DESIGN MODIFICATION OF SLIDING MODE OBSERVERS FOR UNCERTAIN MIMO SYSTEMS WITHOUT AND WITH TIME-DELAY

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ABSTRACT

In this paper, the sliding mode observers design techniques for MIMO and as a simple example for SISO systems are systematically advanced as a first purpose. Design parameters are selected such that on the defined switching surface always is generated asymptotically stable sliding mode. Moreover, observer state error dynamics is globally robustly asymptotically stable. Then, advanced design techniques are generalized to the design of a new modification of sliding mode observers for uncertain MIMO systems with time-delay. Robust sliding and global asymptotic stability conditions are derived by using Lyapunov-Krasovskii V-functional method. By these conditions observer parameters are designed such that an asymptotically stable sliding mode always is generated in observer and observer state error dynamics is robustly globally asymptotically stable. The main results are formulated in terms of Lyapunov matrix equations and inequalities. Design example for AV-8A Harrier VTOL aircraft with simulation results using MATLAB-Simulink show the effectiveness of proposed design approaches.

KeyWords: Time delay systems, sliding mode observer, robust stability, Lyapunov-Krasovskii functional, global stability conditions.

I. INTRODUCTION

The purpose of a state observer is to estimate the unavailable state variables of a plant. The idea of using a dynamical linear system to generate estimates of the plant states can be traced to Luenberger [1], which is the most well known. The Luenberger observer performs well when the plant dynamics are known reasonable well. A full-order observer design method for linear systems with unknown inputs is given by Darouach, Zasadzinski and Xu [2]. However, in the presence of model perturbations and external disturbances, the estimation of the plant states of uncertain time-delay systems may not be sufficiently accurate. From the point of view of robust control, the desirable properties and functional possibilities of variable structure control are very well known (Utkin [3]; De Carlo, Zak and Matthews [4]; Oh and Khalil [5]; Edwards and Spurgeon, [6,7]; Edwards, et al. [8]; Yan, et al. [9]; Garofalo and Glielmo [10]; Jafarov [11]; Choi [12]; Cao and Xu [13]; etc.).

In recent years, the sliding mode observer design problem for uncertain dynamical systems subject to external disturbances has been a topic of considerable interest of several authors. There are several observers successfully designed by Utkin [14], Walcott and Zak [15], Walcott, et al. [16], Zak, et al. [17], Edwards and Spurgeon [6,18,19], Slotine, et al. [20], Watanabe, et al. [21], Hachimoto, et al. [22]; etc., Lyapunov V-function method [23] has been used to formulate sliding mode observers design which guarantees that the state estimation errors converge to zero asymptotically in the presence of matched uncertainties. In other words, this type of discontinuous observers is designed such that observer state error dynamics is globally asymptotically stable or globally uniformly ultimately bounded because the stability region is restricted by some small ball. However, first question arises as to whether these types of observers provide an asymptotically stable sliding mode or not because its
robustness directly is related with the sliding mode. Secondly, could we generalize the design of sliding mode observers for uncertain multivariable systems with time-delay?

It should be noted that in contrast to above mentioned observers, there is a few linear and variable structure observers for time delay systems (Pearson and Fiagbedzi [24]; Fattouh, et al. [25]; Wang and Unbehauen [26]; Wang, et al. [27]; Jafarov [28,29], etc.) by using Lyapunov-Krasovskii functionals [30]. Razumikhin-Hale type theorem (Razumkin [31]; Hale and Verduyn-Lunel [32]) are used for example by (Mahmoud and Muthairi [33]; Shyu and Yan [34]) for control of time-delay systems. Recent advances in analysis and control of time-delay systems using Lyapunov-Krasovskii functionals are presented by Gu, et al. [35]; Niculescu [36]; Richard [37]; Fridman and Shaked [38]; Jafarov [39]; Jing, et al. [40]; etc.

In this paper sliding mode observers design techniques for MIMO and as a simple example for SISO systems are systematically advanced as a first purpose. Design parameters are selected such that on the defined switching surface always is generated asymptotically stable sliding mode. Moreover, observer state error dynamics is globally robustly asymptotically stable. The observer design problem involves estimating the states of the uncertain dynamical system described by the following differential equations:

\[
x(t) = (A_0 + \Delta A_0) x(t) + f_0(t, x(t)) + (B + \Delta B) u(t)
\]
\[
y(t) = C x(t)
\]

where the unknown state \(x(t) \in \mathbb{R}^n\), the control input \(u(t) \in \mathbb{R}^m\), the measurable output \(y(t) \in \mathbb{R}^p\) with \(m = p < n\) and the model uncertainties \(f_0 \in \mathbb{R}^n\) are vectors, and the matrices \(A_0, \Delta A_0, B, \Delta B, \) and \(B\) are compatibly dimensioned. The matrices \(B\) and \(C\) are assumed to be of full rank. The known matrices \(A_0\) and \(B\) represent the nominal linear model parameters of the system; \(\Delta A_0\) and \(\Delta B\) are unknown matrices involving all possible system parameter variations. For solving this problem, we require that the unknown function \(f_0(t, x(t))\) to be continuous in \(x(t)\) and the following conventional matching conditions are assumed to be valid. There exist functions \(h_0, w,\) and \(d_0\) such that

\[
f_0(t, x(t)) = B h_0(t, x(t))
\]
\[
\Delta B u(t) = B w(t)
\]
\[
\Delta A_0 x(t) = B d_0(t, x(t)).
\]

Let \(\xi_0(t, x(t)) = h_0(t, x(t)) + d_0(t, x(t)) + w(t)\).

It is assumed that

\[
\xi_0(t, x(t)) = B \xi_0(t, x(t))
\]

where the function \(\xi_0\) is unknown but bounded, so that

\[
\|\xi_0(t, x(t))\| \leq \rho_0 + \beta_0 \|y(t)\|
\]

\[
\leq \rho_0 + \beta_0 \sqrt{\lambda_{\max}(C^T C)} \|x(t)\|
\]

where \(\rho_0\) and \(\beta_0\) are known constant positive scalars. Note that, second condition of (2) together with...
condition (5) limits the class of available control laws. However, this is a common limitation (Edwards and Spurgeon [7]).

Thus, system (1) can be simplified to:

\[
x(t) = A_0 x(t) + Bu(t) + B \xi_0(t, x(t))
\]
\[
y(t) = Cx(t)
\]

It is also assumed that the pair \((A_0, C)\) is detectable and that there exists a constant feedback gain matrix \(G \in \mathbb{R}^{n \times p}\) such that

\[
A_0 G C = - \text{ has some desirable stable eigenvalues}
\]

There exists a Lyapunov pair \((P_0, Q_0)\) for \(A_0\) such that the conventional structural constraint (Edwards and Spurgeon [6]):

\[
F C = B^T P
\]

is satisfied for some non-singular design matrix \(F \in \mathbb{R}^{m \times m}\).

The problem to be considered is that of reconstructing the state variables using only measured output information in the framework of modern sliding mode control theory.

The observer motion is governed by the following differential equation:

\[
\dot{x}(t) = A_0 \hat{x}(t) + Bu(t) + G[y(t) - C\hat{x}(t)] - Bv
\]
\[
\dot{\hat{x}}(t) = C\hat{x}(t)
\]

where \(v\) is the discontinuous vector term to be formed.

It should be noted that there is various canonical form design of sliding mode observers with different gain matrices:

- Walcott and Zak observer [15]:
  \[
v(\hat{x}, y, \rho) = -\rho(t, u) \frac{P^{-1} C^T Ce(t)}{\|Ce(t)\|}
\]

- Walcott, Corless and Zak observer [16]:
  \[
v(t, \hat{x}, y) = -\rho(t) \frac{P^{-1} C^T Ce(t)}{\|Ce(t)\|}
\]

- Edwards and Spurgeon modification [18]:
  \[
v = -\rho(\frac{P^{-1} C^T F^T F C e(t)}{\|FC e(t)\|})
\]

- Edwards and Spurgeon modification [6, 19]:
  \[
v = -\rho(t, y, u) \frac{F C e(t)}{\|FC e(t)\|}
\]

where \(e(t) = x(t) - \hat{x}(t)\) is the observer state error; \(\rho, P,\) and \(F\) are design parameters. In actual fact, mentioned observers are all equivalent.

Design parameters of these types of observers were determined by using Lyapunov V-function method such that \(e(t) \to 0\) as \(t \to \infty\) or observer motion is uniformly ultimately bounded.

Here we consider another type of observer with modified gain matrix:

\[
v = -[\delta_0 + k_0 \| y(t) \| (B^T PB)^{-1} s(t) - \|s(t)\|]
\]

where, \(\delta_0\) and \(k_0\) are design constants to be selected; \(s(t)\) is a switching function, which can be defined as follows:

\[
s(t) = Fr(t) = F[y(t) - \hat{y}(t)] = F C e(t) = B^T P e(t)
\]

where \(F\) is a design \((m \times m)\)-matrix of full rank, \(r(t) = y(t) - \hat{y}(t)\) is the observer residual.

From Eqs. (6) and (8) the observer state error dynamics can be obtained as follows:

\[
\dot{e}(t) = A_0 e(t) - [\delta_0 + k_0 \| y(t) \| B(B^T PB)^{-1} s(t) - \|s(t)\|]
\]
\[
+ B \xi_0(t, x(t))
\]

where \(A_0 = A_0 - GC\) is a stable matrix.

### 2.1 Sliding conditions

Now, in different from above mentioned observer design approaches we want first to organize on the switching surface \(s(t) = 0\) (14) a sliding mode. For this purpose let us select a Lyapunov V-function candidate as:

\[
V(s(t)) = \frac{1}{2} \|s^T(t) s(t)\|
\]

Then, the time derivative of (16) along observer state error dynamics (15) can be calculated as:

\[
\dot{V}(s) = s^T(t) \dot{s}(t) = s^T(t) B^T P \dot{e}(t) = e^T(t) B B^T P \hat{A}_0 e(t)
\]

\[
- [\delta_0 + k_0 \| y(t) \| (B^T PB)^{-1} s(t) - \|s(t)\|]
\]
\[
+ s^T(t) B^T PB \xi_0(t, e(t))
\]

\[
= \frac{1}{2} e^T(t) (\hat{A}_0 + \hat{A}_0^T P) e(t) - \delta_0 \frac{s^T(t) s(t)}{\|s(t)\|}
\]
\[
- k_0 \| y(t) \| \frac{s^T(t) s(t)}{\|s(t)\|} + s^T(t) B^T PB \xi_0(t, x(t))
\]

\[
\leq \frac{1}{2} e^T(t) (\hat{A}_0 P + \hat{A}_0^T) e(t) - \delta_0 \|s(t)\| - k_0 \| y(t) \| \|s(t)\|
\]
\[
+ \rho_0 \lambda_{\text{max}}(B^T PB) \|s(t)\| + \beta_0 \lambda_{\text{max}}(B^T PB) \| y(t) \| \|s(t)\|
\]

\[
= -\frac{1}{2} e^T(t) (\hat{A}_0 + \beta_0 \lambda_{\text{max}}(B^T PB)) e(t) - [\delta_0 - \rho_0 \lambda_{\text{max}}(B^T PB)] \|s(t)\|
\]

\[
- [k_0 - \beta_0 \lambda_{\text{max}}(B^T PB)] \| y(t) \| \|s(t)\|
\]
where \( \overline{P} = PBB^T P \) is a positive semi-definite matrix satisfying the following Lyapunov matrix equation:

\[
\overline{P} A_0 + \overline{A}_0 \overline{P} = -\overline{Q}_0; \quad \overline{Q} \geq 0; \quad \lambda_{\text{max}}(\overline{Q}) = 0
\]

where \( \overline{Q}_0 \) is in general a positive semi-definite matrix.

Thus, if we select the design parameters \( \delta_0 \) and \( k_0 \) as

\[
\delta_0 \geq \rho_0 \lambda_{\text{max}}(B^T PB)
\]

\[
k_0 = \beta_0 \lambda_{\text{max}}(B^T PB)
\]

then (17) can be evaluated as:

\[
\dot{V} \leq -\frac{1}{2} e^T(t) \overline{Q}_0 e(t) - [\delta_0 - \rho_0 \lambda_{\text{max}}(B^T PB)] || s(t) ||
\]

\[
\leq -[\delta_0 - \rho_0 \lambda_{\text{max}}(B^T PB)] || s(t) || < 0
\]

since \( \lambda_{\text{max}}(\overline{Q}) = 0 \). Therefore, we conclude that if the sliding conditions (18), (19), and (20) are satisfied, then on \( s(t) = 0 \) (14) always is generated a robustly asymptotically stable sliding mode.

### 2.2 Global stability conditions

The next step is to derive the global robust asymptotical stability conditions with respect to the observer state error coordinates.

Choose Lyapunov V-function candidate as

\[
V(e) = \frac{1}{2} e^T(t) Pe(t)
\]

where \( P = P^T > 0 \) (22)

The time derivative of (22) along the observer state error dynamics (15) is given by

\[
\dot{V}(e(t)) = e^T(t) Pe(t) = \frac{1}{2} e^T(t) (A^T P + PA) e(t)
\]

\[
+ e^T(t) PBv + e^T(t) PB \overline{Q}_0 (t, x(t))
\]

\[
= -\frac{1}{2} e^T(t) \overline{Q}_0 e(t)
\]

\[
- [\delta_0 + k_0 || y(t) ||] \frac{e^T(t) PB(B^T PB)^{-1} s(t)}{|| s(t) ||}
\]

\[
+ e^T(t) PB \overline{Q}_0 (t, x(t)) \leq -\frac{1}{2} e^T(t) \overline{Q}_0 e(t)
\]

\[
- [\delta_0 + k_0 || y(t) ||] \frac{s^T(t)(B^T PB)^{-1} s(t)}{|| s(t) ||}
\]

\[
+ \rho_0 || s(t) || + \beta_0 || y(t) || || s(t) || \leq -\frac{1}{2} e^T(t) \overline{Q}_0 e(t)
\]

\[
- [k_0 \lambda_{\text{max}}(B^T PB)^{-1} - \beta_0] || y(t) || || s(t) ||
\]

Then (23) reduces to

\[
\dot{V}(e(t)) \leq -\frac{1}{2} \lambda_{\text{max}}(\overline{Q}_0) || e(t) ||^2 < 0 \quad \text{for} \quad e(t) \neq 0
\]

Therefore, we conclude that if stability conditions (24), (25), (26) are satisfied then observer state error dynamics is robustly globally asymptotically stable, i.e. \( e(t) \) asymptotically converges to zero as \( t \to \infty \).

Note that, since \( \lambda_{\text{max}}(B^T PB) = 1/\lambda_{\text{min}}(B^T PB)^{-1} \) where \( B^T PB \) is a positive definite matrix then the sliding and stability conditions coincide.

### 2.3 Simplified design example for SISO systems

In this subsection finally, let us consider a reduced design of continuous sliding mode observer for the nominal time-invariant SISO systems when \( \Delta A = 0 \), \( \Delta B = 0 \), and \( f_0 = 0 \) as a simple analytical design example.

Then (1) reduces to:

\[
\dot{x}(t) = Ax(t) + bu(t)
\]

\[
y(t) = c^T x(t)
\]

where \( x(t) \in \mathbb{R}^n \) is the unmeasurable state vector, \( u(t) \) is the scalar control input. The measured output \( y(t) \) is scalar. \( A, b \) and \( c \) have the appropriate dimensions.

For this case observer configuration can be selected as follows:

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + \hat{b}u(t)
\]

\[
\dot{\hat{y}}(t) = c^T \hat{x}(t)
\]

where \( c^T = b^T P \) and \( P \) is a positive definite matrix.

It is assumed that the pair \((A, c)\) is completely observable and structural constraint holds:

\[
f c^T = b^T P
\]

Then, the observer-sliding surface can be defined as:

\[
s(t) = fr(t) = f c^T e(t) = b^T Pe(t)
\]

Subtracting (29) from (28) we have the following observer error system:
\[\dot{e}(t) = Ae(t) + b\nu(t)\] (32)

Now, let us select the observer control term according to equivalent control method:
\[\dot{s}(t) = b^TP\dot{e}(t) = b^TPAe(t) + b^TPbv(t) = 0\] (33)

Hence
\[v(t) = v_{eq}(t) = -(b^TPb)^{-1}b^TPAe(t) = -g s(t)\] (34)

where \((b^TPb)^{-1}\) is a positive definite scalar because \((b^TPb) > 0; g\) is a gain scalar.

Substituting (34) into (32) we have observer error system as:
\[\dot{e}(t) = \bar{A}e(t)\] (35)

where \(\bar{A} = [A - b(b^TPb)^{-1}Je_1 A]\). Our goal is to design the parameters \(f\) and \(P\) such that \(\bar{A}\) always is stable or observer error dynamics (35) is globally asymptotically stable.

Now, let us derive the sliding conditions.

Choose a Lyapunov \(V\)-function candidate as follows:
\[V(s(t)) = \frac{1}{2}s^2(t)\] (36)

Then, the time derivative of (36) along (35) is given by:
\[\dot{V}(s(t)) = \dot{s}(t)\dot{s}(t) = s(t) b^TP\dot{e}(t) = s(t) b^TP\bar{A}e(t) = e^T(t)(b^TPb)^{-1}b^TP\bar{A}e(t) = \frac{1}{2}e^T(t)(\bar{P} \bar{A} + \bar{A}^T\bar{P})e(t) = -\frac{1}{2}e^T(t)\bar{Q}e(t) \leq 0\] (37)

where \(\bar{P} = Pbb^TP\) is a positive semi-definite matrix satisfying the following Lyapunov matrix equation:
\[\bar{P}\bar{A} + \bar{A}^T\bar{P} = -\bar{Q}, \quad \bar{Q} \succeq 0\] (38)

where \(\bar{Q}\) is a positive semi-definite matrix.

Thus if (38) is satisfied then on the sliding manifold \(s(t) = 0\) (31) always is generated an asymptotically stable sliding mode.

**Remark 1.** A simpler alternative to (30), (38), sliding conditions which provides a positive-definite solution to Lyapunov matrix equation, can be formulated as:
\[f^T\bar{A}r = r f^T, \quad r < 0\] (39)

where \(r\) is one of the left eigenvalues of the stable matrix \(\bar{A}\) corresponding to the eigenvector \(fc\).

Then (37) becomes
\[\dot{V}(s(t)) = s(t)\dot{s}(t) = s(t) b^TP\bar{A}e(t) = s(t) f^T\bar{A}e(t) = s(t) r f^T e(t) = rs^2(t) < 0\] (40)

And the global asymptotical stability conditions with respect to the observer error state coordinates:
\[P\bar{A} + \bar{A}^T P = -Q, \quad Q = Q^T > 0\] (41)

easily follows from Lyapunov function
\[V(e(t)) = \frac{1}{2}e^T(t)Pe(t)\] (42)

where \(P\) is a positive definite solution of Eq. (41).

Hence
\[\dot{V}(e(t)) = -\frac{1}{2}e^T(t)Qe(t) \leq -\frac{1}{2}\lambda_{\max}(Q)\|e(t)\|^2 < 0\] (43)

Therefore, an asymptotically stable sliding surface is determined through \(f\) and \(P\).

### III. MAIN RESULTS: DESIGN MODIFICATION OF SLIDING MODE OBSERVERS FOR UNCERTAIN MIMO SYSTEMS WITH TIME-DELAY

The purpose of this section is to extend the design techniques advanced in section 2 for the design modification of sliding mode observers for uncertain MIMO systems with time-delay. Direct extension is difficult as well. We overcome these difficulties by using the Lyapunov- Krasovskii \(V\)-functional method.

Consider the uncertain time-delay MIMO system described by the following differential equations with time-delay:
\[\dot{x}(t) = (A_0 + \Delta A_0) x(t) + (A_1 + \Delta A_1) x(t - \tau) + (B + \Delta B)u(t) + f_0(t, x(t)) + f_1(t, x(t - \tau)), \quad t > 0\]

\[x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad y(t) = Cx(t)\] (44)

where the unmeasurable state vector \(x(t)R^n\), the control input \(u(t)R^p\), the measurable output \(y(t)R^q\) and the unknown disturbances \(f_0(R^p), f_1(R^q)\) are vectors, and \(A_0 \in R^{m \times m}, A_1 \in R^{m \times m}, B \in R^{m \times n}, C \in R^{q \times m}\) are known constant matrices with \(m = p < n\). The matrices \(\Delta A_0, \Delta A_1, \Delta B\) are real valued unknown functions representing time-varying parameter uncertainties, \(\tau\) is a known positive time-delay and \(\phi(t)\) is a continuous vector-value initial function with \(\|\phi\| = \sup\|\phi(t)\| \text{ on } -\tau \leq t \leq 0\) and \(x(0) = \phi(0) = x_0\).

We want to design a sliding mode observer modification for uncertain MIMO systems with time-delay such that in which can always be generated a robustly asymptotically stable sliding mode.
In addition to assumptions (2), (4), (7) we now make the following assumptions:

**Assumption 1.** The nominal system of (44) is detectable (Pearson and Fiagbedzi) [24]:

$$\text{rank} \begin{bmatrix} sI - A_0 - A_1 e^{-\tau} \\ C \end{bmatrix} = n$$  \hspace{1cm} (45)

for all complex $s$ with $\text{Re}(s) \geq 0$.

**Assumption 2.** There exist the functions $h_1$ and $d_1$ such that the following conventional matching conditions are satisfied:

$$f_1(t, x(t-\tau)) = Bh(t, x(t-\tau))$$
$$\Delta \xi(t, x(t-\tau)) = Bd_1(t, x(t-\tau))$$

Let $\xi_1(t, x(t-\tau)) = h_1(t, x(t-\tau)) + d_1(t, x(t-\tau))$

It is assumed that

$$f_1(t, x(t-\tau)) = B\xi_1(t, x(t-\tau))$$  \hspace{1cm} (46)

where the function $\xi_1$ is unknown but bounded, so that

$$||\xi_1(t, x(t-\tau))|| \leq \rho_1 + \beta_1 ||y(t-\tau)||$$

\leq \rho_1 + \beta_1 \sqrt{\lambda_{\max}(C^T C)} ||x(t-\tau)||$$  \hspace{1cm} (48)

where $\rho_1$ and $\beta_1$ are the known positive constant scalars.

Then, time-delay system (44) can be represented as

$$\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t) + B\xi_1(t, x(t-\tau)) + B\xi_1(t, x(t-\tau)) \\
\dot{x}(t) &= C \dot{x}(t)
\end{align*}$$

$$\begin{align*}
x(t) &= \phi(t), \quad -\tau \leq t \leq 0 \\
y(t) &= Cx(t)
\end{align*}$$

Now let us construct a new modification of time-delay observer as:

$$\begin{align*}
\dot{\hat{x}}(t) &= A_0 \hat{x}(t) + A_1 \hat{x}(t-\tau) + Bu(t) + G[y(t) - C\hat{x}(t)] - Bv_1 \\
\hat{y}(t) &= C \hat{x}(t)
\end{align*}$$  \hspace{1cm} (50)

The observer design parameters should be determined so that an asymptotically stable sliding mode will be generated on the sliding surface $s(t) = 0$ (14) defined for time-delay system (52) if the following conditions are satisfied:

$$\begin{align*}
\bar{Q} &= -[ A_0^T \bar{P} + \bar{P} A_0 + \bar{R}] \geq 0, \quad \bar{P} = PBB^T P \geq 0
\end{align*}$$  \hspace{1cm} (53)

or

$$0 \leq \bar{R} \leq -[ A_0^T \bar{P} + \bar{P} A_0]$$

$$\begin{align*}
\delta_1 > (\rho_0 + \rho_1)\lambda_{\max}(B^T PB) \\
k_0 &= \beta_0 \lambda_{\max}(B^T PB) \\
k_1 &= \beta_1 \lambda_{\max}(B^T PB)
\end{align*}$$  \hspace{1cm} (54)

where $\bar{Q}$ and $\bar{H}$ are any positive semi-definite matrices.

**Proof.** Choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(t, \theta) = s^T(t) s(t) + \int_{t-\tau}^{t} e^T(\theta) \bar{R} e(\theta) d\theta > 0$$  \hspace{1cm} (58)
where $\overline{R} = \overline{R}^T \geq 0$ is a positive semi-definite matrix to be selected.

The time derivative of (58) along (52) can be calculated as follows:

$$
\dot{V}(t) = 2s^T(t) \dot{s}(t) + e^T(t) \overline{R} e(t) - e^T(t - \tau)\overline{R} e(t - \tau)
$$

$$
= 2e^T(t) PBB^T P \dot{A}_0 e(t) + 2e^T(t) PBB^T P A_1 e(t - \tau)
- 2[\delta_1 + k_0] ||y(t)||
+ k_1 ||y(t-\tau)||
\frac{s(t)}{s(t)}
+ 2s^T(t) B^T P B \overline{\xi}_0(t, z(t))
+ e^T(t) \overline{R} e(t) - e^T(t - \tau)\overline{R} e(t - \tau)
$$

Rearranging (59) similar to (17) advanced in section 2, we get:

$$
\dot{V}(t) \leq e^T(t) \left[ \overline{A}_0 \overline{P} + \overline{P} \overline{A}_0 + \overline{R} \right] e(t) + 2e^T(t) \overline{P} A_1 e(t - \tau)
- e^T(t - \tau) \overline{R} e(t - \tau) - 2[\delta_1] ||s(t)|| - 2k_0 ||y(t)|| ||s(t)||
- 2k_1 ||y(t-\tau)|| ||s(t)|| + 2[\beta_0 \lambda_{\max}(B^T PB)] ||y(t)|| ||s(t)||
+ 2[\beta_1 \lambda_{\max}(B^T PB)] ||y(t-\tau)|| ||s(t)||
= \begin{bmatrix}
\overline{Q}_1 & -\overline{P} A_1 \\
-\overline{A}_1 \overline{P} & \overline{R}
\end{bmatrix}
\begin{bmatrix}
e(t)
e(t-\tau)
\end{bmatrix}
$$

$$
\leq -2[\delta_1 - (\rho_0 + \rho_1) \lambda_{\max}(B^T PB)] ||s(t)||
- 2[k_0 - \beta_0 \lambda_{\max}(B^T PB)] ||y(t)|| ||s(t)||
- 2[k_1 - \beta_1 \lambda_{\max}(B^T PB)] ||y(t-\tau)|| ||s(t)||
$$

If the conditions (53)-(57) hold, then (60) can be evaluated as

$$
\dot{V}(e(t), e(t-\tau)) \leq -e^T(t) \overline{H} e(t) + e^T(t-\tau) \overline{H} e(t-\tau)
- 2[\delta_1 - (\rho_0 + \rho_1) \lambda_{\max}(B^T PB)] ||s(t)||
\leq -2[\delta_1 - (\rho_0 + \rho_1) \lambda_{\max}(B^T PB)] ||s(t)|| < 0
$$

Since $\lambda_{\min}(\overline{H}) = 0$ and

$$
\begin{bmatrix}
e(t)
e(t-\tau)
\end{bmatrix}
\overline{H}
\begin{bmatrix}
e(t)
e(t-\tau)
\end{bmatrix} \leq 0.
$$

Therefore, we conclude that an asymptotically stable sliding motion always is generated on the sliding surface $s(t) = 0$ (14). Although, it should be noted that, as shown by Hong [41] if $V > 0$ and even $V \leq 0$ then a state-delayed system is asymptotically stable also.

### 3.2 Global stability conditions

The following theorem summarizes our stability results.

**Theorem 1.** Suppose that Assumptions 1,2 and the conditions of Lemma 1 are met. Then the time-delay observer error system (52) is robustly globally asymptotically stable if there exist some positive definite matrices $P, R$ and positive constant scalars $\delta_1, k_0$ and $k_1$ such that the following conditions are satisfied:

$$
Q_1 = -\left[ A_0^T P + P A_0 + R \right] > 0
$$

$$
H = \begin{bmatrix}
Q_1 & -P A_1 \\
-A_1^T P & R
\end{bmatrix} > 0
$$

$$
\delta_1 \lambda_{\max}(B^T PB)^{-1} \geq \rho_0 + \rho
$$

$$
\lambda_{\min}(B^T PB)^{-1} = \beta_0
$$

$$
\lambda_{\min}(B^T PB)^{-1} = \beta_1
$$

**Proof.** Choose a Lyapunov-Krasovskii $V$-functional candidate as:

$$
V(t, 0) = e^T(t) Pe(t) + \int_{t-\tau}^{t} e^T(0) Re(0) d\theta > 0
$$

where, $P$ and $R = R^T > 0$ are any positive definite symmetric matrices to be selected.

The time derivative of (67) along the trajectory of observer error system (52) can be calculated and rearranged as follows:

$$
\dot{V}(e(t), e(t-\tau)) = \dot{e}^T(t) Pe(t) + \dot{e}^T(t) Pe(t) + \dot{e}^T(t) Re(t)
- e^T(t - \tau) Re(t - \tau) = e^T(t) \left[ A_0^T P + P A_0 + R \right] e(t)
+ 2e^T(t) P A_1 e(t - \tau) - e^T(t - \tau) Re(t - \tau)
- 2[\delta_1 + k_0] ||y(t)|| ||s(t)|| + 2[\beta_0 \lambda_{\max}(B^T PB)] ||y(t)|| ||s(t)||
+ 2[\beta_1 \lambda_{\max}(B^T PB)] ||y(t-\tau)|| ||s(t)||
$$

$$
\leq -e^T(t) \overline{H} e(t) + e^T(t-\tau) \overline{H} e(t-\tau)
- 2[\delta_1 - (\rho_0 + \rho_1) \lambda_{\max}(B^T PB)] ||s(t)||
\leq -2[\delta_1 - (\rho_0 + \rho_1) \lambda_{\max}(B^T PB)] ||s(t)|| < 0
$$

Therefore, we conclude that an asymptotically stable
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\[-2[\delta_i + k_0 || y(t)|| + k_1 || y(t-\tau)||] || s^T(t) (B^T PB)^{-1} s(t) || \]
\[+ 2s^T(t) \xi_0(t, x(t)) + 2s^T(t) \xi_0(t, x(t-\tau)) \leq - \begin{bmatrix} e(t) \end{bmatrix}^T \begin{bmatrix} e(t) \end{bmatrix} \leq 0 \text{ since } H > 0. \quad (68) \]

Therefore, we conclude that the time-delay observer error system (52) is robustly globally asymptotically stable.

Note that, the sliding and stability conditions are coordinated very well.

IV. OBSERVER DESIGN EXAMPLE FOR AV-8A HARRIER VTOL AIRCRAFT

To illustrate the design modification of a combined time-delay observer, let us consider the observer design example for fault-tolerant control of AV-8A Harrier VTOL aircraft in hovering flight. The nominal parameters of this aircraft is taken from (Calise and Kramer) [42]:

\[\dot{x}(t) = A_0 x(t) + B u(t)\]
\[y(t) = C x(t)\]

where, the state vector is represented by \(x = [\psi \phi \upsilon r p]\), \(\psi\) is the Euler yaw attitude perturbation (rad), \(\phi\) is the Euler roll attitude perturbation (rad), \(\upsilon\) is the velocity perturbation along body y axis (m/s), \(r\) is the body-axis yaw rate (rad/s), \(p\) is the body-axis roll rate (rad/s),

the control inputs are \(u = [\delta_{lat} \delta_{rdd}]\),

\(\delta_{lat}\) is the lateral stick perturbation (cm),
\(\delta_{rdd}\) is the rudder pedal perturbation (cm),

and the system, control and output matrices are given by:

\[
A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\
0 & 9.8 & -0.042 & 0 & 0 \\
0 & 0 & -0.007 & -0.06 & -0.075 \\
0 & 0 & -0.039 & 0.11 & -0.260 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & -0.27 \\
0.0055 & 0.085 \\
0.177 & -0.033 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \end{bmatrix}
\]

For the simulation, the parameter perturbations are selected as follows:

\[\Delta A_0 = 0.2 \sin(t), \Delta A_1 = 0.2 \cos(t), A_4 = 0.3 A_0\]

Aircraft model really has some small time-delay because of pilot’s (or commands) effective time delay (Blakelock [43]) and transports delays of aircraft mechanical and hydraulic servomechanisms. For the simulation purpose we select \(\tau = 0.24s\).

Simplified design procedures for time delay observer (50) and (51) with given parameters can be fulfilled by the following steps:

Find the eigenvalues of matrix \(A_0\):

\[\text{Eig}(A_0) = 0; 0.2715 \pm 0.6239i; -0.8253; -0.0798\]

\(A_0\) is unstable matrix.

Using pole placement Matlab command find gain matrix \(G\) for \(A_0\) and \(C\) such that \(\overline{A_0}\) has some desirable left eigen values:

\[\lambda = [-2.4 -3 -3 + 2i -3 -2i -3.4]\]

\[C = [1 \ 1 \ 0 \ 1 \ 1; 1 \ 1 \ 0 \ 1]\]

\[G = \text{PLACE} (A_0^T, C^T, \lambda)\]

Calculate

\[\overline{A_0} = A_0 - G \ast C\]

\[
\overline{A_0} = \begin{bmatrix} 4.6324 & 4.6324 & 3.1242 & 2.5081 & 4.6324 \\
-2.0361 & -2.0361 & -2.9437 & 0.9076 & -1.0361 \\
-8.1754 & 1.6246 & -5.7264 & -2.4910 & -8.1754 \\
-5.0635 & -5.0635 & 3.1870 & -8.3176 & -5.1385 \\
-3.0923 & -3.0923 & -3.0749 & 0.0536 & -3.3523 \end{bmatrix}
\]
\[
eig(\mathbf{A}_0) = -3.0000 + 2.0000i, -3.0000 - 2.0000i, \\
-2.4000, -3.4000, -3.0000.
\]
which is a stable matrix.

Solve Lyapunov Eq. (62) for \(P\):
\[
Q = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1];
\]
where \(Q = Q_1 + R = I_5\); \(P = \text{LYAP}(\mathbf{A}_0, Q)\)
\[
P = \begin{bmatrix}
1.9024 & -0.0668 & -0.1028 & -1.0030 & -0.6566 \\
-0.0668 & 0.2177 & -0.0055 & 0.1363 & 0.0822 \\
-1.1028 & 0.2177 & 1.4484 & 1.2771 & 0.2407 \\
-0.1028 & -0.0055 & 1.2771 & 1.3908 & -0.2392 \\
-0.0668 & 0.0822 & -0.2407 & -0.2392 & 0.8960
\end{bmatrix}
\]
\[
eig(P) = 0.0712, 0.0906, 0.2763, 1.5689, 3.8483.
\]
which is a positive definite matrix.

\[
\begin{bmatrix}
0.0697 & 0.0047 & 0.0600 & 0.0470 & 0.0156 \\
0.0047 & 0.0004 & 0.0030 & 0.0023 & 0.0020 \\
0.0600 & 0.0030 & 0.0767 & 0.0613 & -0.0097 \\
0.0470 & 0.0023 & 0.0613 & 0.0490 & -0.0088 \\
0.0156 & 0.0020 & -0.0097 & 0.0088 & 0.0250
\end{bmatrix}
\]
\[
eig(P) = 0.1790, 0.0416, 0.0000 + 0.0000i, 0.0000.
\]
which is a positive semi-definite matrix.

The conditions (53) and (62) are independent. Equation (62) has a positive definite solution \(P\). Then (53) always holds because \(0 \leq R \leq -[\mathbf{A}_0 P + P \mathbf{A}_0]\).

Calculate \(B^T PB = \begin{bmatrix} 0.0276 & 0.0015 \\ 0.0015 & 0.0550 \end{bmatrix}\)
\[
eig(B^T PB) = 0.0276, 0.0551;
\]
which is a positive definite matrix.

\[
B^T P = \begin{bmatrix}
-0.1217 & 0.0138 & -0.0356 & -0.0347 & 0.1573 \\
0.2342 & -0.0128 & -0.2746 & -0.2187 & 0.0151
\end{bmatrix}
\]
\[
(B^T PB)^{-1} = \begin{bmatrix}
36.2214 & -0.9658 \\
-0.9658 & 18.1922
\end{bmatrix}
\]
\[
eig((B^T PB)^{-1}) = 36.2730, 18.1406
\]
Select a matrix \(F\) such that condition (7) holds:
\[
F = \begin{bmatrix}
-0.0347 & -0.0356 \\
-0.2187 & -0.2746
\end{bmatrix}; \quad F^{-1} = \begin{bmatrix}
-157.5535 & 20.4257 \\
125.4805 & -19.9093
\end{bmatrix}
\]
Select a matrix \(H_0\) such that a matching condition for external disturbance holds:
\[
D = BH_0 = \begin{bmatrix}
0 & 0 \\
0 & -0.27
\end{bmatrix};
\]
\[
D = \begin{bmatrix}
0.0055 & 0.085 \\
0.177 & -0.033
\end{bmatrix}
\]
\[
ρ_0 = 0.0639.
\]
\[
β_0 = \max_{\sigma} \| ΔA \| = 1.96, β_1 = \max_{\sigma} \| ΔA \| = 0.588
\]
Find from (64), (65), (66):
\[
\delta_1 ≥ 0.0035, k_0 = 0.1080, k_1 = 0.0324
\]
Thus, all parameters of time-delay observer are designed.

For testing the combined time-delay observer (50), (51), and (14) is simulated. Block diagram of which is shown in Fig. 1. For the convenience of simulation the time-delay system model is taken as:
\[
\dot{x}(t) = (A_0 + ΔA_0)x(t) + (A_1 + ΔA_1)x(t - \tau) + Bu(t) + DF(t)
\]
(69)
where \(D\) is the \((n \times n)\)-matrix, \(f(t)\) is a norm bounded \(n\)-vector disturbance \(||f(t)|| \leq f_0\). Equation (69) can easily be transformed to the form of time-delay system (49). Simulation results using MATLAB-SIMULINK are shown in Fig. 2-7 (for original closed-loop system) and Fig. 8-12 (for original unstable open-loop system). As seen from these figures, the combined time-delay observer estimates the state vector satisfactorily (observer residual is satisfactorily small) which show the effectiveness of our observer design approaches.

V. CONCLUSION

In this paper, we have presented two contributions. One of which is to advance a sliding mode observer design techniques for uncertain MIMO and SISO systems, such that on the switching surface can always be generated robustly asymptotically stable sliding mode. The main result is to design a new modification of sliding mode time-
delay observer for uncertain time-delay MIMO systems with parameter perturbations and external disturbances by using Lyapunov-Krasovskii $V$-functional method. Robust stable sliding and global asymptotical stability conditions are obtained and formulated in terms of Lyapunov matrix equations and some matrix inequalities. Design example for AV-8A Harrier VTOL aircraft with simulation results show the effectiveness of our observer design approaches.

Fig. 1. Block Diagram of multivariable sliding mode observer for uncertain time-delay system with parameter perturbations and external disturbances.

Fig. 2. Original state responses.

Fig. 3. Estimated state responses.
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