FEEDBACK STABILIZATION OF NONHOLONOMIC CONTROL SYSTEMS USING MODEL DECOMPOSITION

Fazal-ur-Rehman

ABSTRACT

This paper presents a simple and systematic approach for feedback stabilization of nonholonomic control systems. Its effectiveness is tested on two different nonholonomic control systems such as: a front wheel drive car, and a mobile robot with trailer. The method relies on the decomposition of model into two subsystems. One subsystem is stabilized by using the trajectory interception approach and other subsystem is steered by using sinusoidal inputs. The mixture of both types of control stabilizes the actual system. This approach does not necessitate the conversion of the system model into a “chained form”, and thus does not rely on any special transformation techniques. The approach presented is general and can be employed to control a variety of mechanical systems with velocity constraints.

KeyWords: Systems without drift, nonholonomic systems, nilpotent Lie algebra, locally nilpotent, Lyapunov function.

I. INTRODUCTION

The feedback control strategy presented in this paper applies to systems of the type:

\[
\dot{z} = \sum_{i=1}^{m} Z_i(z) u_i, \quad \text{with i.c. } z(0) = z_0, \quad z \in \mathbb{R}^n, \quad m < n
\]  

(1)

where \(Z_i, i = 1, 2, \ldots, m\), are linearly independent, smooth vector fields in \(\mathbb{R}^n\), \(u_i\) are piece-wise continuous and locally bounded in \(t\), control functions defined on the interval \([0, \infty)\). Such systems arise frequently in practice and typically represent models of mechanical systems with velocity constraints, such as, for example, wheeled vehicles, for which no slipping occurs between the wheels and the contact surface. Such systems are known to be difficult to control as reflected by fact that the linearization of (1) is an uncontrollable system. It is also well known that system (1) cannot be stabilized by continuous, static state feedback see [1]. Hence, a considerable effort has been expended in order to find continuous, time-varying control laws [2-5] discontinuous ones [6-9] as well as mixed strategies [5,10].

Since discontinuous control is practical in many applications, our interest in this paper is to propose a simple method for discontinuous feedback control design for system (1), with the objective of steering the system (1) from any arbitrary initial state to any desired state. Our approach is based on decomposition of system into two subsystems; one subsystem is stabilized by forcing it to intercept the trajectory of its extended system (which is actually asymptotic stable) after each \(T\) second and, the second subsystem is steered by using sinusoidal inputs. This approach does not necessitate conversion of the system model into a “chained form”, and thus does not rely on any special transformation techniques.

II. THE CONTROL PROBLEM AND SOME ASSUMPTIONS

- (SP) Given a desired set point \(z_{des} \in \mathbb{R}^n\), construct a feedback strategy in terms of the controls \(u_i: \mathbb{R}^n \to \mathbb{R}, \quad i = 1, 2, \ldots, m\) such that the desired set point \(z_{des}\) is an attractive set for (1), so that there exists an \(\epsilon > 0\), such that \(z(t; 0, z_0) \to z_{des}\) as \(t \to \infty\) for any initial condition \(z_0 \in B(z_{des}; \epsilon)\).

Without the loss of generality, it is assumed that \(z_{des} = 0\), which can be achieved by a suitable translation of the coordinate system.
The following assumptions are assumed to hold for these types of systems:

- (A1) The vector fields $X_i, i = 1, 2, ..., m$ are linearly independent and contain no singular point for all $z \in \mathbb{R}^r$.
- (A2) The system (1) satisfies the LARC (Lie algebraic rank condition) for controllability (see [12]), namely that the Lie algebra, $L(Z_1, ..., Z_m(z))$ spans $\mathbb{R}^r$ at each point $z \in \mathbb{R}^r$ i.e.

$$\text{span}[Z_i(z), [Z_i, Z_j](z), [Z_i, [Z_j, Z_k]](z)] = \mathbb{R}^r$$ (2)

- (A3) The state variables $z_1, z_2, ..., z_n$ can be steered along the vector fields $Z_1, Z_2, ..., Z_m, Z_{m+1}, ..., Z_r$, where $Z_{m+1}, ..., Z_r$ are the Lie brackets $[Z_i, Z_j]$ of depth one involved in (2). The remaining state variables $z_{r+1}, z_{r+2}, ..., z_r$ can be steered independently along the Lie brackets of depth $\geq 2$ involved in (2).

To solve the stabilization problem for system (1) we decompose the $n$-dimensional system into two subsystems of dimensions $r$ and $n-r$ respectively. The $r$-dimensional subsystem is stabilized by trajectory interception approach and other $n-r$ dimensional subsystem is steered by using sinusoidal inputs. The mixture of both types of control stabilizes the system (1).

### III. DECOMPOSITION OF ORIGINAL SYSTEM INTO TWO SUBSYSTEMS

Under the assumption (A3), we can decompose the original system (1) into two subsystems such as: one subsystem is consist of first $r$ state variables which can be steered along the original vector fields and all Lie brackets with depth one involved in (2), and other subsystem is consist of remaining $n-r$ state variables which can be steered along the Lie brackets with depth $\geq 2$ involved in (2). Then we have the following decomposition:

$$S_1: \quad \dot{x} = \sum_{i=1}^{n} X_i(x) u_i, \quad x = (z_1, ..., z_r)^T \in \mathbb{R}^r \quad (3)$$

$$S_2: \quad \dot{y} = f(z, u), \quad y = (z_{r+1}, ..., z_n)^T \in \mathbb{R}^{n-r} \quad (4)$$

where $X_i(x), i = 1, 2, ..., m$ are vector fields in $\mathbb{R}^r$ and $f(z, u)$ is the right hand side expression of the vector $[z_{r+1}, ..., z_n]^T$ given in original system.

To employ the trajectory interception approach we need the nilpotent Controllability Lie algebra. For this purpose, an approximation to system $S_1$ is considered which gives nilpotent Controllability Lie algebra. Such an approximation is obtained by substituting the nonlinear terms in the expression for the vector field $X_i, i = 1, 2, ..., m$ by their truncated (of order one) Taylor series expansions at zero. In doing so, $\sin x = x$, and $\cos x = 1$, which results in the following approximate system:

$$\dot{\hat{x}} = \sum_{i=1}^{n} Y_i(x) u_i(x), \quad x \in \mathbb{R}^r \quad (5)$$

The sub-models $S_1$ and $\hat{S}_1$ have the following properties:

- [P1] The vector fields $X_1, X_2, ..., X_m$ and $Y_1, Y_2, ..., Y_m$ are smooth and independent.
- [P2] Systems (3) and (5) are completely controllable: both systems satisfy the Lie algebraic rank condition for controllability in that their respective controllability Lie algebras, $L(X_1, X_2, ..., X_m)$, and $L(Y_1, Y_2, ..., Y_m)$ span $\mathbb{R}^r$ at each point $x \in \mathbb{R}^r$.
- [P3] The controllability Lie algebra $L(Y_1, Y_2, ..., Y_m)$ is nilpotent.

### IV. EXTENDED SYSTEMS OF $S_1$ AND $\hat{S}_1$

The extended system (see [11,12]) of the subsystem $S_1$ is defined as:

$$\dot{x} = \sum_{i=1}^{n} X_i(x) v_i(x) + \sum_{i=m+1}^{n} X_i(x) v_i(x), \quad x \in \mathbb{R}^r$$ (6)

where $X_i, i = m+1, ..., r$ are Lie brackets of depth one involve in $L(X_1, X_2, ..., X_m)$ and the extended system of approximate subsystem $\hat{S}_1$ is defined as:

$$\dot{x} = \sum_{i=1}^{n} Y_i(x) v_i(x) + \sum_{i=m+1}^{n} Y_i(x) v_i(x), \quad x \in \mathbb{R}^r$$ (7)

where $Y_i, i = m+1, ..., r$ are Lie brackets of depth one involve in $L(Y_1, Y_2, ..., Y_m)$.

**Theorem 1.** The extended system (7) can be made asymptotically stable by introducing the following feedback control:

$$v_i(x) = -L_i W(x), \quad x \in \mathbb{R}^r, \quad i = 1, ..., r \quad (8)$$

**Proof.** Let $W : \mathbb{R}^r \to \mathbb{R}$ be any smooth, positive definite, decrescent and radially unbounded function with the origin as a unique stationary point. One simple choice is:

$$W(x) = \frac{1}{2} \sum_{i=1}^{n} z_i^2, \quad x \in \mathbb{R}^r$$

then along the controlled extended system trajectories we have

$$\frac{d}{dt} W(x) = \frac{\partial W(x)}{\partial x} \dot{x} = \frac{\partial W(x)}{\partial x} \sum_{i=1}^{n} Y_i(x) v_i(x) = \sum_{i=1}^{n} \frac{\partial W(x)}{\partial x} Y_i(x) v_i(x)$$

$$= \sum_{i=1}^{n} \{L_i W(x)\}^2 < 0, \quad \text{where} \quad L_i W(x) = \frac{\partial W(x)}{\partial x} Y_i(x)$$
for $x \neq 0$ and $\frac{d}{dt} W(x) = 0$ unless $x = 0$. This completes the proof.

The discretization of the above control in time, with sufficiently high sampling frequency $\frac{1}{T}$, does not prejudice stabilization in that if the feedback control (8) is substituted by the discretized control:

$$v_i^T(x(t)) = v_i^T(x(nT)), \quad t \in [nT, (n+1)T], \quad n = 0, 1, \ldots, i = 1, \ldots, r$$

then the latter also stabilizes the system if $T$ is small enough. This leads to a parameterized, asymptotically stable, controlled extended system:

$$\dot{x} = \sum_{i=1}^{r} Y_i(z) a_i$$

where $a_i = v_i^T(x(t)), \quad i = 1, \ldots, r$ are constant over each interval $[nT, (n+1)T]$.

**Theorem 2.** [13] Suppose the controlled extended system (7) is asymptotically stable. Then, for any compact region $R \subset M$ which contains the origin, there exists a constant $T > 0$ such that the discretized controlled extended system (10) is asymptotically stable with region of attraction $R$.

### 4.1 The trajectory interception problem

- **[TIP:]** Find control functions $m_i(a, t), i = 1, 2, \ldots, m$, in the class of functions which are continuous in $a = \{a_1, a_2, \ldots, a_r\}$, and piece-wise continuous and locally bounded in $t$, such that for any initial condition $x(0) = x_0$ the trajectory $x^a(t; x_0, 0)$ of the extended, parameterized system (10) intersects the trajectory $x^m(t; x_0, 0)$ of the approximate system (5) with controls $m, i = 1, 2, \ldots, m$, i.e.

$$\dot{x} = \sum_{i=1}^{m} Y_i(x) m_i(a, t)$$

precisely at time $T$, so that

$$x^a(T; x_0, 0) = x^m(T; x_0, 0)$$

**Theorem 3.** [13] Suppose that a solution to the TIP problem can be found. Then, there exists an admissible time horizon $T_{\text{max}}$ and a neighborhood of the origin $R$ such that for any $T < T_{\text{max}}$ the time-varying feedback controls:

$$u_i(t) = m_i(v^a(x), t), \quad i = 1, 2, \ldots, m, \quad v^a = [v_1^T, \ldots, v_r^T]$$

are asymptotically stabilizing the approximate system (7) with the region of attraction $R$.

### 4.2 The TIP in logarithmic coordinates of flows

To solve the TIP; as the algebra $L(Y_1, Y_2, \ldots, Y_m)$ is nilpotent, it is possible to employ the formalism of [14] by considering a formal equation for the evolution of flows for the approximate model (5):

$$\dot{U}(t) = U(t) \sum_{i=1}^{r} Y_i w_i, \quad U(0) = I$$

where the solution of (13) is known to represent the flow of the dynamic system

$$\dot{x}(t) = \sum_{i=1}^{r} Y_i w_i$$

whose controllability Lie algebra $L(Y_1, Y_2, \ldots, Y_m)$ is nilpotent. Such solution can be expressed locally as:

$$U(t) = \prod_{i=1}^{r} e^{Y_i(t) w_i}$$

where the functions $\gamma_i, i = 1, 2, \ldots, m$ are the logarithmic coordinates for this flow. In the specific case of the approximate system, the latter can be found by solving the following set of differential equations:

$$\dot{\gamma}_1 = w_1$$
$$\dot{\gamma}_2 = w_2$$
$$\vdots$$
$$\dot{\gamma}_m = w_m$$

$$(15)$$

with initial conditions $\gamma_i(0) = 0, i = 1, 2, \ldots, r$ and $i, j, k = 1, 2, \ldots, m$. In a vector form, the above parameter equations are written as:

$$\dot{\gamma}(t) = M(\gamma) w, \quad \gamma(0) = 0$$

$$\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_r], \quad w = [w_1, w_2, \ldots, w_r]$$

The TIP problem is now recast in the logarithmic coordinates as flows:

**[TIP in LC]**

- Find control functions $m_i(a, t), i = 1, 2, \ldots, m$, in the class of functions which are continuous in $a = \{a_1, a_2, \ldots, a_r\}$, and piece-wise continuous and locally bounded in $t$, such that the trajectory $t \rightarrow \lambda^a(t)$ of

$$\dot{\gamma}(t) = M(\gamma) a, \quad \gamma(0) = 0$$

$$(17)$$
intersects the trajectory \( t \mapsto \lambda_I(t) \) of

\[
\dot{\gamma}(t) = M(\gamma) m(a,t), \quad \gamma(0) = 0
\]  

(18)

in which \( m(a,t) = [m_1(a,t), \ldots, m_m(a,t), 0, \ldots, 0] \) at time \( T \), so that

\[
\gamma'(T) = \gamma''(T)
\]  

(19)

A solution to the TIP

Equations (17) and (18) can be treated as two completely controllable control systems:

**CS1:**

\[
\begin{align*}
\dot{\gamma}_1 &= a_1 \\
\dot{\gamma}_2 &= a_2 \\
& \vdots \\
\dot{\gamma}_m &= a_m \\
\dot{\gamma}_{m+1} &= -\gamma_i a_j + a_{m+1} \\
& \vdots \\
\dot{\gamma}_r &= -\gamma_i a_k + a_r
\end{align*}
\]  

(20)

**CS2:**

\[
\begin{align*}
\dot{\gamma}_1 &= m_1 \\
\dot{\gamma}_2 &= m_2 \\
& \vdots \\
\dot{\gamma}_m &= m_m \\
\dot{\gamma}_{m+1} &= -\gamma_i m_j \\
& \vdots \\
\dot{\gamma}_r &= -\gamma_i m_k
\end{align*}
\]  

(21)

both with the same initial conditions \( \gamma_i(0) = 0, i = 1, 2, \ldots, r \) and \( i, j, k = 1, 2, \ldots, m \). Complete controllability of CS1 and CS2 guarantees the existence of solutions to the TIP.

One such solution can be calculated as follows. Motivated by the fact that a flow of \( \{g_i, g_j\} \) can be approximated by the flow of \( \dot{x} = cg_i \sin \frac{2\pi t}{T} + cg_j \cos \frac{2\pi t}{T} \), where \( c \) is some constant, we seek the controls \( m_i(a,t), i = 1, 2, \ldots, m \) in the form

\[
m_i = \left( c_i + c_k \sin \frac{2\pi t}{T} \right), \quad m_j = \left( c_j + c_k \cos \frac{2\pi t}{T} \right)
\]  

(22)

where \( c_k, c_k, c_i, i = 1, 2, \ldots, m, k = m+1, \ldots, r \) are some unknown coefficients.

The above are substituted into CS2, and the systems CS1 and CS2 are integrated symbolically, using Mathematica, to yield respective solutions \( \gamma'(T) \) and \( \gamma''(T) \) in terms of parameters \( a \) and \( c \). The Eq. (19) is then also be solved symbolically to deliver the values for the unknown coefficients \( c_i \) in terms of their counterparts \( a_i \). The values of \( c_i \) are found:

\[
c_i = a_i, i = 1, \ldots, m, \quad c_k = \pm 3.5449 \sqrt{\frac{a_k}{T}}, \quad k = m+1, \ldots, r
\]

Therefore the following controls stabilize the approximate system \( \hat{S}_1 \) as given in (5):

\[
m_i = \left( a_i + c_k \sin \frac{2\pi t}{T} \right), \quad m_j = \left( a_j + c_k \cos \frac{2\pi t}{T} \right)
\]  

(23)

The controls given in (23) can be utilized to stabilize the actual subsystem \( S_1 \) as given in (3), by just replacing \( a_i \rightarrow b_i \), where \( b_i = \dot{v}_i(t) \in \mathbb{R} \), and \( \dot{v}_i = -L_{z_j} V(x) \). In this way the following controls are obtained.

\[
m_i = \left( b_i + d_k \sin \frac{2\pi t}{T} \right), \quad m_j = \left( b_j + d_k \cos \frac{2\pi t}{T} \right)
\]  

(24)

where \( d_k = \pm 3.5449 \sqrt{\frac{b_i}{T}}, \quad k = m+1, \ldots, r \).

**Corollary 1.** If the controlled extended system possesses a sufficiently wide stability margin, the controls given in (23) and (24) provide an asymptotically stabilizing feedback control for the approximate model \( \hat{S}_1 \) and exact model \( S_1 \) respectively.

V. STABILIZING CONTROL ALGORITHM FOR ORIGINAL SYSTEM

The control (24) steers the original system (1), to the manifold \( S_0 = \{ z \in \mathbb{R}^n : z_1 = \ldots = z_r = 0, z_j \neq 0, j = r+1, \ldots, n \} \), and further decrease in the cost function \( V(z) = W(x) \)

\[
1 + \frac{1}{2} \sum_{j=1}^{r} z_j^2 = \frac{1}{2} \sum_{j=1}^{r} z_j^2
\]

where \( W(x) = \frac{1}{2} \sum_{j=1}^{r} z_j^2 \), can be obtained only through system motion in the direction of the Lie brackets with \( \text{depth} \geq 2 \). Such motion can be achieved only indirectly by using sinusoidal control (see [13]) e.g. to move along \( [Z_i, [Z_i, Z_j]] \), the following controls can be used:

\[
u_1 = k_1 \sin \frac{2\pi t}{T}, \quad u_2 = k_2 \cos \frac{2\pi t}{T}, \quad u_3 = 0, i = 3, \ldots, m
\]

where \( k_1, k_2, k_3, k_4 \) are some constant.

Similarly the controls \( u_i = k_1 \sin \frac{2\pi t}{T}, u_2 = k_2 \cos \frac{6\pi t}{T}, u_3 = 0, i = 3, \ldots, m \) generate motion along the Lie bracket \( [Z_i, [Z_i, Z_j]] \).

By introducing the following notations.

\[
S_i = \{ z \in \mathbb{R}^n : z_1 = \ldots = z_i = 0, \quad z_j \neq 0, \quad j = i+1, \ldots, n \}
\]

\[
T_i = \{ z \in \mathbb{R}^n : z_i = f_i(z,u) = 0 \}, \ldots
\]
leads to the following feedback steering algorithm.

5.1 Stabilizing algorithm for original system

Repeat the following until sufficient accuracy is achieved in reaching the origin:

- Data: \( \varepsilon > 0 \)
- Step a: Apply the control (24) to original system (1) until its trajectories converges to \( B(\mathbb{S}^1; \varepsilon) \): where \( \mathbb{S}^1 = \{ z \in \mathbb{R}^n : z_j = 0, \quad j = 1, \ldots, n \} \)
- Step b: Apply the sinusoidal controls until the system trajectories converges to \( B(\mathbb{T}^1; \varepsilon) \), which is an invariant set for the controlled system (1).
- Step c: Again apply the control (24) until system reach at \( \mathbb{S}^1 \). 

Repeat the step (b) and (c) until system reach at origin.

VI. EXAMPLE 1: THE FRONT WHEEL DRIVE CAR

The example considered below represent a non-holonomic four dimensional with control deficiency order two, possessing non-nilpotent controllability algebra which contains Lie brackets of depth one and two. The kinematics model of a front wheel drive car is given as \[3\]:

\[
\begin{align*}
\dot{x} &= \cos \theta \ x_2 \\
\dot{y} &= \sin \theta \ x_2 \\
\dot{\theta} &= \frac{1}{l} \tan \phi \ u_2
\end{align*}
\] (25)

After redefining the states variables as \( (z_1, z_2, z_3, z_4) = (\phi, x, \theta, y) \) in the kinematics model (25) and assuming \( l = 1 \) we have the following:

\[
\dot{z} = Z_1(z) \ u_1 + Z_2(z) \ u_2, \quad z \in \mathbb{R}^4
\] (26)

where \( Z_1(z) = \begin{bmatrix} 1 & 0 \\ 0 & \cos z_3 & 0 \\ 0 & \tan z_3 & \sin z_3 \end{bmatrix} \) and \( Z_2(z) = \begin{bmatrix} 0 \\ \cos z_3 \\ \tan z_3 \end{bmatrix} \).

The following Lie brackets:

\[
Z_3(z) = [Z_1, Z_2](z) = \begin{bmatrix} 0 & 0 & (\sec z_3)^2 \\ 0 & 0 \end{bmatrix},
\] (27)

\[
Z_4(z) = [Z_2, [Z_1, Z_2]](z) = \begin{bmatrix} 0 & \sin z_1(\sec z_3)^2 \\ 0 & 0 \end{bmatrix},
\] (28)

show that the LARC condition is satisfied: \( \text{span}\{Z_1(z), Z_2(z), \ldots, Z_4(z)\} = \mathbb{R}^4 \forall z \in \mathbb{R}^4 \). For this model, the following decomposition is considered:

\[
S_1: \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 & \cos z_3 \\ 0 & \tan z_3 \end{bmatrix} u_2
\] (29)

\[
S_2: \quad \dot{z}_4 = \sin z_3 \ u_2 = f(z) u_2
\] (30)

By defining \( x = (z_1, z_2, z_3) \), subsystem \( S_1 \) can be written as:

\[
\dot{x} = X_1(x) u_1 + X_2(x) u_2, \quad x \in \mathbb{R}^3
\]

where \( X_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos z_3 & 0 \\ 0 & \tan z_3 & 0 \end{bmatrix} \) and \( X_2(x) = \begin{bmatrix} 0 & 0 \\ \sec z_3 & 0 \end{bmatrix} \).

Subsystem \( S_1 \) is controllable as it satisfies: \( \text{span}\{X_1(x), X_2(x), X_3(x)\} = \mathbb{R}^3 \forall x \in \mathbb{R}^3 \)

where \( X_3(x) = [X_1, X_2](x) = \begin{bmatrix} 0 & 0 \\ 0 & (\sec z_3)^2 \end{bmatrix} \).

It can be easily verified that the Lie algebra \( L(X_1, X_2) \) is not nilpotent.

The following approximation to \( S_1 \) is considered:

\[
\hat{S}_1: \quad \dot{x} = \hat{Y}_1(x) u_1 + \hat{Y}_2(x) u_2, \quad x \in \mathbb{R}^3
\] (31)

where \( \hat{Y}_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( \hat{Y}_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Fig. 1. A front wheel drive car model.
The approximate system ˙\hat{S}1 satisfies the LARC condition:
\[ \text{span}\{Y_1(x), Y_2(x), Y_3(x)\} = \mathbb{R}^3, \forall \ x \in \mathbb{R}^3, \]
where  \( Y_i(x) = [Y_i, Y_j](x) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}. \)

The Lie brackets multiplication table for  \( L(Y_1, Y_2) \) is:
\[ [Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = [Y_2, Y_3] = 0, \]
which shows that the controllability Lie algebra  \( L(Y_1, Y_2) \) is nilpotent.

The system for  \( \hat{S}1 \) is given by
\[ \dot{x} = Y_1(x) v_1 + Y_2(x) v_2 + Y_3(x) v_3 \]
where  \( v_i(x) = -L(Y_i, x), \ i = 1, 2, 3, \) and  \( W(x) = \sum_{i=1}^{3} x_i^2 \),
which gives the following extended system with discretized controls:
\[ \dot{x} = Y_1(x) a_1 + Y_2(x) a_2 + Y_3(x) a_3 \]

The formal equation for this system becomes
\[ \dot{U}(t) = U(t) \sum_{i=1}^{3} Y_i \ w_i, \quad U(0) = I \]

The solution of (32) is assumed in the form
\[ U(t) = \prod_{i=1}^{3} e^{\lambda_i(t)Y_i} \]
and the logarithmic coordinates,  \( \gamma_i, i = 1, 2, 3 \) are computed as follows:

Equation (33) is first substituted into (32) which yields:
\[ Y_1 a_1 + Y_2 a_2 + Y_3 a_3 = \gamma_1 Y_1 + \gamma_2 (e^{\delta_1 \Delta t} Y_2) + \gamma_3 (e^{\delta_2 \Delta t} e^{\gamma_2 \Delta t} Y_3) \]
where  \( e^{\delta \Delta t} Y = e^{\delta} Y e^{-\delta} \) and  \( (AdX)Y = [X, Y] \).

Employing the Baker-Hausdorff formula:
\[ (e^{\delta \Delta t} Y)^+ e^{\delta \Delta t} Y e^{-\delta \Delta t} = Y + [X, Y] + [X, [X, Y]]/2! + \ldots \]
\[ (e^{\delta \Delta t} Y)^+ = e^{\delta} Y e^{-\delta} = Y + (\gamma_i / 1)! [Y_i, Y_j] + \ldots \]
\[ + (\gamma_i / 2)! [Y_i, [Y_i, Y_j]] + \ldots \]
gives similarly
\[ (e^{\delta \Delta t} e^{\gamma_2 \Delta t} Y_2)^+ = e^{\delta_1 \Delta t} (e^{\gamma_2 \Delta t} Y_2)^+ = e^{\delta_1 \Delta t} Y_2 \]

Substituting these equations in (34) and comparing the coefficients of  \( Y_i, i = 1, 2, 3 \) yields the following equations for the evaluation of the logarithmic coordinates  \( \gamma_i, i = 1, 2, 3 \):
\[ \dot{\gamma}_1 = a_1 \]
\[ \dot{\gamma}_2 = a_2 \]
\[ \dot{\gamma}_3 + \gamma_2 \dot{\gamma}_2 = a_3 \quad \text{with} \quad \gamma_i(0) = 0, \ i = 1, 2, 3 \quad (35) \]

The TIP in logarithmic now takes the form of a trajectory interception problem for the following two control systems
\[ \text{CS1:} \quad \dot{y}_1 = a_1 \quad \dot{y}_2 = a_2 \quad \dot{y}_3 = -\gamma_1 a_2 + a_3 \quad \text{and} \quad y_i(0) = 0, \ i = 1, 2, 3. \]

Therefore by TIP the following control stabilize the subsystem  \( \hat{S}1 \):
\[ u_1(x) = a_1 + c_3 \sin \frac{2 \pi t}{T}, \quad u_2(x) = a_2 + c_3 \cos \frac{2 \pi t}{T} \]

where  \( c_3 = \pm 3.54491 \sqrt{1/T} \).

Replacing  \( a_i \) by  \( b_i \) and  \( c_i \) by  \( d_i \) in (34) we obtain the following controls which stabilize the subsystem  \( S1 \):
\[ u_1(x) = b_1 + d_3 \sin \frac{2 \pi t}{T}, \quad u_2(x) = b_2 + d_3 \cos \frac{2 \pi t}{T} \]

6.1 Stabilizing algorithm for a front wheel drive car

Repeat the following steps until sufficient accuracy is achieved in reaching the origin:
- [Step a] Apply the controls (37) to original system (26) until its trajectories converge to  \( S_1 = \{ z \in \mathbb{R}^4 : z_1 = z_2 = z_3 = 0, \ z_4 \neq 0 \} \)
- [Step b] To make  \( z_4 = 0 \) is equivalent to generate the motion along the Lie bracket  \( Z_4 = [Z_1, [Z_1, Z_2]] \).
  For this apply the following controls:
\[ u_1 = k_1 \sin \frac{2 \pi t}{T}, \quad u_2 = k_2 \cos \frac{4 \pi t}{T} \]
until the system trajectories converge to
\[ T_4 = \{ z \in \mathbb{R}^4 : z_4 = 0 \} \]
\[ = \{ z \in \mathbb{R}^4 : z_4 \text{ and } \sin z_3 = 0 \} \]
\[ S_4 = \{ z \in \mathbb{R}^4 : z_1 = z_2 = z_3 = z_4 = 0 \} \]

- [Step c] Again apply the control (37) until the system trajectories converge to

\[ S_4 = \{ z \in \mathbb{R}^4 : z_1 = z_2 = z_3 = z_4 = 0 \} . \]

Simulation results are depicted in Figs. 1(a) ~ 1(c), which confirm the applicability of combining strategy. In simulation, the values \( k_1 = -2, k_2 = -3, k_3 = -2.8, k_4 = 5 \), and \( T = 1.2 \) were used.

**VII. THE MOBILE ROBOT WITH TRAILER MODEL**

The example considered below represents a fifth dimensional system with control deficiency order three, possessing a non-nilpotent controllability Lie algebra which contains Lie brackets of depth one, two, and three. Although, the algebraic structure of mobile robot with trailer is more complicated, the decomposition idea can still be employed successfully. The kinematics model of a mobile robot with trailer (see [12]), is given below:

\[
\begin{align*}
\dot{x}_1 &= \cos x_3 \cos x_4 u_1 \\
\dot{x}_2 &= \cos x_3 \sin x_4 u_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= \frac{1}{l} \sin x_3 \ u_1 \\
\dot{x}_5 &= \frac{1}{d} \sin(x_4 - x_3) \cos x_3 \ u_1
\end{align*}
\]

and can be suitably re-written by defining \( (x_1, x_2, x_3, x_4, x_5) \)

\[ \def \quad (z_1, z_4, z_2, z_3, z_5) : \]

\[
\dot{z} = Z_1 (z) u_1 + Z_2 (z) u_2, \quad z \in \mathbb{R}^3
\]

where

\[
\begin{bmatrix}
\cos z_1 \cos z_2 \\
0 \\
\sin z_2 \\
\cos z_1 \sin z_3 \\
\cos z_2 \sin(z_1 - z_3)
\end{bmatrix}
\]

\[ Z_1 (z) = \begin{bmatrix}
-\sin z_1 \cos z_3 \\
0 \\
\cos z_2 \\
-\sin z_3 \sin z_3 \\
-\sin z_2 \sin(z_1 - z_3)
\end{bmatrix}, \quad Z_2 (z) = \begin{bmatrix}
-\sin z_1 \\
0 \\
\cos z_3 \\
\cos(z_3 - z_1)
\end{bmatrix}
\]

The following Lie brackets:

\[
Z_3 (z) = [Z_1, Z_2](z) = \begin{bmatrix}
-\sin z_1 \\
\cos z_3 \\
\sin z_2 \cos z_3 \\
0 \\
\sin z_1 \sin z_3 \\
\sin z_2 \sin(z_1 - z_3) + \cos z_2
\end{bmatrix}
\]

show that the LARC condition is satisfied: \( \text{span} \{ Z_i (z), Z_2 (z), \ldots, Z_5 (z) \} = \mathbb{R}^5 \ \forall \ z \in \mathbb{R}^5 \).

Fig. 1(a). Collective plots of the controlled state trajectories \( t \mapsto (x_1 (t), \ldots, x_4 (t)) \) vs. time.

Fig. 1(b). Plots of the controlled state trajectories \( t \mapsto (x_1 (t), \ldots, x_4 (t)) \) vs. time.
7.1 Decomposition into two subsystems

For this model, the following decomposition is considered:

\[ S_1: \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \cos z_2 \cos z_3 \\ 0 \\ \sin z_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 \] (40)

\[ S_2: \begin{bmatrix} \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} \sin z_4 \cos z_2 \\ \cos z_4 \sin (z_3 - z_5) \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} f_4(z) \\ f_5(z) \end{bmatrix} u_1 \] (41)

By taking \( x = (z_1, z_2, z_3) \), sub-system \( S_1 \) can be written as:

\[ \dot{x} = X_1(x) u_1 + X_2(x) u_2, \quad x \in \mathbb{R}^3, \]

where \( X_1(x) = \begin{bmatrix} \cos z_2 \cos z_3 \\ 0 \\ \sin z_2 \end{bmatrix} \), \( X_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) (42)

Subsystem \( S_1 \) is controllable as it satisfies: \( \text{span} \{X_1(x), X_2(x), X_3(x)\} = \mathbb{R}^3, \forall \, x \in \mathbb{R}^3 \),

where \( X_3(x) = [X_1, X_2](x) = \begin{bmatrix} -\sin z_2 \cos z_3 \\ 0 \\ \cos z_2 \end{bmatrix} \).

It can be easily verified that the Lie algebra \( L(X_1, X_2) \) is not nilpotent. The following approximation to \( S_1 \) is considered:

\[ \hat{S}_1: \dot{x} = Y_1(x) u_1 + Y_2(x) u_2, \quad x \in \mathbb{R}^3 \]

\[ Y_1(x) = \begin{bmatrix} 1 \\ 0 \\ z_2 \end{bmatrix}, \quad Y_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \]

where \( Y_3(x) = [Y_1, Y_2](x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

The approximate system \( \hat{S}_1 \) satisfies the LARC condition: \( \text{span} \{Y_1(x), Y_2(x), Y_3(x)\} = \mathbb{R}^3, \forall \, x \in \mathbb{R}^3 \), where \( Y_3(x) = [Y_1, Y_2](x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and the Lie brackets multiplication table for \( L(Y_1, Y_2): [Y_1, Y_2] = Y_3 \) shows that the controllability algebra \( L(Y_1, Y_2) \) is nilpotent.

The extended system for \( \hat{S}_1 \) is given by

\[ \dot{x} = Y_1(x) v_1 + Y_2(x) v_2 + Y_3(x), \]

\[ v_i(x) = -L_{Y_i} W(x), \quad i = 1, 2, 3, \]

where \( W(x) = \sum_{i=1}^3 z_i^2 \).

The logarithmic coordinates for \( \hat{S}_1 \) satisfy the following differential equations:

\[ \dot{\lambda}_1 = \alpha_1 \]

\[ \dot{\lambda}_2 = \alpha_2 \]

\[ \dot{\lambda}_3 = -\gamma_1 \alpha_2 + \alpha_3, \quad \text{with} \quad \gamma_1(0) = 0, \quad i = 1, 2, 3 \] (44)

Therefore by TIP the following control stabilize the subsystem \( \hat{S}_1 \):

\[ u_1(x) = \alpha_1 + c_3 \sin \frac{2 \pi t}{T}, \quad u_2(x) = \alpha_2 + c_3 \cos \frac{2 \pi t}{T} \] (45)

where \( c_3 = \pm \frac{3.54491}{\sqrt{T}} \).

Replacing \( \alpha_i \) by \( b_i \) and \( c_i \) by \( d_i \) in (45) we obtain the following controls which stabilize the subsystem \( S_1 \):

\[ u_1(x) = b_1 + d_3 \sin \frac{2 \pi t}{T}, \quad u_2(x) = b_2 + d_3 \cos \frac{2 \pi t}{T} \] (46)

with \( d_3 = \pm \frac{3.54491}{\sqrt{T}} \).
7.2 Stabilizing algorithm for a mobile robot with trailer

Repeat the following steps until sufficient accuracy is achieved in reaching the origin:

- [Step a] Apply the controls (46) to original system (39) until its trajectories converge to
  \[ S_1 = \{ z \in \mathbb{R}^3 : z_1 = z_2 = z_3 = 0, \ z_4, z_5 \neq 0 \} \]

- [Step b] To make \( z_4 = 0 \) is equivalent to generate motion along the Lie bracket \( Z_3 = [Z_1, [Z_1, Z_2]] \), apply the following controls
  \[ u_1 = k_1 \sin \frac{2\pi t}{T}, \quad u_2 = k_2 \cos \frac{4\pi t}{T} \]
  until the system trajectories converge to
  \[ T_3 = \{ z \in \mathbb{R}^3 : z_4 = 0 \} \]
  \[ = \{ z \in \mathbb{R}^3 : z_4 \text{ and } \sin z_1 \cos z_2 = 0 \} \]
  \[ = \{ z \in \mathbb{R}^5 : z_4 = z_3 = 0 \} \]

- [Step c] Again apply the control (46) until the system trajectories converge to
  \[ S_5 = \{ z \in \mathbb{R}^5 : z_1 = z_2 = z_3 = z_4 = 0, \ z_5 \neq 0 \} \]

- [Step d] To make \( z_5 = 0 \) is equivalent to generate motion along \( Z_4 = [Z_1, [Z_1, [Z_1, Z_2]]] \) by applying the following controls
  \[ u_1 = k_3 \sin \frac{2\pi t}{T}, \quad u_2 = k_4 \cos \frac{6\pi t}{T} \]
  until its trajectories converge to
  \[ T_5 = \{ z \in \mathbb{R}^3 : z_5 = f_4(z) = 0 \} \]
  \[ = \{ z \in \mathbb{R}^3 : z_5 \text{ and } \sin(z_1 - z_3) \cos z_2 = 0 \} \]
  \[ = \{ z \in \mathbb{R}^3 : z_5 = z_3 = 0 \} \]

- [Step e] Apply the control (46) until the system trajectories converge to
  \[ S_5 = \{ z \in \mathbb{R}^5 : z_1 = z_2 = z_3 = z_4 = z_5 = 0 \} \]

Simulation results are depicted in Figs. 2(a) ~ 2(c) which confirms the applicability of combining strategy. In simulation, the values \( k_1 = -2, k_2 = -3, k_3 = -2.8, k_4 = 5, \) and \( T = 1.2 \) were used.

Fig. 2(a). Collective plots of the controlled state trajectories \( t \mapsto (x_1(t), \ldots, x_5(t)) \) vs. time.

Fig. 2(b). Plots of the controlled state trajectories \( t \mapsto (x_1(t), \ldots, x_5(t)) \) vs. time.
VIII. CONCLUSION

A systematic method for the construction of discontinuous time varying stabilizing feedback controllers for nonholonomic systems with control deficiency \( n - m \geq 2 \) is introduced with out transforming into “chain form”, and the conditions are stated which guarantee that the resulting feedback control strategy yields global asymptotic convergence to the origin. The approach is applied to stabilize a front wheel drive car, and mobile robot with trailer. This method is general and can be employed to control a variety of mechanical systems with velocity constraints.

REFERENCES