QUATERNION FEEDBACK ATTITUDE CONTROL DESIGN: A NONLINEAR $H_\infty$ APPROACH

Long-Life Show, Jyh-Ching Juang, Ying-Wen Jan, and Chen-Tzung Lin

ABSTRACT

The paper presents a quaternion feedback attitude control law for spacecraft attitude maneuver. A nonlinear feedback controller is designed to achieve $L_2$ gain performance, i.e., the resulting closed-loop system is designed such that the $L_2$ gain from the exogenous disturbance to the performance measure is less than a scalar. The solution of the nonlinear $H_\infty$ control problem is known to be related to the existence of a solution to the Hamilton-Jacobi inequality. In the paper, a solution for spacecraft attitude control is conjectured and shown to satisfy the $H_\infty$ criterion. The result generalizes existing methods in two regards: the proposed Hamilton-Jacobi function is more general than existing ones and the resulting controller contains a nonlinear term that can be used to address the nonlinear couplings between quaternion terms. The method is applied to the ROCSAT-3 orbit raising control problem to verify its effectiveness.

KeyWords: Quaternion feedback, $L_2$ gain performance, Hamilton-Jacobi inequality.

I. INTRODUCTION

The large angle attitude control of spacecraft has received extensive attention in recent decades. One characteristic to this kind of attitude control problem is that nonlinear attitude dynamics are involved, restricting the use of linearized control design methods. Many nonlinear or optimal control methods have been proposed to stabilize the attitude dynamics, and hopefully achieve certain performance objectives such as minimal time or minimal control energy. Existing nonlinear control design methods for spacecraft attitude control include the use of sliding mode control [1], model reference adaptive control [2], and quaternion feedback [3-4]. More recently, nonlinear $H_\infty$ control methods have also been proposed [5-7] to address the attitude control problem. The paper adopts the quaternion formulation and proposes a more general Lyapunov (or Hamilton-Jacobi) function to account for the stability and robustness of the attitude control problems. Unlike the results in [5] and [7], the nonlinear $H_\infty$ controller based on the proposed approach is nonlinear.

The organization of the paper is as follows. In Section 2, the nonlinear $H_\infty$ theory is briefly reviewed. The spacecraft dynamics are then analyzed and the control design problem is formulated in Section 3. The nonlinear $H_\infty$ control theory is then applied to spacecraft attitude control. This is followed in Section 4 by an illustrative simulation using the ROCSAT-3 as an example. The concluding remarks are given in Section 5.

II. REVIEW OF NONLINEAR $H_\infty$ CONTROL THEORY

In this section, results in nonlinear $H_\infty$ control are briefly reviewed. Consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)d$$

$$z = h(x)$$

where $x \in \mathbb{R}^n$ is the state vector, $d$ is the exogenous dis-
turbance, and \( \tilde{z} \) is the performance output signal. Assume that \( f(x), g(x), \) and \( h(x) \) are smooth functions and \( x = 0 \) is the equilibrium point of the system, i.e.; \( f(0) = 0 \) and \( h(0) = 0. \) The nonlinear system is said to have an \( L_2 \) gain less than \( \gamma \) if the following relationship holds

\[
\int_0^t \dot{x}(t) z(t) dt < \gamma \int_0^t d^T(t) d(t) dt
\]

for any input \( d \in L_2[0, \infty). \) The \( L_2 \) gain characterizes the relation between the disturbance input energy and performance output energy. A small \( \gamma \) can be interpreted to have a disturbance attenuation property. The following lemma provides a test criterion for the disturbance attenuation property \([8-10]\).

**Lemma 1.** The nonlinear system has an \( L_2 \) gain less than \( \gamma \) if there exists a \( C^1 \) function \( V : \mathbb{R}^n \to \mathbb{R}^+ \) with \( V(0) = 0 \) such that

\[
\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \frac{\partial V}{\partial x} + \frac{1}{2} h^T(x) h(x) < 0
\]

where \( \left( \frac{\partial V}{\partial x} \right) \) is the partial derivative of \( V(x) \), or

\[
\left( \frac{\partial V}{\partial x} \right)^T = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \ldots & \frac{\partial V}{\partial x_n} \end{bmatrix}
\]

The analysis results can be extended for controller synthesis. Consider the nonlinear control design problem in which the system is described by

\[
\dot{x} = f(x) + g_1(x) d + g_2(x) u
\]

\[
\dot{z} = \begin{bmatrix} h_1(x) \\ \rho u \end{bmatrix}
\]

where \( u \) is the control signal and \( \rho \) is a weighting scalar for the control signal. It is desired to synthesize a control law, such that the resulting closed-loop system is asymptotically stable and the \( L_2 \) gain from \( d \) to \( z \) is less than \( \gamma. \) The following celebrated lemma \([8] [9]\) provides a nonlinear \( H_\infty \) control law design method.

**Lemma 2.** The closed-loop system has an \( L_2 \) gain less than \( \gamma \) if there exists a positive \( C^1 \) function with \( V(0) = 0 \) satisfying the following Hamilton-Jacobi partial differential inequality

\[
\left( \frac{\partial V}{\partial x} \right)^T f(x) + \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^T g(x) g^T(x) \left( \frac{\partial V}{\partial x} \right) + \frac{1}{2} h^T(x) h(x) < 0
\]

Furthermore, when the system is zero-state detectable, the closed-loop system is asymptotically stable and indeed a stabilizing feedback control can be constructed

\[
u = -\frac{1}{\rho} g^T(x) \left( \frac{\partial V}{\partial x} \right)
\]

to satisfy the \( L_2 \) gain requirement.

It is clear that the construction of the Hamilton-Jacobi function \( V(x) \) constitutes a crucial step in control performance and control law synthesis.

### III. Spacecraft Dynamics and Design Problem Formula

The equations of motion of the spacecraft are described as follows \([5,6,7]\)

\[
J \ddot{\omega} = -[\omega \times] J \omega + u + d
\]

\[
\dot{\epsilon} = \frac{1}{2} \eta \omega + \frac{1}{2} [\epsilon \times] \omega
\]

\[
\dot{\eta} = -\frac{1}{2} \epsilon \omega + \eta
\]

where \( J \) is the moment of inertia matrix, \( \omega \) is the angular velocity of the spacecraft, \( u \) is the control torque, \( d \) is the environmental disturbance torque, \( [\epsilon \times] \eta \) constitutes the quaternion of the spacecraft. Note that the dynamical equation has two equilibria \( \omega = 0, \eta = 0, \) and \( \eta = \pm 1. \) These two equilibria, however, correspond to the same attitude as the quaternion is a redundant representation of the attitude. Indeed, it is known that the quaternion satisfies

\[
\epsilon \epsilon + \eta^2 = 1
\]

In the following, the notation \([\epsilon \times]\) is used to represent the \( 3 \times 3 \) skew symmetric matrix formed from the vector \( \epsilon. \) More precisely, let \( \epsilon = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \) then

\[
[\epsilon \times] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}
\]
Let \( \mathbf{X} = \begin{bmatrix} \omega \\ \varepsilon \\ \eta \end{bmatrix} \), the design problem is formulated as in (2) in which

\[
\begin{bmatrix}
-J^{-1} (\omega \times) J \omega \\
\frac{1}{2} \eta \omega + \frac{1}{2} (\varepsilon \times) \omega \\
-\frac{1}{2} \varepsilon^T \omega
\end{bmatrix}
\begin{bmatrix}
J^{-1} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f_1(\mathbf{X}) \\
g_1(\mathbf{X})
\end{bmatrix}
= 0,
\]
and

\[
g_2(\mathbf{X}) = \begin{bmatrix} J^{-1} \\ 0 \\ 0 \end{bmatrix}
\]

The objective function \( z \) contains two parts: one pertaining to the attitude control performance and the other is the amount of control. That is,

\[
z = \begin{bmatrix} h_1(\mathbf{X}) \\ \rho u \end{bmatrix}
\]

The function \( h_1(\mathbf{X}) \) is assumed to be of the following form

\[
h_1(\mathbf{X}) = \frac{\sqrt{\rho_1 \omega}}{\sqrt{\rho_2 \varepsilon}}
\]

for some positive scalars \( \rho_1 \) and \( \rho_2 \). Clearly, it is desired to have \( h_1(\mathbf{X}) \) or \( z \) small so that three-axis attitude control performance as characterized by the angular rate and attitude error cannot be kept as small as possible. Also, the control energy is accounted for in the problem formulation.

In summary, given the system as described in (4), (5), and (6), the objective of the control design is to find a smooth control law for \( u \) such that the resulting closed-loop system is stable and the \( L_2 \) gain from \( d \) to \( z \) is bounded by a positive scalar \( \gamma \).

\[
\|\mathbf{z}\|_{L_2} \leq \gamma
\]

### 3.1 Candidate function

To this end, the candidate function is selected as

\[
V(\mathbf{X}) = a \omega^T J \omega + 2(h_1 + 2h_2 \eta) \varepsilon^T J \omega + 2(1 - \eta)(c_1 + c_2 \eta)
\]

for some positive constant \( a, b_1, b_2, c_1, \) and \( c_2 \). Note that via the identity \( 2(1 - \eta) = \varepsilon^T \varepsilon + (1 - \eta)^2 \), the candidate function can be rewritten as

\[
V(\mathbf{X}) = \begin{bmatrix} \omega^T \\ \varepsilon^T \\ \eta - 1 \end{bmatrix}
\begin{bmatrix}
aJ \\
(h_1 + 2h_2 \eta)J \\
(c_1 + c_2 \eta)I
\end{bmatrix}
\begin{bmatrix}
\omega \\
\varepsilon \\
\eta - 1
\end{bmatrix}
\]

where \( J \) denotes the identity matrix with appropriate dimension. The function \( V(\mathbf{X}) \) is positive definite when \( a > 0 \) and

\[
(c_1 + c_2 \eta)I - \frac{1}{a} (h_1 + 2h_2 \eta)^2 J > 0
\]

for all \( \eta \) between \(-1\) and \( 1\).

The nonlinear \( H_\infty \) control design involves the equation in (3). Note that

\[
\frac{\partial V}{\partial \mathbf{X}} = \begin{bmatrix}
2aJ \omega + 2(h_1 + 2h_2 \eta) J \varepsilon\\
2(h_1 + 2h_2 \eta)J \omega \\
2b_2 \omega^T J \varepsilon - 2c_1 + 2c_2(1 - 2\eta)
\end{bmatrix}
\]

This then gives

\[
\left( \frac{\partial V}{\partial \mathbf{X}} \right)^T f(\mathbf{X}) = \omega^T \left( (h_1 + 2h_2 \eta)(\eta I + [\varepsilon \times]) J - b_2 J \varepsilon^T \varepsilon \right) \omega
\]

\[
+ (c_1 + c_2(2\eta - 1)) \varepsilon^T \omega
\]

In the above, the following equality is used

\[
\omega^T [\varepsilon \times] J \omega = -\varepsilon^T [\omega \times] J \omega
\]

It can then be shown that

\[
H_\infty = \left( \frac{\partial V}{\partial \mathbf{X}} \right)^T f(\mathbf{X}) + \frac{1}{2} \left( \frac{\partial V}{\partial \mathbf{X}} \right)^T
\]

\[
\times \left( \frac{1}{\gamma^2} g_1(\mathbf{X})g_1^T(\mathbf{X}) - \frac{1}{\rho_1^2} g_2(\mathbf{X})g_2^T(\mathbf{X}) \right)
\times \left( \frac{\partial V}{\partial \mathbf{X}} \right)
\]

\[
+ \frac{1}{2} h_1(\mathbf{X}) h_1^T(\mathbf{X})
\]

\[
= \omega^T \left( (h_1 + 2h_2 \eta)(\eta I + [\varepsilon \times]) J - b_2 J \varepsilon^T \varepsilon + \frac{1}{2} \rho_1
\]

\[
+ 2a^2 \left( \frac{1}{\gamma^2} - \frac{1}{\rho^2} \right) \right) \omega
\]

\[
+ \varepsilon^T \left( c_1 + c_2(2\eta - 1) + 4a(h_1 + 2h_2 \eta) \left( \frac{1}{\gamma^2} - \frac{1}{\rho^2} \right) \right) \omega
\]
Thus, by selecting \( c_1 \) and \( c_2 \) such that

\[
\begin{aligned}
 & c_1 - c_2 + 4a \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) = 0 \\
& 2c_2 + 4ab \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) = 0
\end{aligned}
\]  

(9a) and

\[
\begin{aligned}
 & 2c_2 + 4ab \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) = 0 \\
& \text{which implies that } (8) \text{ as}
\end{aligned}
\]

\[
-2a^2 (2h_1 + b_1 + b_2 \eta) \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) I - (h_1 + b_1 \eta)^2 J > 0
\]

and the function \( H_\nu \) becomes

\[
H_\nu = \omega^T \left\{ (h_1 + 2b_2 \eta)(\eta I + [\varepsilon \times])J - b_2 \varepsilon \varepsilon^T \right\} \omega
\]

\[
+ \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \left\{ \frac{1}{\gamma^2} \frac{1}{\rho^2} \right\} I \varepsilon
\]

\[
\text{Note that the matrix } \eta I + [\varepsilon \times] \text{ has a norm less than or equal to 1. Thus,}
\]

\[
\omega^T \left\{ (h_1 + 2b_2 \eta)(\eta I + [\varepsilon \times])J - b_2 \varepsilon \varepsilon^T \right\} \omega
\]

\[
\leq \omega^T \left\{ (h_1 + 2b_2 \eta)(\eta I + [\varepsilon \times])J \right\} \omega
\]

\[
\leq (h_1 + 2b_2) \| J \| \| \omega \|^2
\]

Substituting this inequality into \( H_\nu \), a sufficient condition for the system to have an \( L_2 \) gain less than \( \gamma \) can then be established.

\[
H_\nu < \omega^T \left\{ (h_1 + 2b_2) \| J \| \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \right\} \omega
\]

\[
+ \varepsilon^T \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \left\{ \frac{1}{\gamma^2} \frac{1}{\rho^2} \right\} I \varepsilon
\]

**Theorem 1.** There exists a controller such that the \( L_2 \) gain from \( d \) to \( z \) is less than \( \gamma \) if there exist \( a, b_1, \) and \( b_2 \) such that

\[
-2a^2 (2h_1 + b_2 + b_2 \eta) \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) I - (h_1 + b_1 \eta)^2 J > 0
\]

\[
\frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) + b_2 \| J \| + 2b_2 \| J \| < 0
\]

for all \( \eta \) between \(-1\) and \(1\). Furthermore, the control is

\[
u = -\frac{2}{\rho^2} (a \omega + b_2 \varepsilon + b_2 \eta \varepsilon)
\]

(10)

The tests in the above theorem involve the quaternion variable \( \eta \). By applying the worst case analysis, another sufficient condition can be obtained.

**Corollary 1.** A sufficient condition for the existence of the \( H_\infty \) controller of the form (9) is the existence of \( a, b_1, \) and \( b_2 \) such that

\[
\frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) \left( \frac{1}{\gamma^2} \frac{1}{\rho^2} \right) < 0
\]

With some modifications, it can be shown that the proposed method and criteria generalize the results in [5]. Indeed, suppose we set \( b_2 \) and \( c_2 \) as zeros, the conditions in [5] can be recovered using the criteria in the corollary.

In the following, the \( H_\infty \) control design method is applied to the ROCSAT-3 spacecraft to verify its effectiveness.

**IV. SIMULATION RESULTS**

The ROCSAT-3 system contains a constellation of eight micro-satellites for the weather and climate research mission. The eight micro-satellites (about 40 kg for each satellite) will be initially placed on the 400 km circular parking orbit. Figure 1 depicts the ROCSAT-3 on orbit configuration. The Attitude Determination and Control Subsystem (ADCS) is required to transfer the satellite from its parking orbit to an 800 km circular orbit. During this orbit-raising period, four thrusters are employed to provide \( \Delta V \) burn for orbit raising and attitude control. Clearly, control errors in attitudes may result in eventual altitude error and even excessive fuel consumption. Moreover, at the beginning and ending of the orbit-raising, micro-satellite is subject to a large angle maneuver to change from orbit-normal configuration to raising configuration and vice versa.
Fig. 1. ROCSAT-3 nominal on-orbit configuration.

The specifications of the ROCSAT-3 attitude control include (a) the settling time of the three body rates is 100 sec, within 0.1 deg/sec, (b) the settling time of the three attitudes is 100 sec, within 0.1 deg, and (c) the torque commands are within ±1 N-m. In the ROCSAT-3 simulation, the moment of inertia is

\[
J = \begin{bmatrix}
5.5384 & -0.0276 & -0.0242 \\
-0.0276 & 5.6001 & -0.0244 \\
-0.0242 & -0.0244 & 4.2382
\end{bmatrix} \text{ kg m}^2
\]

With \( \gamma = 8 \), \( \rho = 1 \), \( \rho_1 = 1 \) and \( \rho_2 = 1 \), the following controller parameters are designed by solving (11), \( a = 80 \), \( b_1 = 20 \) and \( b_2 = 20 \). The simulation program also contains disturbances due to gravity gradient. In the following illustrative example, the initial condition is

\[
\omega(0) = \begin{bmatrix}
0.53 \\
0.53 \\
0.053
\end{bmatrix} \text{ (deg/sec)} \quad \text{and} \quad \begin{bmatrix}
\xi(0) \\
\eta(0)
\end{bmatrix} = \begin{bmatrix}
0.6842 \\
0.6963 \\
0.1517 \\
0.1544
\end{bmatrix}.
\]

Figure 2 shows the quaternion response of the controller design. In the figure, the proposed design approach is compared with the approach suggested in [5]. Figures 3 and 4 further depict the spacecraft rate response and control input response. It is clear that the proposed approach achieves a faster response with a less amount of control activity. Simulations of other initial conditions have also been performed with similar performance.

V. CONCLUSION

In this paper, a nonlinear \( H_\infty \) controller is designed to achieve global stability and disturbance attenuation for spacecraft under large-angle maneuvers. A solution to the Hamilton-Jacobi inequality is identified and, accordingly, nonlinear \( H_\infty \) controllers are designed to achieve \( L_2 \)-gain performance. The method is applied to
the attitude control of the ROCSAT-3 satellite. Simulation results verify the feasibility of the proposed method.

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