FUZZY SLIDING MODE CONTROL WITH PIECEWISE LINEAR SWITCHING MANIFOLD

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ABSTRACT

Variable Structure Control (VSC) has been successfully established in control systems engineering in the past two decades. Recently, Fuzzy Sliding Mode Control (FSMC) techniques have drawn the attention of the scientific community, due to their effectiveness in reducing the typical chattering phenomenon arising in VSC. In this paper we present a novel FSMC technique for a class of second order dynamical systems based on a piecewise linear sliding manifold. The proposed approach enhances the effectiveness of VSC in the presence of a saturated control input. In addition, employing the proposed VSC with fuzzy rule based algorithms results in smooth dynamics when the trajectory is in the vicinity of the switching manifold, as opposed to the typical chattering arising under classical VSC. After illustrating the proposed strategy on a simple design example, the approach is applied to an inverted pendulum, a well-known benchmark for automatic control techniques. The effectiveness of the technique is shown both by means of simulation tests and experiments on a lab equipment. Future research directions include the extension of the technique in the presence of uncertainties in the plant model.

KeyWords: Nonlinear systems, variable structure control, fuzzy sliding mode control, switching surface, inverted pendulum.

I. INTRODUCTION

Variable Structure Control (VSC) has been successfully established and employed during the past two decades in control systems engineering [24,8,21,7]. It is well known that a VSC regulator with a switching output will (under certain circumstances) result in a sliding mode on a predefined subspace of the state space [11]. Indeed, the purpose of the VSC switching regulator is to drive the state trajectory onto a predefined surface in the state space and maintain the system on that differential geometry. In the literature the former dynamics is called the reaching phase and the latter is named the sliding mode. Ideally speaking, once the so-called sliding surface or switching manifold has been intercepted, the switching control law forces the trajectory to slide along the surface to the desired operating point. Hence, in the related literature VSC is often referred to as Sliding Mode Control (SMC).

The adoption of classical VSC techniques typically produces a continuously switching control action, which may excite unmodeled high frequency dynamics, thus causing an undesired strain to affect the actuator. Recently, the integration of SMC with fuzzy control algorithms [19,20] has drawn the attention of the scientific community. Fuzzy Sliding Mode Control (FSMC) techniques approximate the nonlinear input/output map of SMC by means of a fuzzy inference mechanism applied to a linguistic rule base [29,15]. Benefits resulting from fuzzy rule based algorithms that mimic SMC include smooth dynamics when the trajectory is in the vicinity of the switching manifold [14], as opposed to the typical VSC chattering, and a restriction of the fuzzy rule table dimension with respect to classical fuzzy controllers [18].

The scope of this paper is to introduce a novel FSMC technique for a class of second order dynamical systems based on a piecewise linear sliding manifold that is bent at the extremes of the phase plane. The advantage of the present approach lies in the decrease of the resulting control action magnitude with respect to classical SMC, enhancing the effectiveness of VSC under a satu-
rated control input. After illustrating the proposed FSMC on a simple design example, the approach is employed to perform stabilization and swing up of an inverted pendulum, a well-known benchmark for automatic control techniques. The effectiveness of the strategy is shown both by means of simulation tests and experiments on a lab equipment.

The paper is organized as follows. We briefly review the SMC technique in section 2; in section 3 we report and discuss the numerous FSMC techniques presented in the related literature; in section 4 we introduce a novel FSMC based on a piecewise linear sliding manifold and discuss its benefits; in section 5 we compare SMC and FSMC with a linear sliding surface with the proposed technique for a simple design example; in section 6 we apply the proposed FSMC to an inverted pendulum, performing stabilization and swing-up. Finally, the conclusions are outlined and suggestions for further research are given.

II. SLIDING MODE CONTROL

In this section we briefly recall the general concepts of SMC for second order systems [24,8,21,7]. Consider the following dynamical system in canonical form:

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = f(x) + b(x)u + d \]  

where \( x = [x_1, x_2]^T \) is the observable state vector, \( u \) is the system control input, \( d \) is a time-dependent disturbance with known upper bound \( D \), \( f(x) \) and \( b(x) \) are functions of the state, generally nonlinear, and \( B \) represents the upper bound of \( b(x) \). In addition, consider a given desired trajectory \( x_{id}(t) \) and the tracking error \( e = x_1 - x_{id} \).

The basic idea of SMC is to force the system, after a finite time reaching phase, to a sliding line containing the operating point:

\[ s(x(t)) = e + \lambda e = x_2 - x_{id} + \lambda(x_1 - x_{id}) = 0 \]  

where \( x_{id}(t) = \dot{x}_{id}(t) \) represents the desired trajectory for the second state variable and the sliding constant \( \lambda \) is a strictly positive design parameter. At steady state the system follows the desired trajectory once \( s(x(t_{reach})) = 0 \), i.e., after the trajectory has reached the sliding line for \( t = t_{reach} \) representing the reaching time [21]. Hence, a suitable control action is to be designed for the dynamical system to hit the sliding surface (2).

We select the Lyapunov function [21]:

\[ V = \frac{1}{2} s^2(x) \]  

with the following control action:

\[ u = b^{-1}(x) \cdot (\dot{u} - K \text{sign}(s(x))), \quad K > 0 \]  

where \( K \) is a design parameter or a function of \( x(t) \) such that \( K = K(x) \) [21] and it holds

\[ \dot{u} = -(f(x) - \dot{x}_{id} + \lambda \dot{e}), \]  

while ‘sign’ represents the sign function

\[ \text{sign}(y) = \begin{cases} +1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases} \]  

By derivation of the Lyapunov function (3) we get:

\[ \ddot{V} = s(x)\dot{s}(x) = -Ks(x)\text{sign}(s(x)) + ds \leq -\eta \|s(x)\| \]  

where \( \eta \) is a strictly positive design constant such that \( K \geq (\eta + D) \). Hence, the sliding condition (7) states that \( \dot{V} \) is negative definite in the switching line, such that the sliding surface is attractive. Moreover, condition (7) guarantees that if \( x_1(t = 0) \neq x_{id}(t = 0) \), i.e., if the system trajectory is initially off the sliding line, the surface \( s(x) = 0 \) will nonetheless be reached in a finite time \( t_{reach} \) such that [21]:

\[ t_{reach} \leq \frac{|s(x(t = 0))|}{\eta} \]  

The typical behavior implied by satisfaction of the sliding condition (7) may be described as follows. Starting from an initial condition, the state trajectory reaches the time-varying sliding manifold \( s(x) = 0 \) in a finite time and subsequently slides along the switching line towards the origin of the phase plane \((e, \dot{e})\), where \( x(t) = x_{id}(t) \), with a time constant \( 1/\lambda \) according to (2).

Now, due to the sign function in (4), the SMC exhibits chattering, i.e., high frequency switching phenomena. These may be avoided introducing a boundary layer of width \( \Phi \), i.e., replacing the sign with a saturation function [21]:

\[ u = b^{-1}(x) \cdot (\dot{u} - K\text{sat}(\Phi^{-1}s(x))), \quad K, \Phi > 0 \]  

In (9) \( \dot{u} \) is defined according to (5) and the saturation function is defined as follows:

\[ \text{sat} \left( \frac{y}{\Phi} \right) = \begin{cases} +1 & \text{if } y > \Phi \\ \frac{y}{\Phi} & \text{if } -\Phi \leq y \leq \Phi \\ -1 & \text{if } y < -\Phi \end{cases} \]  

Since the boundary layer width \( \Phi \) indicates the ultimate boundedness of the system trajectory, the steady
state error may arbitrarily be adjusted by proper selection of $\Phi$ [21]. However, a small value of $\Phi$ may result in exciting high-frequency dynamics.

III. FUZZY SLIDING MODE CONTROL

The boundary layer SMC is effective in solving the typical VSC chattering issue, but can compromise the tracking accuracy, since the magnitude of the tracking error directly depends on the boundary layer width. A popular solution to overcome the drawbacks of pure SMC and of the boundary layer SMC consists in combining the SMC methodology with fuzzy logic algorithms [29, 15], thus relaxing the switching surface. Advantages of the resulting Fuzzy Sliding Mode Control (FSMC) strategies are the reduction of the chattering effect and of the computational effort with respect to SMC techniques.

In the recent literature numerous FSMC methodologies have been proposed. These may roughly be classified into two categories: fuzzy boundary layers and fuzzy Lyapunov functions. In the first approach a fuzzy map, rather than the saturation function, is employed as a replacement of the sign function in (4); hence, the boundary layer only is redefined and the overall SMC structure is maintained, with stability and robustness guarantees similar to SMC [17]. Related approaches employ a fuzzy controller to perform on-line tuning of the boundary layer width [6]. The main advantage of fuzzy boundary layers lies in the reduction of the strain on the actuators. The second approach employs a fuzzy control law that completely substitutes the SMC law expressed by (4) or (9), keeping the validity of the sliding condition (7), see [16]. Such a method is generally more heuristic than the former and turns out to be less computationally intense. A particular case is the approach by Wang [27]: after determining a SMC, an equivalent fuzzy control law with fixed accuracy is designed. Moreover, the well-known FSMC strategy proposed by Palm may be placed midway between the above categories [19, 20]. A fuzzy control surface is designed in order to simulate the sign function in the phase plane regions that are located far away from the origin, establishing a sort of variable gain in the remaining zones [4, 5]. Finally, some papers focus on parameters selection in FSMC with special interest on the sliding parameter tuning for tracking performance enhancement. Adaptive FSMC algorithms have been proposed [12, 13]; an alternative solution relies on genetic algorithms for optimal design of the FSMC parameters and structure [22, 9].

Generally speaking, FSMC techniques are characterized by a heuristic rule table in the phase plane, so that the overall control surface mimics a SMC with boundary layer. The typical FSMC for system (1) is governed by the following non linear control law [19, 20, 16, 23]:

$$u = b^{-1}(x) \cdot (\hat{u} + u_{fuzz}(e, \dot{e}, \lambda)),$$

where $\hat{u}$ is defined according to (5) and the fuzzy contribution lies in the term (see Fig. 1):

$$u_{fuzz}(e, \dot{e}, \lambda) = u_{fuzz}(s(x)) = -K_{fuzz}(s(x)) \cdot \text{sign}(s(x)),$$

where $K_{fuzz}(s(x)) \geq 0$. It has been shown that a SMC with boundary layer is mimicked by a FSMC (11)-(12) with the following rule table [16, 19, 20, 18]:

- $R^1$: If $s$ is Negative Big, then $u_{fuzz}$ is Positive Big.
- $R^2$: If $s$ is Negative Small, then $u_{fuzz}$ is Positive Small.
- $R^3$: If $s$ is Zero, then $u_{fuzz}$ is Zero.
- $R^4$: If $s$ is Positive Small, then $u_{fuzz}$ is Negative Small.
- $R^5$: If $s$ is Positive Big, then $u_{fuzz}$ is Negative Big.

In the following the FSMC membership functions for the input $s(x)$ and the output $u_{fuzz}$ [16, 18] are triangular with completeness equal to 1 [15]. If the sup-min compositional rule of inference is considered and the center of area defuzzification is adopted [15], the FSMC is similar to a SMC with boundary layer [16, 19, 20, 18], and equivalence may be achieved by proper selection of the FSMC parameters (i.e., the input/output scaling factors in Fig. 1 and the membership functions bounds), see for instance [16] for a discussion on the subject.

IV. FUZZY SLIDING MODE CONTROL WITH PIECEWISE LINEAR SWITCHING MANIFOLD

The FSMC (11)-(12) with the switching surface (2) and the previously stated rule table comprises two components: the term $b^{-1}(x)\hat{u}$, which is usually called the equivalent control term in the related literature, and a fuzzy contribution. In particular, the latter represents a control action with growing magnitude when the tracking error increases, i.e., when the system trajectory is far from the sliding manifold. Hence, if the initial operating point corresponds to a considerable error, then saturation of the actuator may occur and the available control action may be insufficient to hit the sliding mode. In this section we propose a novel FSMC based on a piecewise linear switching manifold, in order to enhance the effectiveness of FSMC under a saturated control input.
4.1. A novel FSMC methodology

Consider system (1) and define the following variable:

$$s^*(x) = \begin{cases} 
\dot{e} + \lambda e & \text{if } |e| < e_H \\
\dot{e} + \lambda e_H \cdot \text{sign}(e) & \text{if } |e| \geq e_H 
\end{cases}$$

(13)

where the sliding constant \(\lambda\) and the design parameter \(e_H\) are strictly positive. Now, consider the modified piecewise linear sliding surface (see Fig. 2):

$$s^*(x) = 0.$$  

(14)

Hence, according to (13) and (14), the phase plane may be divided into two main zones. First, a region where \(|e| < e_H\) (e.g. point A in Fig. 2) and the original and modified sliding surface overlap, with \(s^*(x) = s(x)\) representing the vertical distance from such a surface. Second, a region containing the points in which \(|e| \geq e_H\), where the two sliding lines and the vertical distances from such manifolds differ, i.e., \(s^*(x) \neq s(x)\). In particular, in the latter case we distinguish the following seven regions in the phase plane (Fig. 2):

- **Zone 1**: points with \(|e| < e_H\), \(s = s^*\) (e.g. A in Fig. 2);
- **Zone 2**: points with \(|e| \geq e_H\), \(|s^*| \leq |s|\) and \(s, s^* \geq 0\) (e.g. B in Fig. 2);
- **Zone 3**: points with \(|e| \geq e_H\), \(|s^*| \leq |s|\) and \(s, s^* \leq 0\) (e.g. C in Fig. 2);
- **Zone 4**: points with \(|e| \geq e_H\), \(|s^*| \geq |s|\) and \(s, s^* \leq 0\) (e.g. D in Fig. 2);
- **Zone 5**: points with \(|e| \geq e_H\) and \(s^* \geq 0, s \leq 0\) (e.g. E in Fig. 2);
- **Zone 6**: points with \(|e| \geq e_H\) and \(s^* \leq 0, s \geq 0\) (e.g. F in Fig. 2);
- **Zone 7**: points with \(|e| \geq e_H\) and \(s^* \leq 0, s \geq 0\) (e.g. G in Fig. 2).

Now, consider system (1) with a FSMC law of type

$$u = b^{-1}(x) \cdot (\dot{u}^* + u^* \text{fuzz}(e, \dot{e}, \lambda)), \quad (15)$$

where a discontinuous equivalent control is defined as follows

$$\dot{u}^* = \begin{cases} 
-(f(x) - \dot{x}_d + \lambda \dot{e}) & \text{if } |e| < e_H \\
-(f(x) - \dot{x}_d) & \text{if } |e| \geq e_H 
\end{cases}$$

(16)

and the fuzzy contribution in (15) is identified by the term

$$u^* \text{fuzz}(e, \dot{e}, \lambda) = u^* \text{fuzz}(s^*(x)) = -K_{fuzz}(s^*(x)) \cdot \text{sign}(s^*(x)). \quad (17)$$

In the sequel the FSMC rule table and design pa-
rameters described in section 3 are maintained, with the
substitution of \( s \) with \( s' \) and \( u_{fuzz} \) with \( u'_{fuzz} \).

In the following we show that system (1) is stable
under the control law (15)-(16)-(17), i.e., the tracking
error converges to zero in a finite time and the surface
\( s'(x) = 0 \) is attractive for the closed-loop system.

**Proposition 1.** The sliding line (14) is attractive for the
state trajectory of system (1) under the FSMC with
piecewise linear sliding manifold (15)-(16)-(17).

**Proof.** Consider the Lyapunov function
\[
V = \frac{1}{2} s'^2(x). 
\]  
(18)

By (1), (15), (16) and (17), for all the operating points
\( x(t) \) such that \( |e| < e_H \) it holds:
\[
\dot{x}_f = f(x) + b(x) \cdot u + d \\
= \dot{x}_{tu} - \lambda \dot{e} - K_{fuzz}(s'(x)) \cdot \text{sign}(s'(x)) + d. 
\]  
(19)

Therefore, we have:
\[
\ddot{e} = -\lambda \dot{e} - K_{fuzz}(s'(x)) \cdot \text{sign}(s'(x)) + d. 
\]  
(20)

Hence, for all points \( x(t) \) such that \( |e| < e_{fuzz} \), by (18) and (20) we get:
\[
\dot{V} = s'(x) \cdot s'(x) = s'(x) \cdot (\ddot{e} - \lambda e) \\
= -K_{fuzz}(s'(x)) \cdot \|s'(x)\| + d \cdot s'(x) \leq -\eta^* \|s'(x)\| 
\]  
(21)

where \( \eta^* \) is a strictly positive design constant such that
\( K_{fuzz} \geq \eta^* + D, \) \( K'_{fuzz} \) being the upper bound of
\( K_{fuzz}(s'(x)). \) On the other hand, for all the points \( x(t) \) such
that \( |e| \geq e_{fuzz} \), by (13) we have \( \dot{s}'(x) = \ddot{e}. \) Now, reasoning
as above, by (15)-(16)-(17) we get:
\[
\ddot{e} = -K_{fuzz}(s'(x)) \cdot \text{sign}(s'(x)) + d. 
\]  
(22)

Hence, for all points \( x(t) \) such that \( |e| \geq e_{fuzz} \), it holds:
\[
\dot{V} = s'(x) \cdot s'(x) = s'(x) \cdot \ddot{e} \\
= -K_{fuzz}(s'(x)) \cdot \|s'(x)\| + d \cdot s'(x) \leq -\eta^* \|s'(x)\| 
\]  
(21)

and we get the modified sliding condition (21) again.

Therefore, \( \dot{V} \) is negative definite in the piecewise
linear switching manifold, and the modified sliding line
(14) is attractive for the FSMC (15)-(16)-(17). In particu-
lar, the modified sliding condition (21) guarantees
reaching of the sliding line in a finite time.

**Proposition 2.** Consider system (1) under the FSMC
(15)-(16)-(17). If the initial state is such that \( x_i(t = 0) \neq x_{id}(t = 0) \) and the system trajectory is off the piecewise
linear sliding line (14), then the surface \( s'(x) = 0 \) will be
reached in a finite time \( t_{reach} \) such that:
\[
t_{reach} \leq \frac{\|s'(x(t = 0))\|}{\eta^*}.
\]  
(23)

**Proof.** Let us call \( t_{reach} \) the time required to hit the attractive
piecewise linear switching manifold (14). Assume
for instance \( s'(t = 0) > 0 \). Integrating the modified sliding
condition (21) between \( t = 0 \) and \( t = t_{reach} \) leads to:
\[
\int_{s'(t = 0)}^{s'(t = t_{reach})} \text{sign}(s') ds' \leq -\eta^* \int_{t = 0}^{t_{reach}} dt, 
\]  
(24)

hence
\[
s'(t = t_{reach}) - s'(t = 0) \leq -\eta^* \cdot (t_{reach} - 0) 
\]  
(25)

and trivial transformations lead to (23). The same result
may be shown for \( s'(t = 0) < 0 \).

A comparison of (23) with (8) shows that for an
initial state in zone 1 in Fig. 2, such that \( s'(x) = s(x) \), the
reaching time upper bound equals the one obtained with
the classical sliding surface (2), under the hypothesis that
the corresponding FSMCs share the same design struc-
ture and parameters. On the other hand, it is worth noting
that if the initial state lies in a region different from zone
1 of the phase plane (see Fig. 2), then \( |s'(x)| \neq |s(x)| \) and
comparison of the reaching time upper bounds (23) and
(8) depends on the choice of the design variables \( \eta \) and
\( \eta^* \) in the corresponding FSMCs.

We have proven that after a reaching phase of finite
duration the system trajectory hits the modified sliding
surface (14). Now, to prove that (1) is stable under
(15)-(16)-(17), we demonstrate that once the system en-
ters the sliding mode it keeps sliding along the switching
manifold to the origin \( (0, 0) \) of the phase plane, hence
tracking is achieved.

**Proposition 3.** Once the system (1) under (15)-(16)-(17)
hits the piecewise linear switching manifold (14), the
system trajectory converges to the origin of the phase
plane.

**Proof.** Let \( x(t = 0) \) be the initial state on the switching
line (14). Hence, \( s'(x(t = 0)) = 0 \). Again, consider two
possible circumstances. If \( x(t = 0) \) is such that \( |e| < e_{fuzz} \),
then \( s'(x(t = 0)) = s(x(t = 0)) = 0 \) and integrating (2)
leads to [21]:
\[
e(t) = e(t = 0) e^{-\lambda t},
\]  
(26)
hence, the system trajectory converges to the origin with a time constant $1/\lambda$.

On the other hand, if $x(t = 0)$ is such that $|e| \geq e_{fl}$, and $s'(x(t = 0)) = 0$, by (13) it holds:

$$\dot{e} = -\lambda e_{fl} \cdot \text{sign}(e(t = 0)).$$

By integrating (27) between $t = 0$ and $t$ such that $|e| \geq e_{fl}$, we get:

$$e(t) = e(t = 0) - \lambda e_{fl} \cdot \text{sign}(e(t = 0)) \cdot t$$

or, equivalently,

$$|e(t)| = |e(t = 0)| - \lambda e_{fl} \cdot t.$$ (29)

Hence, $|e(t)|$ decreases until the trajectory enters zone 1 in figure 2 and (26) holds. We remark that the larger the factor $\lambda e_{fl}$, the faster the reduction of the error magnitude.

The proposed FSMC with piecewise linear switching manifold guarantees tracking of the desired trajectory and represents an alternative to the traditional FSMC discussed in section 3. Moreover, it is worth noting that in numerous applications employing the proposed FSMC with a proper choice of $e_{fl}$ reduces the initial sliding variable magnitude with respect to the corresponding one obtained with a classical FSMC. It is reasonable to suppose that, if $e_{fl}$ has been properly selected, the initial operating point belongs to zones 2 or 3 in Fig. 2. In particular, it often occurs that such an initial state is located on the horizontal Cartesian axis, such that $\dot{e}(t = 0) = 0$. This corresponds to $x_2(t = 0) = x_{id}(t = 0)$, which is often the case (e.g. if $x_2$ represents the plant speed, $x_{id}(t = 0)$ is desired and the initial velocity is equal to zero). If the initial operating point lies in zones 2 or 3 in Fig. 2, then $|s'(x)| \leq |s(x)|$ holds. Now, consider a modified FSMC differing from the one defined in the previous section only with regard to the input $s'(x)$ as a replacement of $s(x)$, while the rule table is the same. Hence, the fuzzy contribution to the control action is smaller than the corresponding one required when $s(x)$ is the input to the FSMC. Summing up, a proper selection of $e_{fl}$ results in an initial state in zones 2-3 such that $|e| \geq e_{fl}$ and a suitable value of $K_{fuzzmax}$ forces the trajectory to hit the sliding mode in a finite time with an upper bound (23) and a reduced control action. In the following we discuss such a feature of the proposed FSMC with respect to the occurrence of saturation in the control input.

4.2 Comparison of the proposed FSMC with classical FSMC under saturation effects

Although in real systems the control input is typically constrained by saturation effects, analyses of VSC techniques taking into account a saturated input are rare in the literature. In this sub-section we discuss the saturation effect occurring when the actuator range is limited with respect to the control action required by operation of a pure SMC or FSMC.

If the actuator saturates, implementing a classical SMC (4) or FSMC (11) results in the following control law:

$$u = \text{sat}\left(\frac{u_i}{U}\right) \cdot U,$$ (30)

where $u$ and $u_i$ represent the actual and ideal control law respectively, $U$ is the maximum available control magnitude and (10) holds. Consider an ideal FSMC:

$$u_i = b^{-1}(x) \cdot (\dot{u} - K_{fuzz}(s(x)) \cdot \text{sign}(s(x))),$$ (31)

where (5) holds. The following discussion is detailed for FSMC, nevertheless we remark that the same results hold for pure SMC if we consider $K$ as a replacement for $K_{fuzz}$ in (31). Now, suppose that saturation occurs at a time instant $t$. By (30) it holds:

$$u(t) = \pm U,$$ (32)

$$u_i(t) = \pm U \pm \Delta(x(t)),$$ (33)

where $\Delta(x(t))$ is the absolute value of the residual control action, depending on the actual system state at the time instant $t$. Hence, by (33) we may re-write (32) as follows:

$$u = u_i - \text{sign}(u_i) \cdot \Delta(x)$$ (34)

where we neglected the time dependencies for sake of simplicity. Now, by derivation of (2) and taking into account the system dynamics (1) we get:

$$\dot{s}(x) = f(x) + b(x) \cdot u + d(x) - x_{id} + \lambda \dot{e}.$$ (35)

Hence, by substitution of (34), (31) and (5) in (35) it holds:

$$\dot{s}(x) = -K_{fuzz}(s(x)) \cdot \text{sign}(s(x)) + d(x) - b(x) \cdot \Delta(x) \cdot \text{sign}(u_i).$$ (36)

Now, consider the derivative of the Lyapunov function (3):

$$\dot{V} = s(x) \dot{s}(x) = -|s(x)| \cdot (K_{fuzz}(s(x)) - d(x) \cdot \text{sign}(s(x)) + b(x) \cdot \Delta(x) \cdot \text{sign}(s(x) \cdot u_i)).$$ (37)

Hence, $V$ is negative definite in the sliding sur-
face (2) if it holds:

\[ K_{\text{fuzz}}(s(x)) - d(x) \cdot \text{sign}(s(x)) \]
\[ + b(x) \cdot \Delta(x) \cdot \text{sign}(s(x) \cdot u_i) > 0. \]  

We remark that by design of the FSMC the sliding condition (7) holds, i.e.:

\[ K_{\text{fuzz}}(s(x)) - d(x)\text{sign}(s(x)) \geq \eta > 0. \]  

Hence, \( V \) remains negative definite in the sliding surface under (34) if it holds:

\[ \eta > - b(x) \cdot \Delta(x) \cdot \text{sign}(s(x) \cdot u_i) \]
\[ = - |b(x)| \cdot \Delta(x) \cdot \text{sign}(s(x) \cdot b(x) \cdot u_i). \]  

Now, if \( \text{sign}(s(x) \cdot b(x) \cdot u_i) \geq 0 \), then (40) is true. On the other hand, if \( \text{sign}(s(x) \cdot b(x) \cdot u_i) = -1 \) a sufficient condition for (40) to hold is

\[ \Delta(x) < \frac{\eta}{|b(x)|} \cdot \text{sign}(s(x) \cdot b(x) \cdot u_i) = -1. \]  

Summing up, the switching manifold is attractive for system (1) under a FSMC with linear sliding surface and saturated input expressed by (30) if one of the following conditions holds:

\[ \text{sign}(s(x) \cdot b(x) \cdot u_i) \geq 0. \]  

(42a)

\[ \Delta(x) < \frac{\eta}{|b(x)|} \cdot \text{sign}(s(x) \cdot b(x) \cdot u_i) = -1. \]  

(42b)

We remark that saturation of the control input occurs if it holds:

\[ -b^{-1}(x) \cdot (f(x) - \bar{x}_{\text{ia}} + \lambda \hat{e} + K_{\text{fuzz}}(s^{*}(x)) \cdot \text{sign}(s^{*}(x))) \geq U. \]  

(43)

Now, consider a FSMC with piecewise linear switching manifold (14) sharing the design parameters of the FSMC with classical sliding surface (31). The actual control action is still expressed by (30), where \( u_i \) is now replaced by:

\[ u_i = b^{-1}(x) \cdot (\hat{u}^{*} - K_{\text{fuzz}}(s^{*}(x)) \cdot \text{sign}(s^{*}(x))) \]  

and \( \hat{u}^{*} \) is expressed by (16).

Reasoning as above, we infer that the switching manifold is attractive for (1) under a FSMC with piecewise linear sliding surface and saturated input (30) if one of the following holds:

\[ \text{sign}(s^{*}(x) \cdot b(x) \cdot u_i) \geq 0. \]  

(45a)

\[ \Delta^{*}(x) < \frac{\eta^{*}}{|b(x)|} \cdot \text{sign}(s^{*}(x) \cdot b(x) \cdot u_i) = -1. \]  

(45b)

We remark that under (44) saturation occurs when a similar expression to (43) holds:

\[ -b^{-1}(x) \cdot (f(x) - \bar{x}_{\text{ia}} + \lambda \hat{e} + K_{\text{fuzz}}(s^{*}(x)) \cdot \text{sign}(s^{*}(x))) \geq U \text{ if } |\hat{e}| < e_H. \]  

(46a)

\[ -b^{-1}(x) \cdot (f(x) - \bar{x}_{\text{ia}} + K_{\text{fuzz}}(s^{*}(x)) \cdot \text{sign}(s^{*}(x))) \geq U \text{ if } |\hat{e}| \geq e_H. \]  

(46b)

Now, consider an initial state \( x(t) \) and a design parameter \( e_H \) such that \( |e| \geq e_H \) and the initial operating point \((\hat{e}, \hat{e})\) belongs to zones 2 or 3 in figure 2, i.e., it holds \(|s^{*}(x)| \leq |s(x)|\). Consider two FSMCs defined by (30)-(31) and (30)-(44), identical for all their design parameters but for the switching manifold. Suppose that the initial condition \( x(t) \) is such that both the FSMCs produce a saturated control input: in other words, (43) and (46b) hold. By (34) we get:

\[ \Delta(x) = u_i \cdot \text{sign}(u_i) - U \]  

and by (30) and (31) it holds:

\[ \Delta(x) = |b^{-1}(x)| \cdot \hat{u} \cdot \text{sign}(b(x) \cdot u_i) \]
\[ - |b^{-1}(x)| \cdot K_{\text{fuzz}}(s(x)) \cdot \text{sign}(s(x) \cdot b(x) \cdot u_i) - U. \]  

(48)

Now, either (42a) holds and \( \dot{V} < 0 \), or else \( \text{sign}(s(x) \cdot b(x) \cdot u_i) = -1 \) and (48) may be re-written as:

\[ \Delta(x) = |b^{-1}(x)| \cdot \hat{u} \cdot \text{sign}(b(x) \cdot u_i) \]
\[ + |b^{-1}(x)| \cdot K_{\text{fuzz}}(s(x)) - U. \]  

(49)

A similar reasoning applies for the FSMC with piecewise linear sliding manifold, hence:

\[ \Delta^{*}(x) = |b^{-1}(x)| \cdot \hat{u}^{*} \cdot \text{sign}(b(x) \cdot u_i) \]
\[ + |b^{-1}(x)| \cdot K_{\text{fuzz}}^{*}(s^{*}(x)) - U. \]  

(50)

Therefore, since \( |e| \geq e_H \), by (5) and (15) the difference between (49) and (50) is:

\[ \Delta(x) - \Delta^{*}(x) = |b^{-1}(x)| \cdot \lambda \hat{e} \cdot \text{sign}(s(x)) \]
\[ + |b^{-1}(x)| \cdot (K_{\text{fuzz}}(s(x)) - K_{\text{fuzz}}^{*}(s^{*}(x))). \]  

(51)

Hence \( \Delta(x) \geq \Delta^{*}(x) \) if the following condition holds:
Fig. 3. Phase plane trajectory (a) and control input (b) under SMC (grey), FSMC (black dashed) and FSMC with piecewise linear switching line (black solid).

\[
\alpha(x) = K_{fuzz}(s(x)) - K^*_fuzz(s^*(x)) + \lambda |e| \cdot \text{sign}(s(x) \cdot e) \geq 0. \tag{52}
\]

On the other hand, by \(|s^*(x)| \leq |s(x)|\) we get (see the rule table in section 3):

\[
0 \leq K_{fuzz}(s^*(x)) \leq K_{fuzz}(s(x)). \tag{53}
\]

Hence, if the FSMC with piecewise linear switching manifold is designed such that it holds:

\[
\alpha(x) \geq 0 \tag{54}
\]

we get \(\Delta(x) \geq \Delta^*(x)\) and we infer that condition (45b), derived under the FSMC with piecewise linear switching manifold, is less restrictive than the corresponding condition (42b) derived for the classical FSMC. This follows from considering that at saturation \(\eta = \eta^*\), since both FSMCs produce their maximum outputs. In particular, we remark that (54) is true for all points in zone 2 belonging to the first quadrant and in zone 3 belonging to the third quadrant (see Fig. 2). On the other hand, if (45a) and (42a) hold, the two FSMCs are both effective in hitting the sliding line, despite the occurrence of saturation.

We conclude that under suitable design of the piecewise linear switching manifold the proposed technique represents an effective alternative to the classical FSMC with linear sliding surface for a dynamical system (1) under saturated control input.

V. A SIMPLE DESIGN EXAMPLE

In this section we illustrate the proposed technique by means of a simple design example in the form (1). Consider the unstable system [2]:

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = x_2 + u \tag{55}
\]

Suppose that the design objective is \(x_{1d}(t) = x_{2d}(t) = 0\), hence \(e(t) = x_1(t)\) and \(\dot{e}(t) = x_2(t)\). To design a VSC, choose \(\lambda = 1\), so that the sliding line is [2]:

\[
s(x) = x_1 + x_2 = 0. \tag{56}
\]

Now, choose \(\eta = K = 0.5\) and the classical SMC as follows [2]:

\[
u = -2x_2 - 0.5 \cdot \text{sign}(s(x)). \tag{57}
\]

In Fig. 3 we report the simulation results (grey solid line) for the trajectory in the phase plane \((e, \dot{e})\) (Fig. 3(a)) and the control input (Fig. 3(b)) with \(x_1(0) = 1.5\) and \(x_2(0) = 0\). We remark that the SMC is effective in stabilizing the system, but the control signal exhibits considerable chattering.

Now, consider a FSMC with sliding surface (56) and the following control law:

\[
u = -2x_2 - K_{fuzz}(s(x)) \cdot \text{sign}(s(x)), \tag{58}
\]

where \(K_{fuzz}\) is defined according to the following simplified set of rules:

R1: If s is Negative, then \(K_{fuzz}\) is Negative.
R2: If s is Zero, then \(K_{fuzz}\) is Zero.
R3: If s is Positive, then \(K_{fuzz}\) is Positive.

The membership functions for the FSMC input and output are triangular with completeness equal to 1 in the normalized ranges \([-10, 10]\) for s and \([-1, 1]\) for \(K_{fuzz}\). The sup-min compositional rule of inference is considered and the center of area defuzzification is adopted [29,15]. Moreover, the selected input scaling gain is \(G_s = \ldots\)
5 and the output normalization factor equals $G_{K_{fuzz}} = 4$. In Fig. 3 the trajectory in the phase plane $(e, \hat{e})$ (Fig. 3(a)) and the control input (Fig. 3(b)) are depicted (black dashed line) under the designed FSMC. We remark that the FSMC is effective in stabilizing the system, while the control signal is exempt from chattering.

Now, consider a FSMC with piecewise linear sliding line as follows:

$$u = b^{-1}(x) \cdot (\hat{u}^* - K_{fuzz}(s^*(x)) \cdot \text{sign}(s^*(x))). \quad (59)$$

Select the following piecewise linear sliding surface with $\lambda = 1$ and $e_{ti} = 0.25$:

$$s^*(x) = \begin{cases} 
  x_2 + x_1 & \text{if } |x_1| < 0.25 \\
  x_2 + 0.25 \cdot \text{sign}(x_1) & \text{if } |x_1| \geq 0.25
\end{cases} \quad (60)$$

Moreover, according to (16), define:

$$\hat{u}^* = \begin{cases} 
  -2x_2 & \text{if } |x_1| < 0.25 \\
  -x_2 & \text{if } |x_1| \geq 0.25
\end{cases} \quad (61)$$

and select the FSMC parameters just like in the previous FSMC, with $s^*(x)$ as a replacement to $s(x)$, $u_{fuzz}$ to $u_{fuzz}$ and $K_{fuzz}$ to $K_{fuzz}$, respectively.

In Fig. 3 the simulation results for the trajectory in the phase plane $(e, \hat{e})$ (Fig. 3(a)) and the control input (Fig. 3(b)) are depicted (black solid line) under the modified FSMC. We remark that the modified FSMC is effective in stabilizing the system, producing a control signal which is not affected by chattering while in the initial transient its magnitude is smaller than the corresponding effort required by the previous FSMC. On the other hand, the system is about 1.5 seconds slower under the modified FSMC.

Now, we repeat the simulations under the previous three control laws when the maximum control input magnitude is $U = 0.3$. Results are reported in Fig. 4 for the trajectory in the phase plane $(e, \hat{e})$ (Fig. 4(a)) and the control input (Fig. 4(b)).

While the SMC and FSMC with linear sliding surface are not effective in controlling the system, under the proposed FSMC the system reaches the point $(0, 0)$ in the phase plane.

In Fig. 5(a) we report the residual control action with saturation under the three control laws. In particular, as regards the two FSMCs it is apparent that $\Delta(x) \geq \Delta^*(x)$ for all time instants.

Accordingly, in Fig. 5(b) we report $\alpha(x)$, which is always positive, as expected.

VI. ILLUSTRATED DESIGN EXAMPLE: INVERTED PENDULUM ON A CART

In the sequel, we illustrate the proposed FSMC technique for an inverted pendulum on a cart.

6.1. Model of the inverted pendulum

In this section we recall the model of an inverted pendulum (see Fig. 6), a well known benchmark for automatic control techniques [28,26,1,25,3]. Two poles, hinged to a cart moving on a track, are balanced upwards by a horizontal force applied to the cart via a DC motor. The cart is simultaneously motioned to an objective position on the track, which is finite. The system state vector is $x = [x_1 \ x_2 \ x_3 \ x_4]^T$, including the cart horizontal distance from the track center, the poles angular distance from the upwards equilibrium point and their derivatives. The force motioning the cart may be expressed as $F = \beta u$ ($\beta = 11.5$), where $u$ is the input, i.e., the motor supply voltage, limited between $-1$ and $+1$. The system model is [10]:
Fig. 5. Residual control action $\Delta(x)$ under SMC (grey), FSMC (black dashed) and FSMC with piecewise linear sliding line (black solid) (a) and factor $\alpha(x)$ (b) with saturation.

Fig. 6. Inverted pendulum on a cart.

\begin{align}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= f_1(x) + b_1(x)u + d_1(x), \\
\dot{x}_4 &= f_2(x) + b_2(x)u + d_2(x),
\end{align}

where $d_1(x)$ and $d_2(x)$ are external disturbances of known bounds $D_1$ and $D_2$ respectively and

\begin{align}
f_1(x) &= \frac{a(-T_c - \mu x_4 \sin x_2) + 1 \cos x_3 (\mu g \sin x_2 - f_p x_4)}{J + \mu_1 \sin^2 x_2}, \\
b_1(x) &= \frac{\alpha}{J + \mu_1 \sin^2 x_2},
\end{align}

with $f_p = 3.7 \times 10^{-4}$.

The cart and poles masses are $m_c = 1.12$ kg and $m_p = 0.12$ kg, $L = 0.25$ m is half the poles length, $g$ is the gravity acceleration, $J$ is the system moment of inertia with respect to its center of mass, $f_p = 3.7 \times 10^{-4}$ is the poles rotational friction coefficient and $T_c$ is the horizontal friction:

\begin{align}
f_2(x) &= \frac{1 \cos x_3 (-T_c - \mu_1 \mu_2 \sin x_4) + \mu g \sin x_2 - f_p x_4}{J + \mu_1 \sin^2 x_2}, \\
b_2(x) &= \frac{1 \cos x_2 \beta}{J + \mu_1 \sin^2 x_2},
\end{align}

with

\begin{align}
1 &= \frac{Lm_p}{2(m_c + m_p)}, \quad J = \frac{L^2 m_p (4m_c + m_p)}{12(m_c + m_p)}, \quad a = l^2 + \frac{J}{m_c + m_p}, \quad \mu = (m_c + m_p)l.
\end{align}

The poles and cart masses are $m_c = 1.12$ kg and $m_p = 0.12$ kg, $L = 0.25$ m is half the poles length, $g$ is the gravity acceleration, $J$ is the system moment of inertia with respect to its center of mass, $f_p = 3.7 \times 10^{-4}$ is the horizontal friction:

\begin{align}
f_c &= 0.31 \text{ N}; \quad f_s = 0.83 \text{ N}; \quad y_c = 0.575 \text{ N}; \quad x_c = 0.19 \text{ m} \cdot \text{s}^{-1}; \quad \text{DZ}_{cv} = 0.008 \text{ m} \cdot \text{s}^{-1}.
Expression (64) computes the cart friction $T_c$, with respect to the cart velocity, as the sum of a quadratic static friction and a linear dynamical friction, where $F_d$, $F_s$, $y_c$ and $x_c$ are parameters depending on the system under study. Moreover, $DZ_{c v}$ indicates the cart dead zone, i.e., the minimum speed with which the cart moves. Similar dead zones apply to the poles velocity and the horizontal force, so that the poles oscillate but the cart stays still when one or more of the following apply: $|x_3| < DZ_{c v}$, $|x_4| < DZ_{p v}$, or $|F| < DZ_{a}$ ($DZ_{p v} = 4.6 \text{ rad} \cdot \text{s}^{-1}$; $DZ_{a} = 0.9 \text{ N}$).

Here, the control task is to swing the poles up from their pendulum position on the track. Hence, the system desired trajectory is $x_{1d} = x_{1d}(t)$, $x_{2d} = 0$, $x_{3d} = x_{3d}(t) = 0$ and $x_{4d} = 0$. There is one control input $u$, which is bounded: the DC motor saturates. We remark that the track is finite, hence $x_2$ is constrained, in particular we consider $x_2 \in [-0.5, 0.5]$

We observe that the inverted pendulum (62) is an under-actuated fourth order system including two second order subsystems, namely, the cart subsystem and the pendulum subsystem, with state vectors respectively $[x_1, x_3]^T$ and $[x_2, x_4]^T$, coupled by way of the control input $u$. For each subsystem the state variables of the other one may be regarded as fictitious uncontrollable inputs. Moreover, the poles dynamics, i.e., the second and fourth equations in (62), are affected by the cart dynamics only by way of friction $T_c$. If we consider this contribution as a disturbance, the poles subsystem is in the form (1) with $d(x)$ incorporating both the actual disturbance $d_2(x)$ and friction. In other words, the poles nonlinearities affect the cart subsystem but the latter does not affect the poles dynamics. Hence, to control the system with one input, we can balance the poles first, and then the cart.

Numerous studies have been developed on the inverted pendulum in the scientific literature [28,26,1,25,3]. In the following we apply the proposed FSMC technique with piecewise linear switching manifold in order to stabilize and swing-up the inverted pendulum on a cart. Moreover, we take into account restrictions such as a finite track and a limited available control action, which are typical for a real system.

6.2. Application of the proposed FSMC to the inverted pendulum: stabilization and swing-up

In this section we apply the proposed FSMC with piecewise linear switching manifold in order to stabilize and swing-up the inverted pendulum. As previously remarked, once the pendulum is balanced it holds $x_2 = x_4 = 0$ and the cart dynamics, i.e., the first and third equations in (62), is linear if we neglect friction. Employing linear state feedback is thus sufficient to control the cart when the poles are in equilibrium. Hence, we apply the following control law:

$$u = u_1 + u_2.$$ (65)

The first contribution in (65) implements linear state feedback in order to control the cart position, with feedback gains determined via pole allocation for the linearized system [10]:

$$u_1 = K_1x_1 + K_3x_3.$$ (66)

In the following we regulate the poles to position $(x_{3d}(t), x_{4d}(t))=(0, 0)$, while the cart is forced to track a square-wave desired trajectory $(x_{1d}(t), x_{2d}(t))$. Hence, the second contribution in (65) implements the control law (15)-(16)-(17), in order to balance the pole upwards:

$$u_2 = b_2^{-1}(x) \cdot (\hat{u}^* - K_{f d}(s^*(x)) \cdot \text{sign}(s^*(x))),$$ (67)

with

$$\hat{u}^* = \begin{cases} - \hat{f}_1(x) - \lambda x_4 & \text{if } |x_2| < e_H, \\ - \hat{f}_2(x) & \text{if } |x_2| \geq e_H, \end{cases}$$ (68)

$$s^*(x) = \begin{cases} x_4 + \lambda e_H \cdot \text{sign}(x_2) & \text{if } |x_2| < e_H, \\ x_4 + \lambda e_H \cdot \text{sign}(x_2) & \text{if } |x_2| \geq e_H, \end{cases}$$ (69)

$$\hat{f}_1(x) = f_1(x)|_{\nu = 0},$$ (70)

where we have taken into account that the error on the poles angle is $e_2 = x_2$.

It can be shown that the previously proposed controller (65) can control the system only for a small initial error on the poles angle (stabilization), but it cannot balance the pendulum upwards starting from its pendulum equilibrium while simultaneously controlling the cart position (swing-up), with the above physical restrictions. Hence, we decompose the control objective into three sub-tasks, namely, swing-up, transition and stabilization. Thus, we define three target zones in the phase plane, corresponding respectively: 1) to swing-up, 2) to a transition mode, 3) to stabilization (see Fig. 7). Accordingly, the controller is defined as follows. First, a controller stabilizing the poles is designed according to (65), with linear feedback in parallel with a FSMC with piecewise linear sliding manifold; next, a similar controller is synthesized both for the swing-up and transition zones, in order to enter the transition region first and the stabilization zone next. Finally, a gain scheduler is introduced in order to select the appropriate controller during operation. The parameters scheduler checks whether the pendulum is in stabilization, transition or swing-up mode and applies the corresponding controller accordingly. Here, we employ a stabilization zone with range $[-0.3,$
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Fig. 7. Inverted pendulum swing-up and stabilization zone.

stabilization zone

initial position
( stable equilibrium)

transition zone

final position
(unstable equilibrium)

swing-up zone

Fig. 7. Inverted pendulum swing-up and stabilization zone.

A first set of tests has been carried out to perform stabilization of the pendulum: the pendulum, initially in its upwards equilibrium, must be balanced while the cart is positioned to an objective position on the track. In particular, the overall controller (65) is operated for initial zero conditions of the plant and a square wave cart set point (Fig. 8, left). With a rail limit of 50 cm from the track center (the track is 1 m long), results for the simulated pendulum and the lab rig are similar. Hence, we report only the experimental outcomes of the stabilization test for the cart (Fig. 8, top) and pole (Fig. 8, bottom), with the following parameters:

\[ K_{1s} = 0.4, K_{3s} = 0.5, \lambda_s = 6, e_H = 0.5, \]
\[ G_{Ss} = 0.7, G_{Us} = 6 \]  

(71)

where the subscript refers to the stabilization zone.

We remark that here the FSMC is employed only in the linear region of its sliding manifold, hence the bending value \( e_H \) is not operational.

Now, consider the task of stabilizing the inverted pendulum with a consistent initial error on the pole angle. In the following we focus on an initial error on the pole position equal to \( \pi \): the pendulum is initially in its pendant equilibrium. Hence, we modify the control law (65) in order to swing-up the poles and simultaneously control the cart to the variable set-point in Fig. 8.

If we employ a FSMC with linear sliding line (2), the available control action is too small to stabilize the poles: the trajectory deviates from the switching manifold and spirally converges to a limit cycle. In other words, due to the actuator saturation, the pendulum permanently oscillates around its pendant equilibrium (see the simulated trajectory in Fig. 9).

Now, let us design a modified FSMC for both the transition and swing-up zones. We have:

\[ K_{1t} = -1, K_{3t} = -2, \lambda_t = 6, e_H = 0.5, G_{St} = 0.7, \]
\[ G_{Ut} = 1.5, \]
\[ K_{1su} = -0.65, K_{3su} = -0.1, \lambda_{su} = 6, e_H = 0.5, \]
\[ G_{Ssu} = 0.7, G_{Usu} = 1.5, \]

(72a)

(72b)

where the subscripts refer respectively to the transition and swing-up zones. We remark that while the sliding surface is kept unchanged in the three zones in Fig. 7,

Fig. 9. Failed attempt of inverted pendulum swing-up with linear sliding line.

0.3], a transition zone with range \([-π/2+1, -0.3] \cup [0.3, π/2+1]\) and a swing-up zone with range \([-π, -π/2+1] \cup [π/2+1, π]\), see Fig. 7.

A first set of tests has been carried out to perform stabilization of the pendulum: the pendulum, initially in its upwards equilibrium, must be balanced while the cart is positioned to an objective position on the track. In particular, the overall controller (65) is operated for initial zero conditions of the plant and a square wave cart set point (Fig. 8, left). With a rail limit of 50 cm from the track center (the track is 1 m long), results for the simulated pendulum and the lab rig are similar. Hence, we report only the experimental outcomes of the stabilization test for the cart (Fig. 8, top) and pole (Fig. 8, bottom), with the following parameters:

\[ K_{1s} = 0.4, K_{3s} = 0.5, \lambda_s = 6, e_H = 0.5, \]
\[ G_{Ss} = 0.7, G_{Us} = 6 \]  

(71)

where the subscript refers to the stabilization zone.

We remark that here the FSMC is employed only in the linear region of its sliding manifold, hence the bending value \( e_H \) is not operational.

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If we employ a FSMC with linear sliding line (2), the available control action is too small to stabilize the poles: the trajectory deviates from the switching manifold and spirally converges to a limit cycle. In other words, due to the actuator saturation, the pendulum permanently oscillates around its pendant equilibrium (see the simulated trajectory in Fig. 9).

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\[ G_{Ut} = 1.5, \]
\[ K_{1su} = -0.65, K_{3su} = -0.1, \lambda_{su} = 6, e_H = 0.5, \]
\[ G_{Ssu} = 0.7, G_{Usu} = 1.5, \]

(72a)

(72b)

where the subscripts refer respectively to the transition and swing-up zones. We remark that while the sliding surface is kept unchanged in the three zones in Fig. 7,
the FSMC output gain is reduced to avoid saturation of the control action. Further, in order to decrease the cart overshoot we employ a variable feedback for the cart, namely a positive feedback while swinging-up and in the transition mode. In Fig. 10 the simulation results for the inverted pendulum swing-up are reported. Finally, results for the real system are depicted in Fig. 11.

VII. CONCLUSIONS

The scope of this paper is to introduce a novel FSMC technique for a class of second order dynamical systems based on a piecewise linear sliding manifold that is bent at the extremes of the phase plane. We demonstrate that a second order nonlinear system in canonical form is stable under the proposed FSMC. We prove that the advantage of the approach lies in the decrease of the control action with respect to SMC and FSMC techniques employing a linear sliding surface, enhancing the effectiveness of VSC in the presence of a saturated control input. In addition, we show that the fuzzy rule based algorithms smooth the system dynamics in the vicinity of the switching manifold with respect to the adoption of classical VSC strategies. It is noteworthy that the proposed approach may be applied with a crisp component replacing the fuzzy contribution in the control law, especially when chattering is not particularly detrimental. We illustrate the proposed technique on a simple design example and on an inverted pendulum, a well-known benchmark for automatic control techniques. Several simulations and laboratory experiments show the effectiveness of the proposed strategy. Future research directions include the extension of the technique in the presence of uncertainties in the plant model.

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