SECOND-ORDER NONSINGULAR TERMINAL SLIDING MODE DECOMPOSED CONTROL OF UNCERTAIN MULTIVARIABLE SYSTEMS

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ABSTRACT

This paper proposes a second-order nonsingular terminal sliding mode decomposed control method for multivariable linear systems with internal parameter uncertainties and external disturbances. First, the systems are converted into the block controllable form, consisting of an input-output subsystem and a stable internal dynamic subsystem. A special second-order nonsingular terminal sliding mode is proposed for the input-output subsystem. The control law is designed to drive the states of the input-output subsystem to converge to the equilibrium point asymptotically. Then the states of the stable zero-dynamics of the system converge to the equilibrium point asymptotically. The method proposed in the paper has advantages for higher-dimensional multivariable systems, in the sense that it simplifies the design and makes it possible to realize a robust decomposed control. Meanwhile, because of the adoption of the second-order sliding mode, the control signal is continuous. Simulation results are presented to validate the design.

KeyWords: Multivariable systems, canonical decomposition, nonsingular terminal sliding mode control, zero dynamics.

I. INTRODUCTION

Research on multivariable linear systems with internal parameter uncertainties and external disturbances is important for both theoretical and practical reasons. Two methodologies have been commonly used: the state-space feedback control and the optimal control [1]. If a system, however, has a relatively high dimension, these two methods may impose a severe computational demand when applied real time control. One approach proposed in [2] alleviates this problem by transforming a multivariable linear uncertain systems into a block controllable canonical form (BC-form), but it does not seem to exhibit robustness to internal parameter uncertainties and external disturbances.

Variable structure systems (VSS) are well known for their robustness to system parameter variations and external disturbances [3,4]. One aspect of VSS of particular interest is the sliding mode control, which is designed to drive and constrain the system states to lie within a neighborhood of prescribed switching manifolds that exhibit the desired dynamics. When the system is in the sliding mode, the closed-loop responses of the system become totally insensitive to both internal parameter uncertainties and external disturbances. One characteristic of the conventional VSS is that the convergence of the system states to equilibrium points usually occurs asymptotically due to the asymptotical convergence of the linear switching manifolds that are commonly chosen. Recently, a terminal sliding mode (TSM) controller was developed [5,6,7]. Compared with the linear hyperplane based sliding mode, TSM offers some superior properties, such as fast, finite time convergence and better static tracking precision. This controller is particularly useful for high precision control as it speeds up the rate
of convergence near the equilibrium point. However, TSM controller design methods suffer from a singularity problem. Based on TSM, some nonsingular terminal sliding mode (NTSM) control systems have been proposed to avoid the singularity in TSM [8,9,10].

It is well known that the chattering phenomenon is a major drawback of the sliding mode approach in the practical realization of VSS. The chattering phenomenon, that is, high-frequency finite amplitude oscillations with finite frequencies caused by system imperfections, results from the discontinuity of the sign function on the sliding manifold. Two popular methods for eliminating chattering replace this sign function with either a saturation function or a sigmoid-like function at the cost of slight deterioration in system performance. In addition, the second-order sliding mode control can be used to smooth the control signal. It is a continuous control, robust to parameter uncertainties and disturbances. Meanwhile, the characteristics of the traditional sliding mode approaches are guaranteed in second-order sliding mode control systems [11].

This paper proposes a second-order nonsingular TSM decomposed control method for linear multivariable systems that is robust to both internal parameter uncertainties and external disturbances. Firstly, we introduce the controllability index and the parameter $r$ of the systems then, the systems are converted into the BC-form, and the coupled state variables are eliminated. As a result, the original systems are converted into $r$ block decoupled BC-form systems consisting of an input-output subsystem and an internal dynamic subsystem. A special second-order nonsingular TSM is proposed for the input-output subsystem. The control law is designed to drive the state variables of the input-output subsystem to the equilibrium points asymptotically. Then, the state variables of the internal dynamic subsystem converge to the equilibrium points asymptotically. Possessing the advantages of a simple controller design and a hierarchical controller structure, the proposed method can be easily applied to linear high-dimensional multivariable systems. Moreover, the second-order sliding mode proposed in this paper can eliminate chattering in systems. Simulation results are presented to validate the method.

II. DECOMPOSITION OF MULTIVARIABLE LINEAR UNCERTAIN SYSTEM

Consider an uncertain multivariable linear system:

$$\dot{x}(t) = (A + A_{ncr}(t))x(t) + Bu(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state variable vector; $u(t) \in \mathbb{R}^m$ is the control input vector; $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the known parameters matrices of the system (1); $A_{ncr}(t) \in \mathbb{R}^{n \times n}$ represents any uncertainties or nonlinearities. Assume that the system (1) satisfies the following assumptions:

1. $A, B$ are constant matrices;
2. $A, B$ are controllable;
3. $A_{ncr}(t)$ satisfies the following matching condition:

$$A_{ncr}(t) = Bd(t), \quad (2)$$

where $d(t) \in \mathbb{R}^{m \times 1}$ is a bounded time-varying matrix and satisfies $\|d(t)\| \leq l_1, l_2 > 0$.

The control objective is to force the system (1) with uncertainty $A_{ncr}(t)$ to converge to the equilibrium points asymptotically or in a finite amount of time from any initial condition $x(0) = 0$.

In order to convert the system (1) into the BC-form, the following nonsingular transformation is performed:

$$x' = Fx, \quad (3)$$

where $x' \in \mathbb{R}^n$ and the state transformation matrix $F$ is defined as

$$F = F_1F_2 \cdots F_{r-1}. \quad (4)$$

By means of the transformation (3), the system (1) is converted into:

$$\dot{x}'(t) = A'x'(t) + B'd(t) + B'u(t), \quad (5)$$

where

$$A' = FAF^{-1}; B' = FB = \left[ \begin{array}{c} 0 \\ B_{0,1}^T \end{array} \right]; B_{0,1} \in \mathbb{R}^{m \times n_1};$$

$$\text{rank}B_{0,1} = n_1, n_1 \leq m,$$ $n_1$ is the controllable index of the system (1). Therefore, the pseudo-inverse matrix of $B_{0,1}$ exists and is given by:

$$B_{0,1}^+ = B_{0,1}^T B_{0,1}^{T+}.$$

Then, the system (5) can be rewritten as

$$\begin{align*}
\dot{x}_j(t) &= A_jx_j(t) + B_{j-1}x_{j-1}(t) \\
\dot{x}_i(t) &= \sum_{j=1}^{i} A_{j}x_j(t) + B_{j}x_{j-1}(t), \quad i = 2, \ldots, r-1 \quad (7) \\
\dot{x}_i(t) &= \sum_{j=1}^{i-1} A_{j}x_j(t) + B_{i}d(t) + B_{i}u(t), \quad i = 1, \ldots, r.
\end{align*}$$

where $x'^T = [x_1'^T \cdots x_r'^T]^T$, $x_i' \in \mathbb{R}^{n_i}$, $i = 1, \ldots, r$, $B_{i-1,i}, i = 1, \ldots, r$, have full rank, i.e., rank $B_{i-1,i}$ = $n_i$; $i = 1, \ldots, r$; $\sum n_i = n$, $(n_1, \ldots, n_r)$ is the controllability index vector of the system (1). $x_{i-1}'$ is regarded as the virtual control vector of the $i$th layer of the system (7).

For the convenience of design, the system (7) is
Further converted into the decoupled BC-form expression.

The following nonsingular state transformation is performed again:
\[ x' = F'z, \]  
(8)
where \( z \in \mathbb{R}^n \) and the nonsingular transformation matrix \( F' \) is
\[
F' = \begin{bmatrix}
I & 0 & 0 & \cdots & 0 & 0 \\
-K_{r-1,r} & I & 0 & \cdots & 0 & 0 \\
-K_{r-2,r} & -K_{r-2,r-1} & I & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-K_{2,r} & -K_{2,r-1} & -K_{2,r-2} & \cdots & I & 0 \\
-K_{1,r} & -K_{1,r-1} & -K_{1,r-2} & \cdots & -K_{1,2} & I
\end{bmatrix}
\]
(9)
with \( K_{ia} \), \( i = 1, \ldots, r \), \( \alpha = 2, \ldots, r \), being the designed parameter, which will be determined by the designed matrix \( N_i \) (11), \( i = 1, \ldots, r \), which in turn will be determined by the system specifications.

Then, the BC-form system (7) is converted into the following form by using nonsingular transformation (8):
\[
\dot{z}(t) = A'z(t) + B'd(t) + B'u(t),
\]
(10)
where
\[
A' = (F')^{-1}A'F', \quad B' = (F')^{-1}B' = [0 \quad B_{10}^T].
\]
Eq. (10) can be rewritten as
\[
\begin{align*}
\dot{z}_i(t) &= N_i(z_i(t) + B_{ija}z_j(t)), \quad i = 2, \ldots, r \\
\dot{z}_1(t) &= \sum_{a=1}^{r} \bar{A}_{ia}z_a(t) + B_{1a}d(t) + B_{1a}u(t),
\end{align*}
\]
(11)
where
\[
z = [z_1^T \cdots z_r^T]^T, \quad z_i \in \mathbb{R}^{n_i}; \quad N_i, i = 2, \ldots, r,
\]
are the matrices designed according to the system specifications. After \( N_i \) is determined, \( F' \) and \( \bar{A}_{ia} \) can also be determined.

### III. SECOND-ORDER NONSINGULAR TSM CONTROL

For the convenience of design, we assume that the uncertain multivariable linear system (1) is already in the canonical form (11) after the transformation (3) and (8). A second-order nonsingular TSM control strategy is then adopted. The design consists of two steps. The first step is to design the second-order nonsingular TSM and ensure that the sliding mode converges to the equilibrium point asymptotically. The second step is to design the robust control so as to ensure that the system is robust to internal parameter uncertainties and external disturbances.

We now propose the following second-order nonsingular TSM manifold for the system (11)
\[
s(t) = \beta z_i + \int_0^t z_i dt,
\]
(12)
where \( s \in \mathbb{R}^n; \quad z_i \in \mathbb{R}^{n_i}; \quad \beta = \text{diag}(\beta_1, \ldots, \beta_{n_i}) \), where \( \beta_i > 0 \) is a constant; \( \int_0^t z_i dt \) are denoted as:
\[
\int_0^t z_i dt = \left[ \int_0^t z_{i1} dt, \ldots, \int_0^t z_{in_i} dt \right]^T.
\]

In order to eliminate chattering, we propose the following second-order nonsingular TSM manifold utilizing the second-order concept [11]:
\[
I(t) = \gamma^{-1}\dot{s}^{p/q} + s,
\]
(13)
where \( I \in \mathbb{R}^n; \quad \gamma = \text{diag}(\gamma_1, \ldots, \gamma_{n_i}) \), where \( \gamma_i > 0 \) remain constant; \( p \) and \( q \) are both odd, \( 1 < p/q < 2; \quad s^{p/q} \) is denoted as
\[
\dot{s}^{p/q} = \left[ \dot{s}_1^{p/q}, \ldots, \dot{s}_{n_i}^{p/q} \right]^T.
\]

The aim of introducing \( s(t) \) is to control the input-output subsystem using the sliding mode, while the purpose of introducing \( I(t) \) is to realize second-order sliding mode control and eliminate chattering effectively. Therefore, the second-order sliding mode is designed as a nonsingular TSM to guarantee that the linear sliding mode of \( s(t) \) will reach the equilibrium points in finite time and have no singularity.

When \( I(t) = 0, \forall t \geq t_e \) (\( t_e \) is the time when \( I(t) \) reaches the nonsingular terminal sliding mode manifold \( I(t) = 0 \)), assume that \( s(t) = [s_1(t), \ldots, s_{n_i}(t)]^T \). Solving Eq. (13), we obtain the time needed to reach \( s(t) = 0 \):
\[
t_e = t_e + \frac{p}{(p - q) \min_{l \in [1, \ldots, n_i]} \gamma_l^{1/(p - q)}} \max_{l \in [1, \ldots, n_i]} \left( s_l(t)^{p/(p - q)} \right)
\]
(14)

Thus, through the suitable control design, \( s(t) \) and \( \dot{s}(t) \) can be driven to reach the second-order sliding mode \( I(t) = 0 \) and then remain at \( I(t) = 0 \) to realize sliding mode motion. Among \( I(t) = 0, s(t) \) will reach the equilibrium points in finite time \( t_e \) (14). After \( s(t) \) reaches zero, the system will remain in linear sliding mode motion (12); that is, the dynamic characteristics of the system (11) can be determined by the design parameters \( \gamma, \beta, p \) and \( q \), and have nothing to do with the system’s parameters (13). The relevant control methodology is given in the following theorem.
Theorem 1. For the uncertain multivariable linear system (1) in the canonical form (13), if the linear sliding mode and second-order nonsingular TSM are chosen as (12) and (13), and if the control law is designed as shown below, then the system (1) is asymptotically stable:

\[ u(t) = u_i(t) + u_i(t), \]  

(15)

where

\[ u_i(t) = -B_p \sum_{a=1}^{l} A_{ia} z_a. \]  

(16)

Here, \( u_i(t) \) is obtained using the low-passed filter:

\[ v(t) = \beta u_i(t) + u_i(t), \]  

(17)

where \( \beta \) is the design parameter (12); in addition, the input \( v(t) \) of the low-passed filter is:

\[ v(t) = v_{eq}(t) + v_{eq}(t), \]  

(18)

\[ v_{eq}(t) = -\frac{q}{p} B_{eq} s^{2+p/q}, \]  

(19)

\[ v_{eq}(t) = -\left( F^{\gamma} \text{diag} (s^{p/q-1}) B_{eq} \right) \]  

(20)

\[ + \alpha (1/2)^{n/2} \frac{q}{p} \int_{t_0}^{t} \| \|^{2p/q} \big( (l_0 + \| \|) \). \]

where \( \gamma, \alpha \) and \( \eta \) are the design parameters, \( \alpha > 0, 0 < \eta < 2 \).

Proof. Consider the following Lyapunov function:

\[ V(t) = \frac{1}{2} I(t)^T I(t). \]

Differentiating \( V(t) \) with respect to time, we get:

\[ \dot{V}(t) = I^T \dot{I} = I^T \left( \frac{p}{q} \gamma^{-1} \text{diag} (s^{p/q-1}) \right) \]  

(21)

\[ \leq -\alpha (1/2)^{n/2} \frac{q}{p} \| \|^{2p/q} \big( (l_0 + \| \|) \). \]

Therefore, according to Lemma 2 [12], the condition for Lyapunov stability is satisfied. The system states can reach the sliding mode \( h(t) = 0 \) within finite time \( t_s \), and the system states can converge to the equilibrium points asymptotically. Since the state transformations between \( x \) and \( z \), (3) and (8), are linear, the variable states \( x \) of the original system (1) will converge to the equilibrium points asymptotically. This completes the proof.

IV. SIMULATIONS

A simulation with a seventh-order system was performed for the purpose of evaluating the performance of the proposed scheme. Consider the following system [13]:

\[ \dot{x} = Ax + Bd(t) + Bu, \]

where the disturbance is \( d(t) = [0.1 \sin(2t) 0.1 \sin(2t) 0.1 \sin(2t)]^T \), and \( A \) and \( B \) are given by

\[
A = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
2 & 1 & 1 \\
0 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

Firstly, the system (21) was transformed into the BC-form (7):
\[
\begin{align*}
    & x' = 2x' + [1 \ 0 \ 0]x,
    \\
    & x' = \begin{bmatrix}
        0 \\
        x' + [2 \ -2 \ 0]x' + [2 \ 1 \ 1]x' \\
        0 \\
    \end{bmatrix},
    \\
    & x' = \begin{bmatrix}
        0 \ 0 \ -2 \\
        0 \ 0 \ 0 \\
        0 \ 0 \ -4 \ 2 \\
    \end{bmatrix} + \begin{bmatrix}
        2 \ 1 \ 1 \\
        0 \ 0 \ 0 \\
        3 \ 2 \ 1 \\
    \end{bmatrix}x' \\
    & + \begin{bmatrix}
        1 \ 1 \ 1 \\
        0 \ 0 \ 0 \\
    \end{bmatrix}d(t).
\end{align*}
\]

Then, Eq. (22) was transformed into the following decoupled block controllable canonical form by means of state transformation (8):

\[
\begin{align*}
    & z_1 = -0.6z_1 + [1 \ 0 \ 0]z_1, \\
    & z_2 = -z_2 + [2 \ 1 \ 1]z_2, \\
    & z_3 = [1 \ 1 \ 1]d(t) + \begin{bmatrix}
        1 \ 1 \ 1 \\
        0 \ 0 \ 0 \\
    \end{bmatrix}u.
\end{align*}
\]

According to Theorem 1, the controller was designed as follows:

\[
u = u_0 + u_1,
\]

According to (17), \( u_1 \) was designed as follows:

\[
v = \begin{bmatrix}
    0.8 & 0 & 0 \\
    0 & 0.8 & 0 \\
    0 & 0 & 0.8
\end{bmatrix} \begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix}.
\]

According to (18), (19) and (20), \( v_{eq} \) and \( v_n \) were designed, respectively, as follows:

\[
v_{eq} = \begin{bmatrix}
    0 & 0 & 1 \\
    1 & 1 & 1 \\
    0 & 1 & -1
\end{bmatrix} \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix},
\]

\[
v_n = \begin{bmatrix}
    0 & 0 & 1 \\
    1 & 1 & 1 \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}.
\]

We assumed that the initial state variables of the system were as follows: \( x_1(0) = 0, x_2(0) = 0, x_3(0) = 0, x_{23}(0) = 1, x_{11}(0) = 2.36, x_{12}(0) = -3.16 \) and \( x_{13}(0) = 2.16. \)
The simulation results are illustrated in Fig. 1 to Fig. 10. The phase planes of $s_1$, $s_2$ and $s_3$ and their differentials are shown in Fig. 1 to Fig. 3 respectively. It is seen that $s_1$, $s_2$ and $s_3$ realized the nonsingular terminal sliding mode. The phase plane of $z_{11}$, $z_{12}$ and $z_{13}$ of the input-output subsystem and their integrals are depicted in Fig. 4 to Fig. 6, respectively. It shows that the input-output subsystem realized the linear sliding mode control. The plots of $z$ vs time and $x$ vs time are depicted in Fig. 7 and Fig. 8, respectively. They both converge to the equilibrium points asymptotically. Fig. 9 shows the
input $v$ of the low-passed filter. The control signals are shown in Fig. 10. It is seen that no chattering phenomenon occurred.

V. CONCLUSIONS

This paper has proposed a special second-order nonsingular TSM decomposed control method for linear uncertain multivariable systems. Through decoupled state transformation, the systems are converted into the BC–form for the convenience of design. A second-order nonsingular TSM has been proposed for the system, which can drive the state variables of the input-output subsystem to the equilibrium points asymptotically; then, the state variables of the stable zero-dynamic subsystem converge to the equilibrium points asymptotically. The method proposed here simplifies the design of the controller and realizes hierarchical control. Moreover, the second-order sliding mode proposed in this paper eliminates chattering.

REFERENCES


