DELAY-DEPENDENT ROBUST CONTROL OF DESCRIPTOR SYSTEMS WITH TIME DELAY

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ABSTRACT

This paper deals with the problem of stability and robust control for both certain and uncertain continuous-time singular systems with state delay. Systems with norm-bounded parameter uncertainties are considered. Robust delay-dependent stability criteria and linear memoryless state feedback controllers based on linear matrix inequality are obtained. By choosing some Lyapunov-Krasovskii functionals, neither model transformation nor bounding for cross terms is required in the derivation of our delay-dependent results. Finally, numerical example is provided to illustrate the effectiveness of the proposed method.

KeyWords: Delay-dependent criteria, linear matrix inequality, continuous singular systems, time-delay systems, linear memoryless state feedback controller.

1. INTRODUCTION

It is well known that the existence of a delay in a dynamical system may induce instability or poor performances in various systems such as electric, pneumatic, and hydraulic networks, chemical processes, long transmission lines, etc. For a survey of time-delay systems, the reader can refer to a recent overview paper [1]. Control of singular systems has been extensively studied in the past years due to the fact that singular systems better describe physical systems than regular ones. Singular systems are also referred to as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems or semi-state systems. A great number of results based on the theory of regular systems have been extended to the area of singular systems (see [2,3] and the references therein). However, to the best of the authors’ knowledge, the delay-dependent stability and robust control of singular continuous-time systems with time-delay have received little attention so far, which remains open and unsolved. The delay-dependent stability problem for singular systems is much more complicated than that for regular systems because it requires to consider not only stability, but also regularity and absence of impulses (for continuous singular systems) and causality (for discrete singular systems) at the same time. The conditions based on which we can guarantee that the system under consideration is regular, impulse free can be found in [3,4-7] and newly publications [8-10].

Recently, in Ref. [2], by decomposing the systems into slow and fast subsystems, the authors proposed a delay-dependent method for stabilization via state feedback for singular linear continuous-time systems. However, the results obtained in [2] are based on some model transformation technique and the assumption that the slow state $x_1(t)$ and the fast state $x_2(t)$ satisfy $\| x_2(t) \|^2 \leq x_1^T(t) M x_1(t)$. We have to estimate the matrix $M$ before designing a controller, and the choice of $M$ would have a significant influence on the performance of the controller, an improper choice of $M$ would induce poor performance of the controller. It is noted that the stability and stabilization conditions were obtained in Ref. [2] were only applicable to singular delay systems without parameter uncertainty. When parameter uncertainty appears, the problems of robust stability and stabilization were considered in Refs. [3,5,11]. Contrast with [2], by introducing the notions of generalized quadratic stability and generalized quadratic stabilization, the stability and stabilization problems are solved in [3] without involving decomposition of the coefficient matrices of
the system, respectively. These results were extended to robust $H_{\infty}$ control for uncertain singular systems with state delay in [5]. However, all the above results are delay-independent. In [6,7], the authors developed delay-dependent stability and $H_{\infty}$ control criteria for singular systems with time delay and polytopic parameter uncertainty. Our goal in this paper is to deal with the delay-dependent stability and robust control of singular continuous time systems with state delay and norm bounded parameter uncertainties. The problem under consideration is then to design a state feedback controller such that, for all admissible uncertainties, the resulting closed-loop system is regular, impulse free and stable, while the closed-loop transfer function from the disturbance to the controlled output meets a prescribed $H_{\infty}$ norm bound constraint.

There are various approaches to reduce the conservatism of delay-dependent conditions. It is well known that existing delay-dependent methods commonly require some kind of model transformation and bounding for cross terms. However, all these techniques entail a considerable conservatism which stems from two main sources. The first cause for conservatism is the model transformation used to describe the system which makes it more amenable for analysis [12-15]. The second reason for conservatism is the bounding method used to derive the bounds on weighted cross products of the state and its delayed version while trying to secure a negative value to the derivative of the corresponding Lyapunov-Krasovskii functional [15,16]. Recently, in [16] and [17] by applying some new methods, neither model transformation nor bounding for cross terms is required to obtain delay-dependent results. In Ref. [16], the authors obtained their results by taking a new Lyapunov-Krasovskii functional and the results turn out to be less conservative than those obtained by model transformation and bounding for cross terms. In [17], the authors devised a new method of dealing with the problem of robust delay-dependent/dependency-stable stability of linear systems with a time-varying delay and polytopic-type uncertainties. In their method, the derivative terms of the state, which is in the derivative of the Lyapunov-Krasovskii functional, are retained and, contrast to bounding for cross terms method, some free weighting matrices are used to express the relationships among the system variables. In this paper, by introducing a certain Lyapunov-Krasovskii functional similarly to that in [16], neither model transformation nor bounding for cross-terms is required in the derivation of our results. In terms of linear matrix inequalities, delay-dependent sufficient conditions under which the systems are stable and satisfy the $H_{\infty}$-norm bound condition are developed for both nominal and uncertain systems. Based on the stability issue, we propose delay-dependent stabilization methods. In order to solve those nonlinear matrix inequalities appear in the controller design, we introduce an iterative algorithm which is presented in the Appendix of this note as well as the methods we use to deal with the LMEs appear in the theorems via Matlab. Finally, numerical example is given to demonstrate the effectiveness of the proposed method.

**Notation.** $R^n$ denotes the $n$-dimensional Euclidean space, $R^{m \times n}$ is the set of $n \times m$ real matrices. $I$ is the identity matrix of appropriate. $\| A \|$ stands for the norm of matrix $A$. $C_0$ denotes the set of all continuous functions from $[-\tau, 0]$ to $R^n$. The space of functions in $R^q$ that are square integrable over $[0, \infty)$ is denoted by $L^2_q[0, \infty)$ with the norm $\| \cdot \|_{L^2_q}$. $Q > 0$ ($Q \geq 0$) stands for $Q$ is a symmetric and positive-definite (positive semi-definite) matrix.

**II. PROBLEM STATEMENT AND PRELIMINARIES**

Consider the singular system with time delay and feedback control law described by

$$
E \dot{x}(t) = (A + \Delta A(t)) x(t) + (A_d + \Delta A_d(t)) x(t-\tau) \\
+ (B + \Delta B(t)) u(t) + B_w w(t) \tag{1}
$$

$$
x(t) = \phi(t), \ t \in [-\tau, 0] \tag{2}
$$

$$
u(t) = K x(t) \tag{3}
$$

$$
z(t) = C x(t) + D u(t) \tag{4}
$$

where $x(t) \in R^n$ is the system state vector, $u(t) \in R^m$ is the control input vector, $w(t) \in L^2_q[0, \infty)$ is the exogenous disturbance signal, $\phi(t) \in C_0$ is the initial condition, $z(t) \in R^q$ is the controlled output. $A$, $A_d \in R^{n \times n}$ are known constant matrices. The matrix $E \in R^{n \times m}$ may be singular, and we assume that rank $E = r \leq n$. $B \in R^{m \times n}$ is a known constant matrix. $C$ and $D$ are known matrices of appropriate dimensions, and $K$ is the unknown design matrix. $\tau \geq 0$ is an unknown but constant delay and $\tilde{\tau}$ is a constant satisfying $\tau \leq \tilde{\tau}$. $\Delta A(t)$, $\Delta A_d(t)$, and $\Delta B(t)$ denote time-varying parameter uncertainties.

**Definition 2.1.** The parameter uncertainties $\Delta A(t)$, $\Delta A_d(t)$, and $\Delta B(t)$ are said to be admissible if both (5) and (6) as described below hold:

$$
[\Delta A(t) \Delta A_d(t) \Delta B(t)] = G F(t) [N_a \ N_d \ N_b] \tag{5}
$$

where $G$, $N_a$, $N_d$, and $N_b$ are known real constant matrices of appropriate dimensions. $F(t)$ is an unknown time-varying matrix with Lebesgue measurable elements bounded by

$$
F^T(t)F(t) \leq I, \ \ \forall t \geq 0 \tag{6}
$$

**Definition 2.2.** The descriptor system (1) with $w(t) = 0$ is said to be regular and impulse free if $(E, A + \Delta A(t))$ is regular and impulse free.
The aim of this note is to develop a delay-dependent robust $H_{\infty}$ control law such that, for all admissible uncertainties and any constant time-delay $\tau$ satisfying $\tau \leq \hat{\tau}$,

1. the closed-loop system is stable;
2. the closed-loop system guarantees, under zero initial condition, $\|z(t)\|_2 \leq \gamma \|w(t)\|_2$ for all nonzero and $w_2 \in L_2[0, \infty)$ some prescribed constant $\gamma > 0$.

In order to handle the regular and impulse free and parameter uncertainties, we introduce the following lemma.

**Lemma 2.1.** [18] Given matrices $Y$, $H$ and $E_1$ of appropriate dimensions and with $Y$ symmetric, then the following matrix inequality $Y + HFE_1 + E_1^TF^TH^T < 0$ holds for all $F$ satisfying $F^TF \leq I$, if and only if there exists a scalar $\zeta > 0$ such that

$$Y + \zeta^{-1}H^TH + \zeta E_1^TE_1 < 0$$

(7)

**Lemma 2.2.** [8] Suppose that piecewise continuous real square matrices $A(t)$, $X$ and $Q > 0$ satisfy $A^T(t)X + X^TA(t) + Q < 0$ for all $t$. Then, the following hold.

1. $A(t)$ and $X$ are invertible.
2. $\|A'(t)\| \leq \delta$ for some $\delta > 0$.

### III. $H_{\infty}$ CONTROL FOR NOMINAL SYSTEM

Let us consider the nominal system with state delay:

$$E \dot{x}(t) = A x(t) + A_d x(t-\tau) + B u(t) + B_u w(t),$$

$$z(t) = C x(t) + D u(t)$$

(8)

with initial condition

$$x(t) = \phi(t), \quad t \in [-h, 0]$$

For some prescribed $\gamma > 0$, we denote the $H_{\infty}$ norm boundedness of the transfer function from $w$ to $z$, $T_{wz}$, by $\|T_{wz}\|_\infty < \gamma$. However, throughout this paper, we prefer the notation $\|z(t)\|_2 \leq \gamma \|w(t)\|_2$, $\forall w(t) \in L_2[0, \infty)$ because the $H_{\infty}$ norm is equal to the induced 2-norm in the time domain. For system without delay the condition for $\|T_{wz}\|_\infty < \gamma$ is stated in the well-known bounded real lemma, which is a necessary and sufficient condition. However, the bounded real lemma we propose here for singular system with state delay is only a sufficient condition.

**Theorem 3.1.** Given $\gamma > 0$ and $\hat{\tau} > 0$, if there exist matrices $Q > 0$, $Z > 0$, symmetric matrix $X$, matrices $Y$ and $X_1$ of appropriate dimensions, such that

$$E^T X_1 = X_1^T E \geq 0$$

(9)

$$\begin{pmatrix} X & Y \\ \ast & Z \end{pmatrix} \geq 0$$

(10)

$$\begin{pmatrix} I_0 & X_1^T A_d - YE & X_1^T B_u & \ast & A^T Z \\ \ast & -Q & 0 & \ast & A_1^T Z \\ \ast & \ast & -\gamma^2 I & \ast & B_1^T Z \\ \ast & \ast & \ast & -\hat{\tau} Z \end{pmatrix} < 0$$

(11)

then the unforced system (8), namely, with $u(t) \equiv 0$ is regular, impulse free, stable and satisfies the $H_{\infty}$ norm bound condition $\|z(t)\|_2 < \gamma \|w(t)\|$ for any time-delay in the interval $[0, \hat{\tau}]$, where $L_0 = A^T X_1 + X_1^T A + Q + \hat{\tau} X + YE + E^T Y^T + C^T C$ and $\ast$ denotes the terms that are induced by symmetry.

**Proof.** Following [16], [13] and [14], we consider the following index:

$$J_{wz} = \int_0^\infty \|z^T(t)z(t) - \gamma^2 w^T(t)w(t)\| dt$$

(12)

and the Lyapunov-Krasovskii functional as follows:

$$V(x(t)) = V_1(x_1) + V_2(x_2) + V_3(x_3) + V_4(x_4)$$

where

$$V_1(x_1) = x_1^T(t) E_1^T X_1 x(t)$$

$$V_2(x_2) = \int_{-\hat{\tau}}^0 \int_{\tau < t \leq \hat{\tau}} x_2^T(t) E_2^T Z E \dot{x}(\alpha) d\alpha d\beta$$

$$V_3(x_3) = \int_{-\hat{\tau}}^0 x_3^T(t) Q x(t) d\alpha$$

$$V_4(x_4) = \int_0^\beta \int_{\beta - \tau}^\beta \left( x(\beta) \right)^T \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix}^T \left( x(\beta) \right) d\alpha d\beta$$

(13)

with $E^T X_1 = X_1^T E \geq 0$, $Q > 0$ and

$$\begin{pmatrix} X & Y \\ \ast & Z \end{pmatrix} \geq 0 .$$

Hence, differentiating of (13) along the trajectory of (8) with respect to $u(t) \equiv 0$ gives us,

$$\dot{V}_1(x_1) = (E \dot{x}(t))^T X_1 x(t) + x_1^T(t) E_1^T X_1 \dot{x}(t)$$

$$+ (Ax(t) + A_d x(t-\tau) + B_u w(t))^T X_1 x(t)$$

$$+ x_1^T(t) X_1^T (Ax(t) + A_d x(t-\tau) + B_u w(t))$$

$$\dot{V}_2(x_2) = \tau x_2^T(t) E_2^T Z E \dot{x}(t) - \int_{-\hat{\tau}}^0 x_2^T(s) E_2^T Z E \dot{x}(s) ds$$

$$\dot{V}_3(x_3) = x_3^T(t) Q x(t) - x_3^T(t-\tau) Q x(t-\tau)$$

$$\dot{V}_4(x_4) = \tau x_4^T(t) X x(t) + \int_{-\hat{\tau}}^\beta \int_{\beta - \tau}^\beta x_4^T(s) E \dot{x}(s) ds$$

Thus

$$\dot{J}_{wz} = \int_0^\infty \left( \tau x_4^T(t) X x(t) + \tau x_2^T(t) E_2^T Z E \dot{x}(t) - \gamma^2 \|w(t)\|^2 \right) dt$$

$$+ \int_{-\hat{\tau}}^\beta \int_{\beta - \tau}^\beta x_4^T(s) E \dot{x}(s) ds$$

$$+ \int_{-\hat{\tau}}^\beta \int_{\beta - \tau}^\beta x_2^T(s) E_2^T Z E \dot{x}(s) ds$$

$$- \gamma^2 \int_0^\infty \|w(t)\|^2 dt.$$
\[ \dot{V}(x) = x^T(t) (A^T X_1 + X_1^T A + \tau X + Q + Y E + E^T Y^T) x(t) \]
\[ - x^T(t - \tau) Q x(t - \tau) + 2 x^T(t) (X_1^T A_d - YE) x(t - \tau) \]
\[ + \tau x^T(t) E^T Z E x(t) + 2 x^T(t) X_1^T B_w x(t) \]

Note that \( J_{\infty} \) can be represented as
\[ J_{\infty} = \int_0^\infty [z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(x)] \, dt \]
\[ + V(x) \bigg|_{t=0} - V(x) \bigg|_{t=\infty} \tag{14} \]

Since \( V(x) \bigg|_{t=0} = 0 \) under zero initial condition and \( V(x) \bigg|_{t=\infty} \geq 0 \), it follows that
\[ J_{\infty} \leq \int_0^\infty [z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(x)] \, dt \]
\[ = \int_0^\infty \begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix}^T \begin{bmatrix} L_0 & X_1^T A_d - YE & X_1^T B_w \\ * & -Q & 0 \\ * & * & -\gamma I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \\ w(t) \end{bmatrix} \, dt \]

Define \( \Phi = \begin{bmatrix} L_0 & X_1^T A_d - YE & X_1^T B_w & \hat{\tau} A^T Z \\ * & -Q & 0 & \hat{\tau} A_d^T Z \\ * & * & -\gamma I & \hat{\tau} B_w^T Z \\ * & * & * & -\hat{\tau} Z \end{bmatrix} \)

According to Schur complement, \( \Phi < 0 \) implies that \( J_{\infty} < 0 \). In case of \( w(t) = 0 \), consider the following matrix inequality:
\[ \begin{bmatrix} L_0 & X_1^T A_d - YE \\ * & -Q \end{bmatrix} < 0 \tag{15} \]

which guarantees \( \dot{V}(x) < 0 \), where \( L_0 = A^T X_1 + X_1^T A + Q + \hat{\tau} X + Y E + E^T Y^T \). Noting that the LMI (11) is feasible implies the LMI (15) is also feasible. Thus, system (8) is stable. Next we are in position to prove that the system is regular and impulse free. To simplify the proof, we note that LMI (15) implies that
\[ \begin{bmatrix} L_0 & X_1^T A_d - YE \\ * & -Q \end{bmatrix} < 0 \tag{16} \]

From (16) and using Lemma 2.2, we can conclude that \( X_1 \) is invertible. Pre-multiplying \( \text{diag}\{X_1^{-1}, X_1^{-1}\} \) and post-multiplying \( \text{diag}\{X_1^{-1}, X_1^{-1}\} \) to (16) yields
\[ \begin{bmatrix} X_1^{-T} L_0 & X_1^{-T} A_d - X_1^{-T} Y E X_1^{-1} \\ * & -X_1^{-T} Q X_1^{-1} \end{bmatrix} < 0 \tag{17} \]

Let \( T = X_1^{-1}, \hat{A} = A + T^T Y E, \hat{A}_d = A_d - T^T Y E, \)
\[ \hat{Q} = X_1^{-T} Q X_1^{-1}, \hat{X} = X_1^{-T} X X_1^{-1}, \]
yields
\[ (T^T \hat{A} + \hat{A} T + \hat{Q} + \hat{X} \hat{A}_d T - \hat{Q}) < 0 \tag{18} \]

Noting that (10) implies that \( X > 0 \), thus \( \hat{Q} = \hat{Q} + \hat{X} > 0 \).

The proof of the system is regular and impulse free is then reduced to a similar format of the proof of Corollary 1 (LMI (23) in [8]) and Theorem 1 (LMI (21) in [8]), one can prove the system is regular and impulse free under the conditions of Theorem 3.1 by making full use of the method in [8]. It is therefore we conclude that the system is regular and impulse free and omit the detail proof here.

**Theorem 3.2.** For some prescribed \( \gamma > 0 \) and \( \hat{\tau} > 0 \), if there exist matrices \( R_1 > 0, R > 0, \) symmetric matrix \( P, R_2 \), and matrices \( R_4, W \) of appropriate dimensions, such that
\[ PE^T = EP \geq 0 \tag{19} \]
\[ \begin{bmatrix} R_2 & R_3 \\ * & PR^{-1} P \end{bmatrix} \geq 0 \tag{20} \]
\[ \begin{bmatrix} L_4 & A_d P - R_3 E^T B_w & (CP + DW)^T \hat{X} (AP + BW)^T \\ * & -R_3 & 0 & 0 & \hat{X} A_d^T \\ * & * & -\gamma I & 0 & \hat{X} B_w^T \\ * & * & * & -I & 0 \\ * & * & * & * & -R \end{bmatrix} < 0 \tag{21} \]
then the system with controller \( u(t) = W P^{-1} x(t) \) is regular, impulse free, stable and satisfies the \( H_\infty \) norm bound condition \( \|z(t)\| \leq \gamma \|w(t)\| \) for any time-delay in the interval \([0, \hat{\tau}]\), where \( L_4 = (AP + BW)^T + R_1 + \hat{\tau} R_2 + R_1 E^T + R_1 (R_1 E^T)^T \) and \( W = KP \).

**Remark 3.1.** Although the iterative LMI algorithm we use to solve our numerical examples gives a more approving suboptimal solution than one obtained by setting \( P = R \) in (20). However, as will be shown in the Appendix, it requires that \( P \) is positive-definite, whereas this is not necessary to Theorem 3.2 and Theorem 4.2 in section 4. That is to say the iterative LMI algorithm we used in this note induce conservatism not only when it is applied to replace the original non-convex feasibility problem of Theorem 3.2 and Theorem 4.2, but also the attached condition \( P > 0 \).

**Proof.** As indicated in the Theorem 3.1 and Lemma 2.2, we can conclude that the matrix \( P \) is invertible. Pre- and post-multiplying \( P^{-1} \) to (19), we have
\[ E^T P^{-1} = P^{-1} E \tag{22} \]

Now, pre- and post-multiplying \( \text{diag}\{P^{-1}, P^{-1}, I, I, R^{-1}\} \) to
stable and satisfies the \( \gamma \) and \( \varepsilon \) and unforced system under consideration satisfies robust stabil-
ity and provides a sufficient condition to determine whether the problem for uncertain systems. The following theorem
completes the proof.

In light of Schur complement, (25) is equivalent to

\[
\begin{pmatrix}
L_3 & X_1^T A_d - YE & X_1^T B_w & \hat{t} A_d Z
\\
* & -Q & 0 & 0 & \hat{t} A_d Z
\\
* & * & -\gamma^2 I & \hat{t} B_w Z & < 0
\\
* & * & * & -I & 0
\\
* & * & * & -\hat{t} Z & 0
\end{pmatrix}
\]

\( \text{(26)} \)

where \( L_3 = A_d^T X_1 + X_1^T A_d + Q + \hat{t} X + Y E + E^T Y^T + C_c^T C_c \).

Pre- and post-multiplying diag \( \{P^1, P^1\} \) to (20) yield

\[
\begin{pmatrix}
X & Y \\
* & Z
\end{pmatrix} \geq 0.
\]

According to Theorem 3.1, we conclude that the system with controller \( u(t) = WP^1 x(t) \) is regular, impulse free, stable and satisfies the \( H_\infty \) norm bound condition \( \| z(t) \| < \gamma \| w(t) \| \) for any time-delay in the interval \([0, \hat{t}]\). This completes the proof.

\section*{IV. ROBUST \( H_\infty \) CONTROL FOR UNCERTAIN SYSTEMS}

In this section, we consider the robust \( H_\infty \) control problem for uncertain systems. The following theorem provides a sufficient condition to determine whether the unforced system under consideration satisfies robust stability and \( H_\infty \) norm bound constraint.

**Theorem 4.1.** Given \( \gamma > 0 \) and \( \hat{t} > 0 \), if there exist matrices \( Q > 0 \), \( Z > 0 \) symmetric matrix \( X \), matrices \( Y \) and \( X_1 \) of appropriate dimensions, a scalar \( \zeta > 0 \) such that (9), (10) and

\[
\begin{pmatrix}
P^1 L_1 P^1 & P^1 (A_d P - R_1 E^T) P^1 & P^1 B_w & P^1 (C P + D W)^T & \hat{t} P^1 (A P + B W)^T
\\
* & -P^1 R_1 P^1 & 0 & 0 & \hat{t} A_d^T R_1
\\
* & * & -\gamma^2 I & 0 & \hat{t} b_w^T R_1
\\
* & * & * & -I & 0
\\
* & * & * & -\hat{t} R_1 & < 0
\end{pmatrix}
\]

\( \text{(23)} \)

are feasible. Then the unforced system (1) with \( u(t) = 0 \) is regular, impulse free, robustly stable and satisfies the \( H_\infty \) norm bound condition \( \| z(t) \| < \gamma \| w(t) \| \) for any time-delay in the interval \([0, \hat{t}]\) and all admissible uncertainties.

**Proof.** Replace \( A \) and \( A_d \) in (11) with \( A + GF(t) N_a \) and \( A_d + GF(t) N_d \), respectively. Thus (11) is then changed into

\[
\begin{pmatrix}
X_1^T G \\
0
\end{pmatrix} = \begin{pmatrix}
F(t) & N_a & N_d & 0 & 0
\\
\hat{t} Z G
\end{pmatrix}
\]

\( \text{(28)} \)

It follows from Lemma 2.1 that (28) holds for all admissible uncertainties if and only if there exists a scalar \( \zeta > 0 \) such that

\[
\begin{pmatrix}
X_1^T G \\
0
\end{pmatrix}^T = \begin{pmatrix}
0 & 0
\\
\hat{t} Z G
\end{pmatrix}
\]

\( \text{(29)} \)

Applying Schur complement to (29), we obtain (27), which completes the proof.
Theorem 4.2. For some prescribed \( \gamma > 0 \) and \( \hat{\tau} > 0 \), if there exit symmetric matrices \( R_1 > 0 \), \( R_2 > 0 \), symmetric matrix \( P \), \( R_3 \), matrices \( R_4 \), \( W \) of appropriate dimensions and a scalar \( \zeta > 0 \), such that (19), (20) and

\[
\begin{bmatrix}
L_t + \zeta GG^T & A_jP - R_tE & B_w & (CP + DW)^T \tau (AP + BW)^T + \zeta GG^T & (N_jP + N_jW)^T \\
\cdot & -R_t & 0 & 0 & 0 \\
\cdot & \cdot & -\gamma^2 I & 0 & 0 \\
\cdot & \cdot & \cdot & -I & 0 \\
\cdot & \cdot & \cdot & \cdot & 0 \\
\end{bmatrix}
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{bmatrix}
\begin{bmatrix}
\tau P A_j^T & 0 \\
\hat{\tau} B_w^T & 0 \\
0 & 0 \\
-\zeta R & -\zeta I \\
0 & 0 \\
\end{bmatrix}
< 0
\]  

(30)

then the system (1) with the controller \( u(t) = WP^{-1}x(t) \) is regular, impulse free, stable and satisfies the \( H_\infty \) norm bound condition \( \|z(t)\|_2 < \gamma \|w(t)\|_2 \) for all admissible uncertainties and any time-delay in the interval \([0, \hat{\tau}]\).

Proof. As in the proof of Theorem 4.1, replace \( A, A_d \) and \( B \) in (21) with \( A + GF(t)N_r, A_d + GF(t)N_d \) and \( B + GF(t)N_r \), respectively. Define the left side of (21) to be \( \Omega \), then (21) is changed into

\[
\Omega + \zeta^{-1}
\begin{bmatrix}
(G) \\
0 \\
F(t) \\
\hat{\zeta}G
\end{bmatrix}
\begin{bmatrix}
(N_jP + N_jW)^T \\
(N_dP)^T \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
G^T \\
F^T(t) \\
\hat{\zeta}G
\end{bmatrix}
< 0
\]  

(31)

In light of Lemma 2.1, (31) holds for all admissible uncertainties if and only if there exists a scalar \( \zeta > 0 \) such that

\[
\begin{bmatrix}
(G) \\
0 \\
0 \\
0 \\
\hat{\zeta}G
\end{bmatrix}
< 0
\]  

(32)

In the sense of Schur complement, (32) is equivalent to (30). This completes the proof.

V. NUMERICAL EXAMPLE

We consider the following system.

\[
E = \begin{bmatrix} 1 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\
1 & -2 \\
\end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\
0.1 & -0.1 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix} 2 \\
1 \\
\end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\
-0.2 \\
\end{bmatrix},
\]

\[
C = (-1, 2), \quad \hat{\tau} = 6.41, \quad \gamma = 0.35
\]

(33)

Suppose the parameter uncertainties are given below

\[
\Delta A(t) = \Delta A_d(t) = \begin{bmatrix} r \sin \theta & r \sin \theta \\
0 & 0 \\
\end{bmatrix}
\]

and

\[
\Delta B(t) = \begin{bmatrix} 0 \\
q \sin \theta \\
\end{bmatrix}
\]

where \( r, q \leq 0.1 \). We represent the uncertainties as

\[
G = \begin{bmatrix} 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix},
\]

\[
N_r = \begin{bmatrix} 0.1 & 0.1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad N_h = \begin{bmatrix} 0 \\
0.1 & 0 \\
0 & 0.1 \\
0 & 0 \\
\end{bmatrix}, \quad N_d = \begin{bmatrix} 0 & 0 \\
0 & 0.3528 \\
\end{bmatrix}
\]

(34)

By applying Theorem 4.2, more exactly the iterative LMI algorithm presented in the Appendix, we obtain the following controller

\[
K = (-0.8190, 0.1765) \quad \text{with} \quad P = \begin{bmatrix} 0.2794 & 0 \\
0 & 0.3528 \\
\end{bmatrix}
\]

after 5 iterations. To verify that the system is stabilized by the controller, we simulate it via Matlab and the result is illustrated by Fig. 1.
VI. CONCLUSION

In this paper, the robust stability and stabilization problem of singular linear continuous time systems with a state delay and parameter uncertainties have been studied. Firstly, a delay-dependent real bounded lemma has been developed for nominal systems in terms of linear matrix inequalities, based on which a memoryless state feedback stabilization control law is presented in terms of nonlinear matrix equalities. Secondly, we extend our results for nominal systems to systems with parameter uncertainties. Since no model transformation or bounding for cross terms is used, our method is less conservative than some existing ones, which have been demonstrated by numerical examples and Matlab based simulation. Our result can be extended to the class of uncertain linear singular systems with time-varying delay.

REFERENCES

APPENDIX
THE ITERATIVE LMI ALGORITHM FOR
THIS NOTE [16,19]

We introduce this algorithm here to deal with the nonlinear term $PR^{-1}P$ in (20).

First, we define a new variable $S$ such that $PR^{-1}P > S$ and replace the constraint (20) with

$$\begin{pmatrix} R_2 & R_3 \\ * & S \end{pmatrix} \geq 0, \quad PR^{-1}P > S \quad (A1)$$

Note that $PR^{-1}P > S$ is equivalent to $P^{-1}RP^{-1} \prec S^{-1}$, thus condition (A1) is equivalent to, in light of Schur complement,

$$\begin{pmatrix} R_2 & R_3 \\ * & S \end{pmatrix} \geq 0, \quad \begin{pmatrix} S^{-1} & P^{-1} \\ * & R^{-1} \end{pmatrix} > 0 \quad (A2)$$

Let $T = S^{-1}, T_1 = P^{-1}$ and $T_2 = R^{-1}$, then condition (A2) can be represented as

$$\begin{pmatrix} R_2 & R_3 \\ * & S \end{pmatrix} \geq 0, \quad \begin{pmatrix} T & T_1 \\ * & T_2 \end{pmatrix} > 0 \quad (A3)$$

As in [16] and [19], we can find a suboptimal maximum delay for some prescribed $\gamma$ or can find a suboptimal minimum $\gamma$ for some prescribed $\hat{\tau}$ relatively easily using an iterative algorithm presented below.

Algorithm.

1. Choose a sufficiently small initial $\hat{\tau} > 0$ (Choose a sufficiently large initial $\gamma > 0$). Set $\hat{\tau}_{SO} = \hat{\tau}$ (Set $\gamma_{SO} = \gamma$).

2. Find a feasible set $(P_0, R_0, R_0^2, W_0, S_0, T_0, T_0^2)$ satisfying (A3). Set $k = 0$.

3. Solve the following LMI problem for the variables $(P, R, R_1, R_2, W, S, T, T_1, T_2)$

Minimize

$$\text{Tr} (S^2 \hat{T} + T^3 S + P^3 \hat{T}_1 + \hat{T}_1^3 P + R^3 \hat{T}_2 + \hat{T}_2^3 R)$$

Subject to (A3).

Set $S^{k+1} = S, T^{k+1} = T, P^{k+1} = P, T_1^{k+1} = T_1, T_2^{k+1} = T_2$ and $R^{k+1} = R$.

4. If condition (20) is satisfied, then set $\hat{\tau}_{SO} = \hat{\tau}$ (Set $\gamma_{SO} = \gamma$) and return to step 2 after increasing $\hat{\tau}$ (decreasing $\gamma$) to some extent. If condition (20) is not satisfied within a specified number of iterations, say $k_{\text{max}}$, then exist. Otherwise, set, $k = k + 1$ and go to step 3.

Using the above algorithm, we can obtain a suboptimal maximum of $\hat{\tau}$ (a suboptimal minimum of $\gamma$).

Remark. Since the LMI control toolbox in Matlab deals with strict inequality only, (19), referred to as “linear matrix equality” (LME), cannot be handled by Matlab LMI toolbox directly. In order to solve the conditions presented in theorem 3.2 via Matlab, we need to perform some preparation:

(P1) We note that the regularity and the absence of impulses of the pair $(E, A)$ implies that there exist two invertible matrices $M$ and $N \in \mathcal{R}^{n \times n}$ such that

$$\bar{E} \triangleq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} \triangleq \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{R}^{n \times n}$$

where $I_r \in \mathcal{R}^{r \times r}$ is identity matrix. Then we obtain
the following equivalent system:

\[ E \ddot{x}(t) = A \ddot{x}(t) + \tilde{A}_d \dot{x}(t - \tau) + B u(t) + B_w w(t) \tag{A4} \]

where \( \tilde{A}_d \triangleq MA_N \), \( \tilde{B} = MB \), \( \tilde{B}_w = MB_w \), \( \tilde{C} = CN \), and \( x(t) = N \bar{x}(t) \).

(P2) We now replace \( P \) in the original conditions with \( \bar{P} \).

As it is required in (19) that \( EE = \bar{P} E' \geq 0 \), noting that

\[ P = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \]

therefore we conclude that \( \bar{P} \)

must of the form \( \bar{P} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \) with \( R_1 > 0 \). As in

(A3), it is required that \( \bar{P} > 0 \). Thus, the condition (19) reduced to search a matrix \( \bar{P} > 0 \) of the form \( \bar{P} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \). We denote the corresponding state feedback controller obtained by

\[ u(t) = K \bar{P}^{-1} \bar{x}(t) \]

(P3) The \( H_\infty \) state feedback controller of the original system is then given by

\[ u(t) = K \bar{P}^{-1} N^{-1} x(t) \]

We can handle condition (9) in Theorem 3.1 in a similar manner. Suppose

\[ \bar{X}_1 \triangleq M X_r (N^{-1})^T = \begin{pmatrix} \bar{X}_{11} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{pmatrix} \]

Noting the expression of \( E \) and using condition (9), we can deduce that \( \bar{X}_{11} = \bar{X}_{11}^T \geq 0 \) and \( \bar{X}_{12} = 0 \), therefore condition (9) reduces to search a matrix \( \bar{X}_1 \) of the form

\[ \bar{X}_1 = \begin{pmatrix} \bar{X}_{11} & 0 \\ \bar{X}_{21} & \bar{X}_{22} \end{pmatrix} \] with \( \bar{X}_{11} \geq 0 \).