STABILITY ANALYSIS OF UNCERTAIN DISCRETE-TIME SYSTEMS WITH TIME-VARYING STATE DELAY: A PARAMETER-DEPENDENT LYAPUNOV FUNCTION APPROACH

Huijun Gao, James Lam, Tongwen Chen, and Changhong Wang

ABSTRACT

This paper presents several new robust stability conditions for linear discrete-time systems with polytopic parameter uncertainties and time-varying delay in the state. These stability criteria, derived by defining parameter-dependent Lyapunov functions, are not only dependent on the maximum and minimum delay bounds, but also dependent on uncertain parameters in the sense that different Lyapunov functions are used for the entire uncertainty domain. It is established, theoretically, that these robust stability criteria for the nominal and constant-delay case encompass some existing result as their special case. The delay-dependent and parameter-dependent nature of these results guarantees the proposed robust stability criteria to be potentially less conservative.

KeyWords: Discrete-time systems, networked control systems, robust stability, time-delay systems.

I. INTRODUCTION

Stability of linear systems with uncertain parameters has been well recognized to be a fundamental problem due to its important role in analysis and synthesis of such systems. During the past few decades, the quadratic stability notion has been widely used for robust analysis and synthesis for uncertain linear systems. The quadratic stability idea, however, may lead to sometimes conservative results due to the fact that a fixed Lyapunov function is used for the entire uncertainty domain. In recent years, to improve the conservativeness of the quadratic stability based results, parameter-dependent Lyapunov functions have been introduced for testing robust stability as well as performance of uncertain systems. Very recently, some interesting LMI-based conditions using parameter-dependent Lyapunov functions to evaluate the robust stability of systems with parameter uncertainties residing in a polytope have been reported. One may cite [1] for continuous-time systems and [3] for discrete-time systems. The idea behind these results is to decouple the product terms between the Lyapunov matrix and system matrices by introducing additional slack matrix variables, such that parameter-dependent Lyapunov functions can be obtained when testing robust stability for polytopic uncertain systems. It is noted that although being
able to provide less conservative testing than quadratic approaches, these criteria still impose some common matrices for the entire uncertainty domain in order to obtain parameter-dependent Lyapunov functions.

On another research front, time-delay systems have continued as a hot research area over the last few decades [2,5,9,11,13]. One of the fundamental problems concerning time-delay systems is their stability. For systems with time delay in the state and parameter uncertainties residing in a polytope, an advanced research topic is to develop robust delay-dependent and parameter-dependent stability conditions. As is mentioned above, the notion of parameter-dependence is introduced in order to overcome the conservativeness of the quadratic stability condition. Very recently, the parameter-dependent idea was further extended to time-delay systems, and some less conservative robust stability conditions have been proposed [4,6,12]. The techniques used to realize the parameter-dependent ideas for time-delay systems in these papers are quite similar to those in [3,10]. That is, by introducing one or more additional slack matrix variables (without any structural restriction), the product terms between the Lyapunov matrices (positive matrices) and system matrices are eliminated, thus the Lyapunov matrices are exempt from being fixed for the entire uncertainty polytope. It is worth mentioning that all these results are concerned with continuous-time systems, and to the authors' knowledge, no robust delay-dependent and parameter-dependent parameter-dependent stability condition has been reported for uncertain discrete time-delay systems.

In this paper, the authors investigate the robust stability for linear discrete-time systems with time-varying delay in the state and parameter uncertainties residing in a polytope. Since the state delay is assumed to be time-varying, with known minimum and maximum delay bounds, it is not possible to transform the system into a delay-free system and then use well-established stability conditions. Therefore, the Lyapunov-Krasovskii approach is adopted to solve the problem concerned. By defining a parameter-dependent Lyapunov-Krasovskii Function, several LMI based robust stability criteria are obtained, which are not only dependent on the maximum and minimum delay bounds, but also dependent on the uncertain parameters in the sense that different Lyapunov Functions are used for the entire uncertainty polytopic domain. It is established, theoretically, that these robust stability criteria for the nominal and constant-delay cases encompass some existing result as their special case. Numerical examples are provided to show the advantage of the proposed stability conditions.

Notations: The notations used throughout the paper are fairly standard. The superscript “$^T$” stands for matrix transposition; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; the notation $P > 0$ means that $P$ is real symmetric and positive definite; $I$ and 0 represent identity matrix and zero matrix. In symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry, and for a matrix $A$, sym($A$) denotes $A + A^T$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. For a matrix $U \in \mathbb{R}^{m \times m}$ with rank $k$, we denote $U^\perp$ as the orthogonal complement, which is defined as a (possibly nonunique) $(m - k) \times m$ matrix with rank $(m - k)$, such that $U^\perp U = 0$.

II. PROBLEM DESCRIPTION

Consider the following discrete-time system $\Sigma$ with a time-varying delay in the state:

$$
\begin{align*}
\Sigma: & \quad x(k+1) = A_x x(k) + B_x x(k-d(k)) \\
& \quad x(k) = \phi(k), \quad k = -d_M, -d_M+1, \ldots, 0
\end{align*}
$$

where $x(k) \in \mathbb{R}^n$ is the state vector, and $d(k)$ is a time-varying state delay satisfying the following assumption:

Assumption 1. The time delay $d(k)$ is assumed to be time-varying and satisfy $1 \leq d_n = d(k) \leq d_M$, where $d_n$ and $d_M$ are constant positive scalars representing the minimum and maximum delays, respectively.

In system $\Sigma$, $\{\phi(k), k = -d_M, -d_M+1, \ldots, 0\}$ is a given initial condition sequence. $A_x, B_x$ represent uncertain system matrices belonging to a given convex polytope $\mathcal{R}$, that is, matrices $A_x, B_x$ can be written in the following form:

$$
[A_x \ B_x] = \sum_{i=1}^\ell \lambda_i [A_i \ B_i]
$$

where $1 \leq \lambda \leq [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_\ell]^T$ denotes an uncertain vector satisfying

$$
\sum_{i=1}^\ell \lambda_i = 1, \quad \lambda_i \geq 0
$$

The appropriately dimensioned matrix $[A_i \ B_i]$ denotes the $i$th vertex of the polytope $\mathcal{R}$. Throughout the paper, $\tau$ denotes $d_M - d_n + 1$.

Remark 1. The assumption about the time delay $d(k)$ in Assumption 1 characterizes a real situation in many practical applications. A typical example containing time delays that can be characterized by Assumption 1 can be found in a networked control system, where the delays induced by the network transmission (either from the sensor to the controller or from the controller to the actuator) are actually time-varying, and can be assumed to have minimum and maximum delay bounds without loss of generality. It is worth noting that by assuming $d_n = d_M = d$, the time-varying delay $d(k)$ reduces to a constant delay $d$, which has been widely studied in a vast amount of literature. For the constant delay case, since the delay $d$ is ex-
actly known a priori, it is possible to transform the delay system to a delay-free system by using state augmentation techniques. However, such treatment is not possible for the time-varying delay case.

In this paper, the researchers are interested in the robust stability analysis of system $\Sigma$ in (1) with parameter uncertainty (2) and time-varying delay $d(k)$ satisfying Assumption 1.

III. MAIN RESULTS

The following theorem presents a new robust stability analysis result based on parameter-dependent Lyapunov functions.

**Theorem 1.** System $\Sigma$ in (1) with parameter uncertainty (2) and time-varying delay $d(k)$ satisfying Assumption 1 is robustly asymptotically stable if there exist matrices $P_\lambda > 0$, $Q_\lambda > 0$, $M_\lambda > 0$, $X_\lambda$, $Y_\lambda$, $Z_\lambda$, $R$, $S$, $T$ satisfying

$$
\begin{bmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & -X_i \\
* & \Pi_{22} & \Pi_{23} & -Y_i \\
* & * & \Pi_{33} & -Z_i \\
* & * & * & -d_{kl}^{-1} M_i
\end{bmatrix} < 0, \quad (i = 1, \ldots, s) \quad (4)
$$

where

$$
\Pi_{11} \triangleq \tau Q + \text{sym}(X_i - R(A_1 - I))
$$

$$
\Pi_{12} \triangleq -X_i + Y_i^T - RB_i - (A_1 - I)^T S^T
$$

$$
\Pi_{22} \triangleq -Q_i - \text{sym}(Y_i + S B_i)
$$

$$
\Pi_{13} \triangleq \bar{P} + Z_i^T + R - (A_1 - I)^T T^T
$$

$$
\Pi_{23} \triangleq -Z_i^T + S - B_i^T T^T
$$

$$
\Pi_{33} \triangleq P_i + d_{kl} M_i + T + T^T
$$

**Proof.** Define the following Lyapunov Function which is dependent on the uncertain parameter $\lambda$:

$$
V(k, \lambda) \triangleq V_1 + V_2 + V_3 + V_4 \quad (5)
$$

where

$$
V_1 \triangleq x^T(k) P_\lambda x(k),
$$

$$
V_2 \triangleq \sum_{i=0}^{k-1} x^T(i) Q_\lambda x(i),
$$

$$
V_3 \triangleq -\sum_{i=0}^{k-1} \sum_{j=0}^{d_{kl}} x^T(i) Q_\lambda x(i),
$$

$$
V_4 \triangleq \sum_{i=0}^{k-1} \sum_{j=0}^{d_{kl}} \delta^T(m) M_\lambda \delta(m),
$$

$$
\delta(k) \triangleq x(k+1) - x(k),
$$

$$
P_\lambda \triangleq \sum_{i=1}^{s} \lambda_i P_i, \quad Q_\lambda \triangleq \sum_{i=1}^{s} \lambda_i Q_i,
$$

$$
M_\lambda \triangleq \sum_{i=1}^{s} \lambda_i M_i \quad (6)
$$

and $P_i > 0, Q_i > 0, M_i > 0$ are matrices to be determined. Define $\Delta V = V(k+1, \lambda) - V(k, \lambda)$, then along the solution of system $\Sigma$ in (1) we have

$$
\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 \leq -\frac{1}{d(k)_{m=k-d(k)}} \Xi(m, \lambda) \quad (7)
$$

where

$$
\Xi(m, \lambda) \triangleq [\delta(k) + x(k)]^T P_\lambda [\delta(k) + x(k)] - x^T(k) P_\lambda x(k) + \tau x^T(k) Q_\lambda (x(k) - x^T(k) (k - d(k)) Q_\lambda x(k - d(k))
$$

$$
+ d_{kl} \delta^T(k) M_\lambda \delta(k) d_{kl} \delta^T(m) M_\lambda \delta(m)
$$

Since we have

$$
x(k - d(k)) = x(k) - \sum_{m=k-d(k)}^{k-1} \delta(m) \quad (8)
$$

Then, for any matrices

$$
X_\lambda \triangleq \sum_{i=1}^{s} \lambda_i X_i, \quad Y_\lambda \triangleq \sum_{i=1}^{s} \lambda_i Y_i, \quad Z_\lambda \triangleq \sum_{i=1}^{s} \lambda_i Z_i \quad (9)
$$

we have

$$
\Lambda_1 \triangleq \frac{1}{d(k)_{m=k-d(k)}} \left[ x^T(k) X_\lambda x(k) + x^T(k - d(k)) Y_\lambda \right.
$$

$$
+ \delta^T(k) Z_\lambda \left[ x(k) - x(k - d(k)) - d_{kl} \delta(m) \right] = 0 \quad (10)
$$

In addition, based on (1), for any matrices $R, S, T$, we have

$$
\Lambda_2 \triangleq \frac{1}{d(k)_{m=k-d(k)}} \left[ x^T(k) R + x^T(k - d(k)) S
$$

$$
+ \delta^T(k) T \left[ \delta(k) - (A_1 - I) x(k) - B_x x(k - d(k)) \right] = 0 \quad (11)
$$

Adding $2\Lambda_1$ in (10) and $2\Lambda_2$ in (11) to (7) yields

$$
\Delta V \leq \frac{1}{d(k)_{m=k-d(k)}} \left[ \eta^T(k, m) \Phi_\lambda \eta(k, m) \right] \quad (12)
$$

where

$$
\eta^T(k, m) \triangleq \left[ x^T(k) \quad x^T(k - d(k)) \quad \delta^T(k) \quad \delta^T(m) \right]
$$
Authors will prove that LMIs (4) guarantee stable for all uncertain parameter \( \lambda \) in (1) with parameter uncertainty (2) and time-varying delay. Then, according to the standard Lyapunov theory, system (9) is robustly asymptotically stable if there exist matrices \( P_\lambda > 0 \) and \( M_\lambda \) such that \( \Delta V < -\epsilon |x(k)|^2 \), then according to the standard Lyapunov theory, system (1) is robustly asymptotically stable for all uncertain parameter \( \lambda \). In the following, the authors will prove that LMIs (4) guarantee \( \Phi_\lambda < 0 \).

First note that \( \Phi_\lambda < 0 \) if

\[
\Phi_\lambda \triangleq \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & -X_\lambda & -d(k)X_\lambda \\
\Phi_{12}^* & \Phi_{22} & \Phi_{23} & -Y_\lambda & -d(k)Y_\lambda \\
\Phi_{13}^* & \Phi_{23}^* & \Phi_{33} & -Z_\lambda & -d(k)Z_\lambda \\
\Phi_{11}^* & \Phi_{22}^* & \Phi_{33}^* & -d(M_\lambda)Y_\lambda \\
\end{bmatrix} < 0
\]

Now, substitute \( P_\lambda, Q_\lambda, M_\lambda \) defined in (6), \( X_\lambda, Y_\lambda, Z_\lambda \) defined in (9) into \( \Psi_\lambda \). We know that \( \Psi_\lambda < 0 \) if and only if (4) holds, therefore we have \( \Phi_\lambda < 0 \), and the proof is completed.

Theorem 1 presents a delay-dependent and parameter-dependent robust stability condition for discrete-time systems with time-varying delay in the state and polytopic parameter uncertainties. An important feature of this criterion lies in the fact that the positive definite matrices (Lyapunov matrices) \( P_\lambda, Q_\lambda, M_\lambda \) are not involved in the product terms with the system matrices \( A_\lambda \) and \( B_\lambda \), therefore these matrices are not required to be the same for different vertices of the polytope \( \mathcal{R} \). It is noted that the derivation of this condition is quite different from that of previous ones, identified by the following two aspects:

- No model transformation is performed in order to obtain the delay-dependent condition. It should be noted that in deriving delay-dependent stability and performance conditions, a common approach is to transform the original system into another one using the Newton-Leibniz formula. In this framework, usually one has to employ some bounding techniques to find upper bounds for the inner product between two vectors. These bounding techniques involve some matrix inequalities. Employing these inequalities will inevitably introduce some overdesign into the derived conditions. However, it is worth emphasizing that in the proposed derivation, no system transformation has been performed to the original system thus no inequality is needed to seek the upper bounds of the inner product between two vectors. This feature has the potential to enable one to obtain less conservative results.

- The parameter-dependent idea is realized by introducing the three slack matrix variables \( R, S, T \), whose relationships with system \( \Sigma \) in (1) are expressed in (11). By doing so, one does not directly substitute \( x(k+1) \) by \( A_\lambda x(k) + B_\lambda x(k-d(k)) \) for the term \( \Delta V = x(k+1)^T P_\lambda x(k+1) - x(k)^T P_\lambda x(k) \) in (6) as treated in previous references, which allows one to avoid the appearance of product terms between the positive definite matrix \( P_\lambda \) and system matrices \( A_\lambda \) and \( B_\lambda \).

It is worth noting that, although Theorem 1 presents a delay-dependent robust stability condition which is also dependent on uncertain parameters in the sense that a \( \lambda \)-dependent Lyapunov function has been used in the derivation, the slack matrix variables \( R, S, T \) are still required to be fixed for the entire uncertainty domain. This feature is similar to the parameter-dependent stability conditions presented in [3,10] for delay-free systems. In the following, the authors will propose another robust stability condition which does not require any fixed matrix variable for the uncertainty polytope.

**Theorem 2.** System \( \Sigma \) in (1) with parameter uncertainty (2) and time-varying delay \( d(k) \) satisfying Assumption 1 is robustly asymptotically stable if there exist matrices \( P_i > 0, Q_i > 0, M_i \) defined in (9), \( X_i, Y_i, Z_i \), \( R_i, S_i, T_i \) satisfying

\[
\Psi_i - \Psi_{ij} \leq 0, \quad (1 \leq i < j \leq s)
\]

where

\[
\Psi_i \triangleq \begin{bmatrix}
X_i & Y_i^T & R & B_i & -(A_i - I)^T S_i^T & P_i + Z_i^T + R_i - (A_i - I)^T T_i^T & -X_i \\
* & -Q_i - \text{sym} (Y_i + S_i B_i) & -Z_i^T + S_i - B_i^T T_i^T & -Y_i \\
* & * & -d(M_i)Y_i & -Z_i \\
* & * & * & -d(M_i)Y_i
\end{bmatrix}
\]
where $\Xi$ is given in (14). On the other hand, by considering $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$, (13) guarantees $\check{P}_\lambda < 0$, therefore one has $\check{F}_\lambda < 0$, and the proof is completed.

**Remark 2.** An important feature of Theorem 2 lies in the fact that no matrix variable is required to be fixed for the entire uncertainty polytope $\mathcal{R}$, which constitutes the most significant distinction from the quadratic stability and Theorem 1.

**Remark 3.** It is worth mentioning that Theorem 2 encompasses Theorem 1 as a special case. That is, by imposing additional constraints on the matrix variables, Theorem 2 will recover Theorem 1. To this end, let us set $R_i = R$, $S_i = S$, and $T_i = T$, and LMIs (13) as equivalent to condition (4) in Theorem 1.

### IV. CONNECTION WITH AN EXISTING RESULT

In the above section, the authors have developed some parameter-dependent robust stability conditions by adopting new techniques. One may be interested in the following question: can one establish the relationship of these criteria with some existing stability conditions theoretically? Since there seems to be no result for discrete time-delay systems with polytopic uncertainties, the authors will make the comparison for systems without parameter uncertainties. To this end, the system matrices $A_i$, $B_i$ in (1) are assumed to be exactly known, that is,

$$ [A_i, B_i] = [A, B] $$

(18)

It is found that a delay-dependent stability condition for discrete-time systems with constant delay was reported in Theorem 1 of [8]. To facilitate the comparison, this stability condition will be restated in the following proposition.

**Proposition 1.** System $\Sigma$ in (1) with (18) and $d(k) = d$ is asymptotically stable if there exist matrices $P > 0, Q > 0, E, X, M > 0$ satisfying

$$
\begin{bmatrix}
-P & PA & PB & 0 \\
* & dE + X + X^T - P + Q & -X & (A - I)^T M \\
* & * & -Q & B^T M \\
* & * & * & -d^{-1} M
\end{bmatrix} < 0
$$

(19)

$$
\begin{bmatrix}
E & X \\
X^T & M
\end{bmatrix} \succeq 0
$$

(20)

The conditions in both Theorems 1 and 2 of this paper for the constant delay case becomes the following proposition.
Proposition 2. System \( \Sigma \) in (1) with (18) and \( d(k) \equiv d \) is asymptotically stable if there exist matrices \( P > 0, Q > 0, M > 0, X, Y, Z, R, S, T \) satisfying

\[
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & -X \\ * & \Omega_{22} & \Omega_{23} & -Y \\ * & * & \Omega_{33} & -Z \\ * & * & * & -d^{-1}M
\end{bmatrix} < 0
\]

where

\[
\begin{align*}
\Omega_{11} & \triangleq Q + \text{sym} \left( X - R(A-I) \right) \\
\Omega_{12} & \triangleq -X + Y^T - RB - (A-I)^T S^T \\
\Omega_{22} & \triangleq -Q - \text{sym} \left( Y + SB \right) \\
\Omega_{13} & \triangleq P + Z^T + R - (A-I)^T T^T \\
\Omega_{23} & \triangleq -Z^T + S - B^T T^T \\
\Omega_{33} & \triangleq P + dM + T + T^T
\end{align*}
\]

Now, we have the following theorem, which characterizes the relationship between Propositions 1 and 2.

Theorem 3. Proposition 2 implies Proposition 1.

Proof. Rewrite (21) in the following form

\[
W + UG^T Y + (UG^T)^T < 0
\]

where

\[
W = \begin{bmatrix}
Q + X + X^T & -X + Y^T & P + Z^T & -X \\
* & Q - Y - Y^T & -Z^T & -Y \\
* & * & P + dM & -Z \\
* & * & * & -d^{-1}M
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}, \quad G = \begin{bmatrix}
R \\
S \\
T
\end{bmatrix}
\]

\[
V = \begin{bmatrix}
-I & -B & I & 0
\end{bmatrix}
\]

If one selects

\[
U^⊥ = \begin{bmatrix}
0 & 0 & 0 & I
\end{bmatrix}, \quad V^⊥ = \begin{bmatrix}
I & 0 & (A-I)^T & 0 \\
0 & I & B^T & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\]

Then, by using Projection Lemma [7], (21) is solvable for \( G \) if and only if

\[
\begin{bmatrix}
\Theta_{11} & \Theta_{12} & -X - (A-I)^T Z \\
* & \Theta_{22} & -Y - B^T Z \\
* & * & -d^{-1}M
\end{bmatrix} < 0
\]

where

\[
\begin{align*}
\Theta_{11} & \triangleq Q + \text{sym} \left( X - R(A-I) \right) - P + A^T PA + (A-I)^T dM(A-I) \\
\Theta_{12} & \triangleq -X + Y^T - (A-I)^T Z + Z^T B + A^T PB + (A-I)^T dMB \\
\Theta_{22} & \triangleq -Q - \text{sym} \left( Y + B^T Z \right) + B^T PB + B^T dMB
\end{align*}
\]

By introducing three matrix variables \( E, F, G \) with

\[
\begin{bmatrix}
E & F \\
F^T & G
\end{bmatrix} \geq 0 , \quad (22)
\]

\[
(22)
\]

By Schur complement, (23) and (24) are equivalent to

\[
\begin{bmatrix}
\Theta_{11} + dE & \Theta_{12} + dF \\
* & \Theta_{22} + dG
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
-X - (A-I)^T Z \\
-Y - B^T Z
\end{bmatrix} M^{-1} \begin{bmatrix}
-X^T - Z^T (A-I) & -Y^T - Z^T B
\end{bmatrix}
\]

\[
(26)
\]

By selecting

\[
F = 0, \quad Y = 0, \quad Z = 0, \quad G = \epsilon I
\]

then, for a sufficiently small positive \( \epsilon \), (25) and (26) imply (19) and (20) respectively, which means that Proposition 2 implies Proposition 1, and the proof is completed. ■
Remark 4. According to Theorem 3, one knows that Proposition 1 is actually a special case of Proposition 2. In other words, if one can find feasible solutions by Proposition 1, one must also be able to find feasible solutions by Proposition 2, which means that the stability condition derived in this paper for the nominal constant-delay case is no more conservative than the result in [8]. In fact, by trying the first numerical example in [8], that is,

\[
A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.91 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}
\]

The maximum delay size found by Proposition 1 is 41. However, by Proposition 2, the maximum delay size is found to be 42.

V. ILLUSTRATIVE EXAMPLES

In this section, the authors will provide two numerical examples to show the advantage of the proposed parameter-dependent stability conditions.

Example 1. Consider system \( \Sigma \) in (1) with the following matrices:

\[
A = \begin{bmatrix} 0.8 + \alpha & 0 \\ 0 & 0.91 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 - \alpha \end{bmatrix}
\]

where \( \alpha \) is an uncertain parameter satisfying \( |\alpha| \leq \overline{\alpha} \). The purpose is to find the maximum value of \( \overline{\alpha} \) for which the system is robustly asymptotically stable. To facilitate the comparison (since the previous result in [8] cannot deal with the time-varying delay case), the time delay is assumed to be constant, that is, \( d_M = d_m = d \). The calculation results obtained by different approaches for different delay \( d \) are listed in Table 1.

<table>
<thead>
<tr>
<th>( d )</th>
<th>Quadratic Approach based on [8]</th>
<th>Parameter-Dependent Approach (Theorem 3)</th>
<th>Parameter-Dependent Approach (Theorem 12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.1308</td>
<td>0.1900</td>
<td>0.1900</td>
</tr>
<tr>
<td>6</td>
<td>0.1006</td>
<td>0.1635</td>
<td>0.1635</td>
</tr>
<tr>
<td>8</td>
<td>0.0781</td>
<td>0.1138</td>
<td>0.1138</td>
</tr>
<tr>
<td>10</td>
<td>0.0615</td>
<td>0.0838</td>
<td>0.0838</td>
</tr>
<tr>
<td>20</td>
<td>0.0211</td>
<td>0.0255</td>
<td>0.0255</td>
</tr>
</tbody>
</table>

It can be seen from the table that the parameter-dependent stability conditions yield less conservative stability bounds than the quadratic approach as expected. It is also noted that the two parameter-dependent stability conditions (Theorems 1 and 2) yield the same results for this specific example. To further illustrate the advantage of Theorem 2, consider another example.

Example 2. Consider system \( \Sigma \) in (1) with the following matrices:

\[
A_1 = \begin{bmatrix} 0 & 1.9460 & 0 \\ 0 & 0 & 1.9460 \\ 0.0664 & -0.5003 & 1.5118 \end{bmatrix},
A_2 = \begin{bmatrix} 0 & 1.9460 & 0 \\ 0 & 0 & 1.9460 \\ -0.0664 & -0.5003 & -1.5118 \end{bmatrix}
\]

\[
B_1 = B_2 = \begin{bmatrix} 0.0088 & 0.0083 & 0.0071 \\ 0.0061 & 0.0046 & 0.0103 \\ 0.0014 & 0.0018 & 0.0084 \end{bmatrix}
\]

By solving the convex feasibility problem in Theorem 2, one can conclude that the above polytopic uncertain system is robustly asymptotically stable for \( 2 \leq d(k) \leq 5 \). However, under the condition that \( 2 \leq d(k) \leq 5 \), the quadratic approach based on [8] and Theorem 1 all fail to find feasible solutions.

VI. CONCLUSIONS

Some new robust stability criteria in terms of linear matrix inequalities have been proposed for linear discrete-time systems with polytopic parameter uncertainties and time-varying delay in the state. These stability criteria are not only dependent on both the maximum and minimum delay bounds, but also dependent on uncertain parameters in the sense that different Lyapunov Functions are used for the entire uncertainty polytope. The relationship of these stability conditions with some existing ones has been established theoretically. Numerical examples have shown the reduced conservativeness of the proposed stability conditions.

REFERENCES


