
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ABSTRACT

This paper considers the problem of designing PID controllers that achieve the guaranteed gain and phase margin specifications for a given but arbitrary single-input single-output plant. Using definitions of gain and phase margins, the given specifications are translated into simultaneous stabilization of two families of polynomials. Thereafter, an extension of the results of PID stabilization to the case of complex polynomials is shown to be useful in providing a solution to this simultaneous stabilization problem. A computational characterization of all admissible PID controllers is given. This characterization for all admissible PID controllers involves the solution of a linear programming problem. An example is presented to illustrate use of the design procedure in detail.

KeyWords: PID control, gain margin, phase margin, linear programming.

I. INTRODUCTION

The primary requirement of any control system is that closed-loop stability be guaranteed. In addition to possessing closed-loop stability, a control system must be designed to meet certain robustness and/or performance specifications. In classical control design, gain and phase margins are important frequency-domain measures used to assess robustness and performance [1]. Thus, the problem of designing a controller to ensure that the resulting control system achieves the guaranteed gain and satisfies the phase margin specifications arises in practical classical control design.

The PID controller is the most widely used structure in industrial applications. This prevalence of the PID controller is due to its simplicity of controller structure and ability to solve many practical control problems. Despite the fact that many PID tuning methods are available to achieve the specified gain and phase margins, most of the methods proposed so far are either narrow in scope because they are limited to some specific classes of plants [2-4] or based on less efficient graphical approaches [5,6]. The major obstacle in the design of PID controllers to achieving the desired specifications, in any sense whatsoever, has been the difficulty of characterizing the entire set of stabilizing PID controllers. Solving this problem is a necessary first step in any rational design of PID controllers based on achievable specifications. This PID stabilization problem has been solved using the methods presented in [7,8], which provide a computational characterization of all stabilizing PID controllers for a given but arbitrary plant. This characterization is based on a fundamental and new result generalizing the classical Hermite-Biehler Theorem [9] to the case of not necessarily Hurwitz real polynomials. Recently, based on a complex version of the generalized Hermite-Biehler Theorem, an extension of the results of PID stabilization has been developed in [10] for the case with an even more general setting, one which involves complex polynomials. In this paper, we show that this extension of the results of PID stabilization can be exploited to determine PID gain values, if any exist, for which the closed-loop system is internally stable and the specified gain and phase margins are guaranteed for a given plant. From definitions of gain and phase margins, we first formulate the problem to be solved in terms of simultaneous stabilization of two families of polynomials. Then the extension of the results of PID stabilization to the case of complex polynomials is used to provide a computational characterization of all admissible PID controllers. This characterization for all admissible PID controllers involves the solution of a linear programming problem.

The paper is organized as follows. In Section 2, we show that the problem of interest to us in this paper can be translated into simultaneous stabilization of two families of polynomials, and that our extension of the results of PID stabilization to the case of complex polynomials can be used to solve the simultaneous stabilization problem in question. This extension of the results of PID stabilization is based on a complex version of the generalized Hermite-Biehler Theorem. In Section 3, we present an appropriate generalization of the Hermite-Biehler Theorem applicable to complex polynomials. In Section 4, we consider the extended PID stabilization problem of finding admissible values of \((k_p, k_i, k_d)\), if any exist, for which a complex polynomial of the form \(L(s) + (k_p s^2 + k_p s + k_i)M(s)\) is Hurwitz. A computational characterization of all admissible \((k_p, k_i, k_d)\) values is provided. These results immediately lead to a solution to the problem of synthesizing PID controllers to achieve the guaranteed gain and phase margins. In Section 5, we present the detailed synthesis procedure, and a simple example is given to illustrate the procedure. Finally, Section 6 contains some concluding remarks.

II. PROBLEM FORMULATION

To this end, consider the standard feedback control system shown in Fig. 1.

Here \(r\) is the command signal, \(y\) is the output, \(G(s) = \frac{N(s)}{D(s)}\) is the plant to be controlled, \(N(s)\) and \(D(s)\) are coprime polynomials, and \(C(s)\) is the controller used to make the closed-loop system stable and to satisfy the desired design specifications. In this paper, the chosen controller \(C(s)\) is a PID controller, \(i.e.,\)

\[
C(s) = k_p + \frac{k_i}{s^2} + k_d s = \frac{k_p + k_i s + k_d s^2}{s}.
\]

(1)

\(A_m\) and \(\theta_m\) denote the specified gain and phase margins, respectively. The objective of this paper is to determine the admissible PID gain values \((k_p, k_i, k_d)\), if any exist, for which the gain margin is at least \(A_m\) and the phase margin is at least \(\theta_m\). From the definition of the gain margin, the closed-loop stability must be preserved even if the loop gain is increased by a factor of up to \(A_m\). The phase margin tells us that a phase lag of up to \(\theta_m\) can be added to control the loop before the closed-loop system becomes unstable. Thus, the admissible PID gain values \((k_p, k_i, k_d)\) must satisfy the following two conditions:

1. \(s D(s) + A (k_p s^2 + k_p s + k_i) N(s)\) is Hurwitz for all \(A \in [1, A_m]\);
2. \(s D(s) + e^{\jmath \theta} (k_p s^2 + k_p s + k_i) N(s)\) is Hurwitz for all \(\theta \in [0, \theta_m]\).

We will now formulate the problem to be solved as simultaneous stabilization of two families of polynomials. In general, for a fixed \(\theta\), \(s D(s) + e^{\jmath \theta} (k_p s^2 + k_p s + k_i) N(s)\) is a complex polynomial. The results of [7,8], which are based on a generalized Hermite-Biehler Theorem for real polynomials [8,12], cannot be used to determine the values of \((k_p, k_i, k_d)\) for which \(s D(s) + e^{\jmath \theta} (k_p s^2 + k_p s + k_i) N(s)\) is Hurwitz. Hence, our approach to solving this simultaneous stabilization problem is based on an extension of the results of PID stabilization to the case of complex polynomials [10]. This extended PID stabilization problem can be used to determine the real values of \((k_p, k_i, k_d)\), if any exist, for which \(L(s) + (k_p s^2 + k_p s + k_i) N(s)\) is Hurwitz, where \(L(s)\) and \(M(s)\) are the specified complex polynomials. In [10], a computational characterization of all admissible \((k_p, k_i, k_d)\) values was provided. This characterization is based on a new result generalizing the classical Hermite-Biehler Theorem [9] to the case of not necessarily Hurwitz complex polynomials.

III. A GENERALIZATION OF THE HERMITE-BIEHLER THEOREM FOR COMPLEX POLYNOMIALS

Several generalizations of the Hermite-Biehler Theorem for real polynomials were presented in [8,12]. Complex versions of the generalized Hermite-Biehler Theorem for complex polynomials with real leading coefficients were given in [13]. To solve the complex polynomial stabilization problem described in the previous section, we need an appropriate generalization of the Hermite-Biehler Theorem for complex polynomials with complex leading coefficients. In this section, we will present such a generalization of the Hermite-Biehler Theorem. This complex version of the generalized Hermite-Biehler Theorem is a minor extension of the results presented in [13] to the case of a more general setting. It can be derived in a straightforward manner from the results presented in [13]. The result is, therefore, simply stated without proof. We will now first establish some notations. The standard signum function \(\text{sgn}: \mathbb{R} \to \{-1, 0, 1\}\) is defined by

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]
Now, consider a complex polynomial \( \delta(s) \) of degree \( n \):
\[
\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \ldots + \delta_n s^n,
\]
\( \delta_i \in \mathbb{C}, i = 0,1,\ldots,n, \delta_n \neq 0. \)

Define the signature of the polynomial \( \delta(s) \) denoted by \( \sigma(\delta) \), as
\[
\sigma(\delta) := \text{the number of open left half plane roots of } \delta(s) - \text{the number of open right half plane roots of } \delta(s).
\]

Now, for every frequency \( \omega \in \mathbb{R} \), \( \delta(j\omega) \) is a point in the complex plane. Let \( p(\omega) \) and \( q(\omega) \) be two functions defined pointwise by \( p(\omega) = \text{Re}[\delta(j\omega)] \), \( q(\omega) = \text{Im}[\delta(j\omega)] \). With this definition, we have
\[
\delta(j\omega) = p(\omega) + jq(\omega) \forall \omega \in \mathbb{R}.
\]

Furthermore, define
\[
\delta_\omega(\omega) = p_\omega(\omega) + jq_\omega(\omega),
\]
where
\[
p_\omega(\omega) = \frac{p(\omega)}{f(\omega)}, \quad q_\omega(\omega) = \frac{q(\omega)}{f(\omega)}; \quad f(\omega) = (1 + \omega^2)^{n/2}
\]
We will now state a generalization of the Hermite-Biehler Theorem concerning \( \sigma(\delta) \).

**Theorem 3.1.** Let \( \delta(s) \) be a given complex polynomial of degree \( n \). Let \( \delta_i \) denote the leading coefficient of \( \delta(s) \). Let \( \omega_1 < \omega_2 < \ldots < \omega_{m-1} \) be the real, distinct finite zeros of \( q(\omega) \) with odd multiplicities. Also, define \( \omega_0 = -\infty \) and \( \omega_m = +\infty \). Then
\[
\sigma(\delta) = \begin{cases} 
\frac{1}{2} \{ \text{sgn}(p_\omega(\omega_0)) \cdot (-1)^{m-1} + 2 \sum_{i=1}^{m-1} \text{sgn}(p_\omega(\omega_i)) \} \\
\cdot (-1)^{m-1-i} \cdot \text{sgn}(q(\infty)) \\
\text{if } n \text{ is even and } \delta_\omega \text{ is purely real, or if } n \text{ is odd and } \delta_\omega \text{ is purely imaginary.} 
\end{cases}
\]

IV. AN EXTENSION OF PID STABILIZATION TO COMPLEX POLYNOMIALS

Now, we will consider a complex polynomial of the form
\[
\delta(s,k_p,k_i,k_d) = L(s) + (k_p s^2 + k_i s + k_d)M(s), \tag{3}
\]
where \( L(s) \) and \( M(s) \) are two arbitrary complex polynomials. In this section, we will focus on the problem of determining those real values of \( (k_p, k_i, k_d) \), if any exist, for which (3) is Hurwitz stable. Note that by setting \( L(s) = sD(s) \) and \( M(s) = N(s) \), the problem stated above becomes that of PID stabilization. We will, therefore, refer to stabilization of (3) as the extended PID stabilization problem. In this section, we will make use of Theorem 3.1 to provide a complete solution to the extended PID stabilization problem.

To this end, we consider
\[
L(s) = (a_0 + j b_0) + (a_1 + j b_1)s + \cdots + (a_{n-1} + j b_{n-1})s^{n-1} + (a_n + j b_n)s^n, \quad a_i + jb_i \neq 0; \tag{4}
\]
\[
M(s) = (c_0 + j d_0) + (c_1 + j d_1)s + \cdots + (c_{n-1} + j d_{n-1})s^{n-1} + (c_n + j d_n)s^n, \quad c_m + j d_m \neq 0. \tag{5}
\]

It is clear from (3) that, in general, both the real and imaginary parts of \( \delta(j\omega, k_p, k_i, k_d)M(j\omega) \) will depend on all three parameters, \( k_p, k_i \), and \( k_d \). Proceeding as in [7,8], we first construct a polynomial, say \( M'(s) \), such that the real part of \( \delta(j\omega, k_p, k_i, k_d)M'(j\omega) \) depends on \( k_p, k_i \) and the imaginary part of \( \delta(j\omega, k_p, k_i, k_d)M'(j\omega) \) depends on \( k_d \). To do so, we consider the following “real-imaginary” decompositions\(^1\) of \( L(s) \) and \( M(s) \):
\[
L(s) = L_\ell(s) + L_\imath(s), \quad M(s) = M_\ell(s) + M_\imath(s),
\]
where
\[
L_\ell(s) = a_0 + j b_1 s + a_2 s^2 + j b_3 s^3 + \cdots, \quad L_\imath(s) = j b_0 + a_1 s + j b_2 s^2 + a_3 s^3 + \cdots, \tag{6}
\]
\[
M_\ell(s) = c_0 + j d_1 s + c_2 s^2 + j d_3 s^3 + \cdots, \quad M_\imath(s) = j d_0 + c_1 s + j d_2 s^2 + c_3 s^3 + \cdots. \tag{7}
\]

Define
\[
M_\ell^*(s) = M_\ell(s) - M_\imath(s).
\]

\(^1\)This terminology refers to the fact that if the polynomial being decomposed is evaluated at \( s = j\omega \), then one of the components evaluates out to the real part, while the other component evaluates out to the imaginary part.
Also let \( n \) and \( m \) be the degrees of \( \delta(s, k_p, k_i, k_d) \) and \( M(s) \), respectively. Multiplying \( \delta(s, k_p, k_i, k_d) \) by \( M(s) \) and considering the resulting polynomial, it is clear that

\[
\sigma(\delta(s, k_p, k_i, k_d)M'(s)) = \sigma(\delta(s, k_p, k_i, k_d)) + \sigma(M'(s)).
\]

Now, \( \delta(s, k_p, k_i, k_d) \) of degree \( n \) is Hurwitz if and only if \( \sigma(\delta(s, k_p, k_i, k_d)) = n \). Therefore, we have the following lemma.

**Lemma 4.1.** \( \delta(s, k_p, k_i, k_d) \) is Hurwitz if and only if

\[
\sigma(\delta(s, k_p, k_i, k_d)M'(s)) = n + \sigma(M'(s)).
\]  

(4)

Our task now is to determine those values of \( k_p, k_i, \) and \( k_d \), if any exist, for which (4) holds. Evaluating \( \delta(s, k_p, k_i, k_d) \) and \( M'(s) \) at \( s = j\omega \), and making use of the real-imaginary decompositions introduced earlier, we obtain

\[
\delta(j\omega, k_p, k_i, k_d) = \left[ L_n(j\omega) + L_i(j\omega) \right] + \left[ (k_i - k_d\omega^2) + jk_p\omega \right][M_n(j\omega) + M_i(j\omega)] + \sigma_{\delta}(\delta(s, k_p, k_i, k_d)).
\]

Since \( L_n(j\omega) \) and \( M_n(j\omega) \) are purely real, while \( L_i(j\omega) \) and \( M_i(j\omega) \) are purely imaginary, we can write

\[
\delta(j\omega, k_p, k_i, k_d)M'(j\omega) = p(\omega, k_p, k_i, k_d) + jq(\omega, k_p, k_i, k_d),
\]

where

\[
p(\omega, k_p, k_i, k_d) = p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega),
q(\omega, k_p, k_i, k_d) = q_1(\omega) + k_pq_2(\omega),
\]

\[
p_1(\omega) = L_n(j\omega)M_n(j\omega) - L_i(j\omega)M_i(j\omega),
q_1(\omega) = \frac{1}{j}[L_i(j\omega)M_n(j\omega) - L_n(j\omega)M_i(j\omega)],
\]

\[
p_2(\omega) = M_n^2(j\omega) - M_i^2(j\omega),
q_2(\omega) = \omega[M_n^2(j\omega) - M_i^2(j\omega)].
\]

Also, define

\[
p_j(\omega, k_p, k_i, k_d) = \frac{p(\omega, k_p, k_i, k_d)}{(1 + \omega^2)^{\frac{m+n}{2}}},
q_j(\omega, k_p, k_i, k_d) = \frac{q(\omega, k_p, k_i, k_d)}{(1 + \omega^2)^{\frac{m+n}{2}}}.
\]

We first note that \( k_i, k_d \) appear affinely in \( p(., ., .) \), while \( k_p \) appears affinely in \( q(., .) \). Furthermore, for every fixed \( k_p \), the zeros of \( q(\omega, k_p) \) will not depend on \( k_i \) or \( k_d \), so Theorem 3.1 can be used to determine \( \sigma(\delta(s, k_p, k_i, k_d)M'(s)) \). Furthermore, since \( \sigma(M'(s)) \) is known, the admissible values of \( k_i \) and \( k_d \) can be determined from (4).

Since there are two variables here, a linear programming problem has to be solved for each fixed \( k_p \). As \( k_p \) is varied, we will have a one parameter family of linear programming problems to solve. The formal statement of our main result for extension of the PID stabilization problem involves certain strings of the real numbers 0, 1 and -1. These strings are used to essentially capture all the different possibilities for the signs of \( p(j\omega, k_p, k_i, k_d) \) at the real zeros of \( q(\omega, k_p, k_d) \) (with odd multiplicities) for a fixed \( k_p \). Among these possibilities, we are interested in only those that, when substituted into the expression for \( \sigma(\delta(s, k_p, k_i, k_d)M'(s)) \) and calculated using formula (2), yield value for which (4) holds.

For clarity of presentation, we will first introduce these strings and provide some definitions before formally stating our main result of the extended PID stabilization problem.

**Definition 4.1.** Let \( m, n, q(\omega, k_p) \) be as already defined. Let \( \xi \) denote the leading coefficient of \( \delta(s, k_p, k_i, k_d)M'(s) \). For a given fixed \( k_p \), let \( \omega_0 < \omega_1 < \ldots < \omega_{m-1} \) be the real, distinct finite zeros of \( q(\omega, k_p) \) with odd multiplicities. Also, define \( \omega_0 = -\infty \) and \( \omega_\infty = +\infty \). Define a sequence of numbers \( i_0, i_1, i_2, \ldots, i_l \) as follows:

\[
A_{\xi} = \begin{cases}
\{i_0, i_1, \ldots, i_l\} & \text{if } m + n \text{ is even and } \xi \text{ is purely real, or if } m + n \text{ is odd and } \xi \text{ is purely imaginary,} \\
\{i_1, i_2, \ldots, i_{l+1}\} & \text{if } m + n \text{ is even and } \xi \text{ is not purely real, or if } m + n \text{ is odd and } \xi \text{ is not purely imaginary,}
\end{cases}
\]

where

\[
A_{\xi} = \begin{cases}
\{0\} \text{ if } j\omega_t \text{ is not a } j\omega \text{–axis root of } M'(s), \\
\{-1, 1\} \text{ otherwise.}
\end{cases}
\]

\[
\text{for } t = 0, 1, \ldots, l, i_t \in A_{\xi}. \tag{5}
\]

**Remark 4.1.** It can be easily shown that the length of strings in \( A_{\xi} \) is independent of the values of \( (k_p, k_i, k_d) \).

Next we introduce the set \( A_{\xi}(\lambda) \) of strings in \( A_{\xi} \), with a prescribed “signature” \( \lambda \). To do so, we first need to define the “signature” \( \lambda(\tilde{\lambda}) \) associated with any element \( \tilde{\lambda} \in A_{\xi} \). This definition follows from Theorem 4.1.

**Definition 4.2.** Let \( m, n, \xi, q(\omega, k_p) \) be as already defined. For a given fixed \( k_p \), let \( \omega_0 < \omega_1 < \ldots < \omega_{m-1} \) be the real, distinct finite zeros of \( q(\omega, k_p) \) with odd multiplicities. Also, define \( \omega_0 = -\infty \) and \( \omega_\infty = +\infty \). For
Definition 4.3. The set of strings in $\mathcal{A}_p$ with a prescribed signature $\lambda = \psi$ is denoted by $A_p(\psi)$. For a given fixed $k_p$, we also define the set of admissible strings for the extended PID stabilization problem as

$$F_{**}^p = A_p(\psi + \sigma(M'(s))).$$

We are now ready to state the main result of this section.

Theorem 4.1. (Main Result on Extended PID Stabilization) [10] The extended PID stabilization problem, with a fixed $k_p$, is solvable for given complex polynomials $M(s)$ and $M(s)$ if and only if the following conditions hold:

(i) $F_{**}^p$ is not empty where $F_{**}^p$ is as already defined, i.e., at least one feasible string exists, and

(ii) there exists a string $I$, either $I = \{i_0, i_1, \ldots, i_l\}$ or $\{i_1, i_2, \ldots, i_{l-1}\} \in F_{**}^p$, and values of $k_i$ and $k_j$ such that either $\forall \, t = 0, 1, \ldots, \frac{m+n}{2}$, $p(\omega, k_i, k_j) < 0$.

where $p(\omega, k_i, k_j)$ is as already defined. Furthermore, if there exist values of $k_i$ and $k_j$ such that the above condition is satisfied for the feasible strings $I_1, I_2, \ldots, I_s \in F_{**}^p$, then the set of stabilizing $(k_i, k_j)$ values corresponding to the fixed $k_p$ is the union of the $(k_i, k_j)$ values satisfying (7) for $I_1, I_2, \ldots, I_s$.

Proof. From (4), we know that $\delta(s, k_p, k_i, k_j)$ is Hurwitz if and only if

$$\sigma(\delta(s, k_p, k_i, k_j)M'(s)) = n + \sigma(M'(s)).$$

Thus, for a fixed $k_p$, it follows that $\delta(s, k_p, k_i, k_j)$ is Hurwitz if and only if there exists $\bar{\omega} \in F_{**}^p$, where $\bar{\omega} = \{i_0, i_1, \ldots, i_l\}$ when $m + n$ is even and $\xi$ is purely real, or when $m + n$ is odd and $\xi$ is purely imaginary, and $I$ satisfies

$$i_t = sgn[p_t(\omega, k_i, k_j)], t = 0, 1, \ldots, l.$$  

Otherwise, $\bar{\omega} = \{i_1, i_2, \ldots, i_{l-1}\}$ when $m + n$ is even and $\xi$ is not purely real, or when $m + n$ is odd and $\xi$ is not purely imaginary, and $I$ satisfies

$$i_t = sgn[p_t(\omega, k_i, k_j)], t = 1, 2, \ldots, l - 1.$$  

Let us now consider three different cases:

Case (I) $i_t > 0$:

If $i_t > 0$, then the stability requirement is

$$p(\omega, k_i, k_j) > 0.$$  

It follows that $p(\omega, k_i, k_j) < 0$.

Case (II) $i_t < 0$:

If $i_t < 0$, then the stability requirement is

$$p(\omega, k_i, k_j) < 0.$$  

Once again, it follows that $p(\omega, k_i, k_j) > 0$.

Case (III) $i_t = 0$: This case can only happen when $j\omega$ is a $\omega$-axis root of $M'(s)$ (see the definition of $\mathcal{A}_p$). Hence, it follows that $M'(j\omega) = M(j\omega) = 0$, so $p_1(\omega) = p_2(\omega) = p(\omega, k_i, k_j) = 0$ for all $(k_i, k_j)$. Thus, this case does not lead to any constraints on $(k_i, k_j)$.

This completes the proof of the necessary and sufficient conditions for the existence of the stabilizing $(k_i, k_j)$ for a fixed $k_p$. The set of all stabilizing $(k_i, k_j)$ values is now obtained by taking the union of the $(k_i, k_j)$ that are determined from the feasible strings that satisfy condition (ii).

V. DESIGN PROCEDURE

In Section 2, we formulated the problem of synthesizing PID controllers that ensure closed-loop stability and achieve the guaranteed gain and phase margins through the simultaneous stabilization of two families of polynomials. In this section, we now use the results of extended PID stabilization given in the previous section to solve the simultaneous stabilization problem in question. The design procedure is presented in the following:
For any fixed $k_p$ and any fixed $A^* \in [1, A_n]$, by setting $L(s) = sD(s)$ and $M(s) = A^*N(s)$, and using Theorem 4.1, we can solve a linear programming problem to determine the admissible set of $(k_i, k_d)$ values, if any exist, for which $sD(s) + A^*(k_p s^2 + k_d s + k_i)N(s)$ is Hurwitz. Let this set be denoted by $\mathcal{G}M(k_p, A^*)$. By keeping $k_p$ fixed and letting $A^*$ vary in $[1, A_n]$, we can determine the set of admissible $(k_i, k_d)$ values such that $A^*$ is the guaranteed gain margin of the resulting control systems. This set is denoted by $\mathcal{G}M(k_p)$ and defined as

$$\mathcal{G}M(k_p) = \bigcap_{A \in [1, A_n]} \mathcal{G}M(k_p, A).$$

Again with such a fixed $k_p$ and any fixed $\theta' \in [0, \theta_n]$, by setting $L(s) = sD(s)$ and $M(s) = e^{j\theta'}N(s)$, and using Theorem 4.1, we can solve a linear programming problem to determine the admissible set of $(k_i, k_d)$ values for which $sD(s) + e^{j\theta'}(k_p s^2 + k_d s + k_i)N(s)$ is Hurwitz. Let this set be denoted by $\mathcal{P}M(k_p, \theta')$. By keeping $k_p$ fixed and letting $\theta'$ vary in $[0, \theta_n]$, we can determine the set of admissible $(k_i, k_d)$ values such that $\theta_n$ is the guaranteed phase margin of the resulting control systems. This set is denoted by $\mathcal{P}M(k_p)$ and defined as

$$\mathcal{P}M(k_p) = \bigcap_{\theta \in [0, \theta_n]} \mathcal{P}M(k_p, \theta).$$

Now for a fixed $k_p$, the set of all admissible $(k_i, k_d)$ values for which $A_n$ and $\theta_n$ are, respectively, the guaranteed gain and phase margins of the resulting control systems, denoted by $\mathcal{S}(k_p)$, is given by

$$\mathcal{S}(k_p) = \mathcal{G}M(k_p) \cap \mathcal{P}M(k_p).$$

Then, the set of all admissible $(k_p, k_i, k_d)$ values can now be found by simply sweeping over $k_p$ and determining $\mathcal{S}(k_p)$ at each stage. The necessary admissible range of $k_p$ values over which the sweeping has to be carried out can be a priori narrowed down by using the root locus ideas presented in [8].

Instead of given tuning formulas [2-4] based on simple characterizations of process dynamics, such as the characterization by a first order model with time delay, the proposed design method explicitly characterizes the entire set of admissible PID gain values for an arbitrary plant. Also, this characterization facilitates the optimal design of any additional time-domain specifications. To characterize the entire admissible set, the proposed method requires more computational effort that involves sweeping over proportional gain, gain, and phase margin specifications. However, it requires only moderate computational effort. Although the graphical approach [6] can also provide the entire set of admissible PID gain values, the proposed method is more efficient due to the superior computational efficiency of its linear programming characterization.

**Remark 5.1.** Note that from Theorem 4.1, since the constraint set is linear, the admissible set for (7) is either a convex polygon or an intersection of half planes. Therefore, for a fixed $k_p$, the admissible $(k_i, k_d)$ values achieving the guaranteed gain and phase margin specifications is a union of convex sets.

**Remark 5.2.** The design procedure presented here is only applicable to plants described by rational transfer functions. For a plant with time delay, the time delay factor can be approximated by a properly chosen Padé approximation. Based on the resulting rational transfer function model, the design procedure can then be carried out directly. It is reasonable to expect that the higher the order of the Padé approximation chosen, the more accurate the admissible region will be in the space of PID parameters.

We will now present a simple example to illustrate the procedure involved in determining all admissible $(k_p, k_i, k_d)$ values needed to achieve the guaranteed gain and phase margins. For detailed calculations used to determine the admissible $(k_i, k_d)$ gain values based on Theorem 4.1, the interested reader is referred to the analogous procedures presented in [7,8].

**Example 5.1.** Consider an unstable and non-minimum phase plant:

$$\frac{G(s)}{D(s)} = \frac{2s - 1}{s^4 + 3s^3 + 4s^2 + 7s + 9}$$

and the PID controller

$$C(s) = \frac{k_ds^2 + k_ps + k_i}{s}.$$

In this example, we will consider the problem of determining the admissible $(k_p, k_i, k_d)$ gain values needed to achieve a gain margin $A_n \geq 3.0$ and a phase margin $\theta_n \geq 40^\circ$. From Section 2, we know that the admissible $(k_p, k_i, k_d)$ values exist if and only if the following conditions hold:

1. $s(s^4 + 3s^3 + 4s^2 + 7s + 9) + A_k(s^2 + k_i s + k_d)(2s - 1)$ is Hurwitz for all $A \in [1, 3.0]$;
2. $s(s^4 + 3s^3 + 4s^2 + 7s + 9) + e^{j\theta}(k_p s^2 + k_d s + k_i)(2s - 1)$ is Hurwitz for all $\theta \in [0^\circ, 40^\circ]$.

Under condition (1), with a fixed $k_p = 2.0$, by setting $L(s) = s(s^4 + 3s^3 + 4s^2 + 7s + 9)$ and $M(s, A) = A(2s - 1)$, sweeping over $A \in [1, 3.0]$, and using the results of Theorem 4.1 at each stage, we obtained the admissible set $\mathcal{G}M_{(2,0)}$ sketched in Fig. 2. In Fig. 2, for different values of $A \in [1, 3.0]$, the boundaries.
of the admissible regions $\mathcal{G}_M(2.0, A)$ are indicated using solid lines. The shaded portion is $\mathcal{G}_M(2.0)$, which is the intersection of $\mathcal{G}_M(2.0, A)$ as $A$ varies over the interval $[1, 3.0]$. Under condition (2), with the same fixed $k_p = 2.0$, by setting $L(s) = s(s^4 + 3s^3 + 4s^2 + 7s + 9)$ and $M(s, \theta) = e^{-j\theta}(2s - 1)$, sweeping over $\theta \in [0^\circ, 40^\circ]$, and using the results of Theorem 4.1 at each stage, we obtained the admissible set $\mathcal{P}_M(2.0)$ sketched in Fig. 3. In Fig. 3, for different values of $\theta \in [0^\circ, 40^\circ]$, the boundaries of the admissible regions $\mathcal{P}_M(2.0, \theta)$ are indicated using solid lines. The shaded portion is $\mathcal{P}_M(2.0)$, which is the intersection of $\mathcal{P}_M(2.0, \theta)$ as $\theta$ varies over the interval $[0^\circ, 40^\circ]$. Then, for $k_p = 2.0$, the admissible set of $(k_i, k_d)$ values is the intersection of $\mathcal{G}_M(2.0)$ and $\mathcal{P}_M(2.0)$. The intersection is $S_{2.0}$ sketched in Fig. 4. Using root locus ideas [8], it was determined that a necessary condition for the existence of admissible $(k_i, k_d)$ values is that $k_p \in (1.14746, 3.0)$. Thus, by sweeping over $k_p \in (1.14746, 3.0)$, and repeating the above procedure, we obtained the admissible set of $(k_p, k_i, k_d)$ values for which the resulting control systems simultaneously achieved the gain margin $A_m \geq 3.0$ and the phase margin $\theta_m \geq 40^\circ$. The entire admissible set is sketched in Fig. 5.

VI. CONCLUDING REMARKS

In this paper, we have provided a computational characterization of all admissible PID controllers that ensure the closed-loop stability and achieve the guaranteed gain and phase margins for a given plant. The characterization was based on an extension of the results of PID stabilization to the case of complex polynomials. This characterization of all admissible PID controllers reveals important structural properties of admissible PID controllers. It has been shown that for a fixed proportional gain, the set of admissible integral and derivative
gains lie in a union of convex sets. Also, this characterization facilitates the optimal design of any additional time-domain specifications.

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REFERENCES


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